Lec 20

* Discrete-time LTI system
  (Linear Time Invariant)
  - impulse response function
* z-transform.

* LTI system

continuous time

$$\underset{\text{input}}{x(t)} \longrightarrow \boxed{\text{circuit}} \longrightarrow \underset{\text{output}}{y(t)}$$

Assume the circuit does not change in time (system)

e.g. RLC is LTI when R, L, C are constant (in time).

$$LC\, y''(t) + RC\, y'(t) + y(t) = x(t)$$

$$\hat{y}(\omega) = \hat{H}(\omega)\hat{x}(\omega)$$

$$y(t) = (H * x)(t) \qquad H: \text{impulse response}$$

* Discrete-time LTI system

$$\underset{\text{Input}}{x[n]} \longrightarrow \boxed{\text{LTI}} \longrightarrow \underset{\text{output}}{y[n]}$$

e.g.

LTI systems

- $y[n] = K\,x[n]$. $\quad K$ constant.
- $y[n] = x[n-m]$ $\quad m$ fixed integer
- $y[n] = x[n] - x[n-1]$
- $y[n] - a\,y[n-1] = x[n]$. $\quad a$ constant.

- $\cancel{y[n] - 2^n y[n-1] = x[n]}$ not LTI (especially, not time-invariant)
- $\cancel{y[n] - y[n-1]^2 = x[n]}$ not LTI (especially, not linear).

	input		output
- linear : (superposition)	$a_1 x_1[n] + a_2 x_2[n] + \cdots$ $a_1, a_2, \cdots$ constants.	$\rightsquigarrow$	$a_1 y_1[n] + a_2 y_2[n] + \cdots$ $y_1[n]$ = output from $x_1[n]$ $y_2[n]$ = " " $x_2[n]$
- Time-invariant.	$x[n-n_0]$	$\rightsquigarrow$	$y[n-n_0]$

e.g.

Input	output
$\delta[n]$	$h[n]$
$\delta[n-m]$	$h[n-m]$

"impulse response function"

Consequence of LTI property.

$h[n]$ is the impulse response.

	Input		output
⊛	$x[n]$	$\rightsquigarrow$	$y[n] = (h * x)[n]$

reason for ⊛

Can write.

$$x[n] = (x * \delta)[n].$$

$$= \sum_{m=-\infty}^{\infty} x[m]\, \delta[n-m]$$

any $x[n]$ can be written as linear combination of $\delta[n-m]$'s.

Then, by linearity. the output is

$$y[n] = \sum_{m=-\infty}^{\infty} \underbrace{x[m]}_{x[m] \text{ is constant in } n.} \; \underbrace{h[n-m]}_{\text{output for } \delta[n-m]}$$

So,

$$y[n] = (x * h)[n]$$

$$= (h * x)[n]. \quad \blacksquare$$

"The impulse response $h[n]$ determines the LTI system"

i.e. for any input $x[n]$, the output $y[n]$ is determined by $h[n]$ as $y[n] = (h * x)[n]$.

* How to determine $h[n]$?

This is an example of "difference equations" which are discrete version of differential eqns.

Ex* $y[n+1] - y[n] = x[n]$. $\forall n \in \mathbb{Z}$.

$h[n] = ?$

<sol>. Input $x[n] = \delta[n]$. output $y[n] = h[n]$.

$\therefore \quad h[n+1] - h[n] = \delta[n]$

$n=0: \quad h[1] - h[0] = \delta[0] = 1$

$\therefore \quad h[1] = h[0] + 1$

$n \neq 0: \quad h[n+1] - h[n] = \delta[n] = 0$ for $n \neq 0$.

$\therefore \quad h[n+1] = h[n]$

Therefore, $\cdot \; h[0] \underset{n=-1}{=} h[-1] = h[-2] = \cdots$

$\cdot \; h[0] + 1 = h[1] = h[2] = \cdots$

But $h[0]$ is undetermined.

$$\therefore \; h[n] = \begin{cases} \alpha & \text{for } n \leq 0 \\ \alpha + 1 & \text{for } n \geq 1 \end{cases} \qquad \alpha : \text{undetermined constant}$$

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"Given a difference equation, to determine the solution, we need an additional condition"

e.g. "decay at infinity" $\lim_{n \to \pm\infty} h[n] = 0$

important notion! (details later)

e.g. $\boxed{\text{"Causality"} \quad h[n] = 0 \text{ for all } n < 0.}$

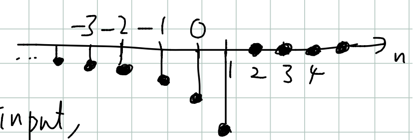
- In the previous example, if we assume causality, then $h[n] = u[n-1]$.

Ex

Find $y[n]$ when $y[n+1] - 2y[n] = x[n]$ for $n \in \mathbb{Z}$.
with $x[n] = \delta[n]$,
in addition to $y[n] = 0$ for $n < 0$.

note In Lec 19, we used F.T. to find

$$y[n] = -\left(\frac{1}{2}\right)^{-n+1} u[-n]$$



This satisfies the difference eqn with given input,
Here, we have $\lim_{n \to \pm\infty} y[n] = 0$

But, $y[n] \neq 0$ for $n < 0$,

So this is NOT what we look for in this problem.

"F.T method may not give us the solution with given additional conditions."

(mostly F.T. method gives us solutions that "decay at infinity")

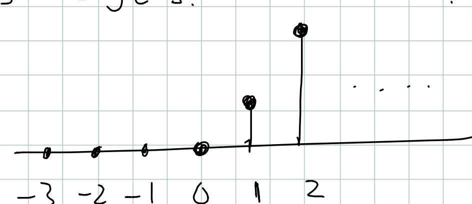
<sol>. $x[n] = \delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$

Case $n \neq 0$: $y[n+1] - 2y[n] = x[n] = 0$.

$\therefore \quad y[n+1] = 2y[n]$. for $n \neq 0$.

Case $n = 0$: $y[1] - 2y[0] = x[0] = 1$

$y[1] = 2y[0] + 1$.



additional condition: $\boxed{y[n] = 0 \text{ for } n < 0.}$

$\therefore \quad y[0] = y[-1+1] = 2y[-1] = 0 \quad \leftarrow n = -1 \neq 0.$

Thus $y[1] = 2y[0] + 1 = 2 \cdot 1 + 1 = 3$

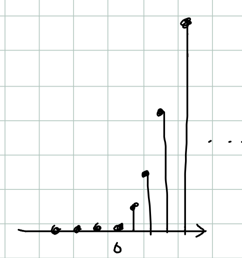
Now, using $y[n+1] = 2y[n]$ for $n \neq 0$.

$$y[2] = 2y[1] = 2 \cdot 3$$
$$y[3] = 2y[2] = 2^2 \cdot 3$$
$$y[4] = 2y[3] = 2^3 \cdot 3$$

In this pattern,

for $n \geq 1$, $y[n] = 3 \cdot 2^{n-1}$.

$$\therefore \quad y[n] = \begin{cases} 3 \cdot 2^{n-1} & \text{for } n \geq 1 \\ 0 & \text{for } n \leq 0 \end{cases}$$



$$= \underline{3 \cdot 2^{n-1} u[n-1]} \quad \blacksquare$$

Note the signal $3 \cdot 2^{n-1} u[n-1] \to +\infty$ as $n \to +\infty$.

For signals that do <u>not</u> decay at infinity,

F. T. method may not work well.

e.g. the sum $\sum_{n=1}^{\infty} 3 \cdot 2^{n-1} e^{-i\omega \cdot n}$ does not make sense.

There is a transform that works even in such situations.

* Z-transform

$h[n] \quad n \in \mathbb{Z}. \xrightarrow{\mathcal{Z}} H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$

$z \in \mathbb{C}$ where the series

$\sum_{n=-\infty}^{\infty} |h[n]| |z|^{-n}$ converges.

"Region of Covergence (ROC)".

☆ Important to know ROC
especially for <u>inverse</u> z-transform

<u>Rmk</u> DTFT is a special case where $z = e^{i\omega}$.

(DTFT make sense if the unit circle $\{z = e^{i\omega}, \omega \in \mathbb{R}\}$ is inside ROC)

<u>Ex*</u> $h[n] = a^n u[n] \quad a \neq 0$ any fixed constant.

$H(z) = ?$

$\sum_{n=0}^{\infty} r^n = \begin{cases} \frac{1}{1-r} & \text{for } |r|<1 \\ \text{diverges} & \text{for } |r| \geq 1 \end{cases}$

<sol>. $H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$

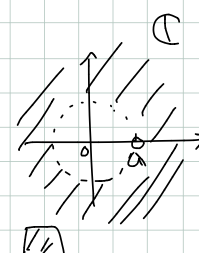
$= \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n}$

otherwise the series diverges.

$= \sum_{n=0}^{\infty} (a z^{-1})^n$

$= \frac{1}{1 - \frac{a}{z}}$ ← Require $\left|\frac{a}{z}\right| < 1$

i.e. $|z| > |a|$.



$\therefore H(z) = \frac{1}{1 - \frac{a}{z}} = \frac{z}{z-a}$; ROC : $|z| > |a|$ $\square$