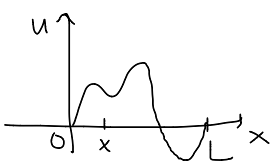
Lecture 2 Separation of Variables

(for solving partial differential equations)

Wave equations :
- describe
  - string motion
  - sound
  - electro-magnetic propagation
  - etc
- are related to
  - quantum mechanics (Schrödinger eqn)
  - relativity (Einstein eqn)

1D: 

t : time

x : position

$u(x,t)$: vertical distribution at (x,t)

(WE) $$\frac{\partial^2 u}{\partial t^2}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t) \quad \text{for} \quad 0<x<L, \quad 0<t$$

$c>0$: constant depending on the material/medium

- Boundary condition (or constraint)

(BC) … $\begin{cases} u(0,t)=0 & \forall t>0 \\ u(L,t)=0 & \forall t>0 \end{cases}$ ← at end points $x=0$ & $x=L$

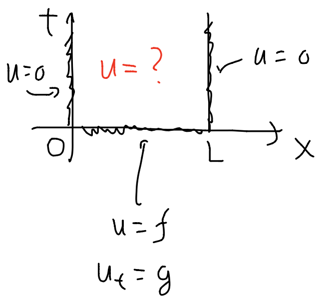
※ other BC are also possible :

e.g. $\frac{\partial u}{\partial x}(0,t)=0$ & $\frac{\partial u}{\partial x}(L,t)=0$

e.g. $u(0,t)=u(L,t)$ & $\frac{\partial u}{\partial x}(0,t)=\frac{\partial u}{\partial x}(L,t)$

Initial condition (or data)

(IC) … $\begin{cases} u(x,0)=f(x) & \text{for } 0<x<L \\ \frac{\partial u}{\partial t}(x,0)=g(x) & 0<x<L \end{cases}$ ← at initial time $t=0$



notation $u_t = \frac{\partial u}{\partial t}$

$u_{tt} = \frac{\partial^2 u}{\partial t^2}$

WE: $u_{tt} = c^2 u_{xx}$

Q. How to find $u(x,t)$ solving the WE with the given BC & IC

Method of separation of variables

· background idea

I. Find as many as possible different (linearly independent) nonzero (nontrivial) solutions $u_1, u_2, \dots$ of simple type, satisfying (BC) $\begin{cases} u(0,t) = 0 \\ u(L,t) = 0 \end{cases}$

II. Use "superposition" $\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \cdots$

to find more general solution

especially to satisfy (IC): $u(x,0) = f(x)$, $\frac{\partial u}{\partial t}(x,0) = g(x)$

For I,

"Separation of variables"

key1 Try: $u(x,t) = X(x)\,T(t)$

$$u_{tt} = c^2 u_{xx} \implies X(x)\,T''(t) = c^2 X''(x)\,T(t)$$

key 2

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = \text{const.} \underset{\text{say}}{=} -\gamma, \quad \gamma \in \mathbb{R}$$

independent variables t & x.

Reason: $f(t) = g(x) \Rightarrow \frac{\partial f}{\partial t}(t) = \frac{\partial}{\partial t} g(x) = 0$

$\frac{\partial g}{\partial x}(x) = \frac{\partial}{\partial x} f(t) = 0$

$\therefore \quad f(t) = \text{const.} = g(x).$

$\therefore$ Get TWO ODE's

- $T''(t) + \gamma c^2 T(t) = 0$
- $X''(x) + \gamma X(x) = 0$

Use BC : $\begin{matrix} u(0,t) = 0 \\ u(L,t) = 0 \end{matrix} \Rightarrow \begin{matrix} X(0)\,T(t) = 0 \\ X(L)\,T(t) = 0 \end{matrix} \Rightarrow$ Can set $X(0) = 0 = X(L)$

We have to find NONZERO solutions for

1 ... $\begin{cases} X''(x) + \gamma X(x) = 0 \\ X(0) = 0 = X(L) \end{cases}$ "eigen value problem".

- γ is to be determined.

only special value of γ will allow nonzero $X(x)$.

2 $T''(t) + \gamma c^2 T(t) = 0$

nonzero

Solution for [1]

$$\gamma = \frac{k^2\pi^2}{L^2} \qquad k = 1, 2, 3, \cdots$$

&

✡ ... $$X(x) = C_1 \left[e^{i\frac{k\pi}{L}x} - e^{-i\frac{k\pi}{L}x} \right]$$

C_1 = arbitrary constant in $\mathbb{C}$

Reason for ✡ : next lecture.

NOTE: different BC will give different γ & $X(x)$!

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nonzero

Solution for [2]

from [1], $\gamma = \dfrac{k^2\pi^2}{L^2}$

we have to use the same γ as ①.

$$\therefore\ T'' + \frac{k^2\pi^2}{L^2} c^2 T = 0$$

sol: $$T(t) = C_2 e^{i\frac{k\pi}{L}t} + C_3 e^{-i\frac{k\pi}{L}t}$$

C_1, C_2 arbitrary constant in $\mathbb{C}$

1 & 2 together (with common γ)

$$X(x)T(t) = \left(e^{i\frac{k\pi}{L}x} - e^{-i\frac{k\pi}{L}x}\right)\left[A_k e^{ic\frac{k\pi}{L}t} + B_k e^{-ic\frac{k\pi}{L}t}\right]$$

A_k, B_k arbitrary constants in $\mathbb{C}$

$k = 1, 2, 3, 4, \cdots\cdots$

To make more general solution we apply.

II. superposition.

$$u(x,t) = \sum_{k=1}^{\infty} X_k(x)T_k(t)$$

i.e.

$$u(x,t) = \sum_{k=1}^{\infty}\left[e^{i\frac{k\pi}{L}x} - e^{-i\frac{k\pi}{L}x}\right]\left[A_k e^{ic\frac{k\pi}{L}t} + B_k e^{-ic\frac{k\pi}{L}t}\right]$$

- Here, A_k, B_k can be chosen.
- This $u(x,t)$ satisfies (WE) $u_{tt} = c^2 u_{xx}$

& (BC) $u(0,t) = 0$ & $u(L,t) = 0$

- Another expression

$$u(x,t) = \sum_{k=1}^{\infty}\sin\left(\frac{k\pi}{L}x\right)\left[\alpha_k\cos\left(\frac{k\pi}{L}ct\right) + \beta_k\sin\left(\frac{k\pi ct}{L}\right)\right]$$

(Use $e^{i\theta} = \cos\theta + i\sin\theta$)

III Matching (IC)

EX Solve
$$\begin{cases} u_{tt} = 4u_{xx} \\ u(0,t) = 0 = u(2,t) \\ u(x,0) = 2\sin(\pi x) - 3\sin(3\pi x) \\ u_t(x,0) = 0 \end{cases}$$

<sol>

$c^2 = 4 \Rightarrow c = 2, \quad L = 2$

$c=2,\ L=2$

$$u(x,t) = \sum_{k=1}^{\infty} \sin\left(\frac{k\pi}{2}x\right)\left[\alpha_k \cos(k\pi t) + \beta_k \sin(k\pi t)\right]$$

satisfies WE & BC.

To match IC

① $2\sin(\pi x) - 3\sin(3\pi x) = u(x,0) = \sum_{k=1}^{\infty} \sin\left(\frac{k\pi}{2}x\right)\cdot \alpha_k$

② $0 = u_t(x,0) = \sum_{k=1}^{\infty} \sin\left(\frac{k\pi}{2}x\right)\left(-\beta_k k\pi\right)$

$\frac{\partial}{\partial t}\Big|_{\text{at } t=0}\left(\alpha_k \cos(k\pi t) + \beta_k \sin(k\pi t)\right)$

② $\Rightarrow \beta_k = 0$

① $\Rightarrow \alpha_2 = 2, \ \alpha_6 = -3$, for other k, $\alpha_k = 0$

Therefore

$$u(x,t) = 2\sin(\pi x)\cos(2\pi t) - 3\sin(3\pi x)\cos(6\pi t).$$

▨

<Another sol>

$c^2=4 \Rightarrow c=2, \quad L=2$

We know

$$u(x,t)=\sum_{k=1}^{\infty}\left[e^{i\frac{k\pi}{2}x}-e^{-i\frac{k\pi}{2}x}\right]\left[A_k e^{i2\frac{k\pi}{2}t}+B_k e^{-i2\frac{k\pi}{2}t}\right]$$

satisfies WE & BC

To match IC.

① $2\sin(\pi x)-3\sin(3\pi x)=u(x,0)=\sum_{k=1}^{\infty}\left[e^{i\frac{k\pi}{2}x}-e^{-i\frac{k\pi}{2}x}\right]\left[A_k+B_k\right]$

② $0=u_t(x,0)=\sum_{k=1}^{\infty}\left[e^{i\frac{k\pi}{2}x}-e^{-i\frac{k\pi}{2}x}\right]\left[i\,k\pi A_k-i\,k\pi B_k\right]$

② $\Rightarrow A_k=B_k$

To use ①, rewrite it as

$$2\frac{1}{2i}\left(e^{i\pi x}-e^{-i\pi x}\right)-\frac{3}{2i}\left(e^{i3\pi x}-e^{-i3\pi x}\right)=\sum_{k=1}^{\infty}2A_k\left[e^{i\frac{k\pi}{2}x}-e^{-i\frac{k\pi}{2}x}\right]$$

$k=2 \Rightarrow A_2=\frac{1}{2i}$

$k=6 \Rightarrow A_6=-\frac{3}{4i}$

for other k, $A_k=0$

Therefore,

$$u(x,t)=\left(e^{i\pi x}-e^{-i\pi x}\right)\frac{1}{2i}\left[e^{i2\pi t}+e^{-i2\pi t}\right]+\left(e^{i3\pi x}-e^{-i3\pi x}\right)\cdot\left(-\frac{3}{4i}\right)\left[e^{i6\pi t}+e^{-i6\pi t}\right]$$

(in real form) $=$ $\boxed{2\sin(\pi x)\cos(2\pi t)-3\sin(3\pi x)\cos(6\pi t)}$ ▨