Lec 19.

* Discrete Time Fourier Transform (DTFT) for non-periodic (i.e. infinite length) signals.

  − Def. Ex. Some properties

Next lecture. ⟶ * linear time invariant (LTI) systems.

Def (DTFT)

discrete signal $n \in \mathbb{Z}$ 2π-periodic function

$$x[n] \xrightarrow{\mathcal{F}} \hat{x}(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-i\omega n}$$

$\omega \in \mathbb{R}$.

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{x}(\omega) e^{in\cdot\omega} d\omega \xleftarrow{\mathcal{F}^{-1}} \hat{x}(\omega)$$

2π-periodic function

explanation for these formulas

short explanation: View $x[n]$ as Fourier coeff at $-n$ (instead of n) of a 2π-periodic function $\hat{x}(\omega)$

long explanation: $\boxed{\text{optional}}$

" discrete-time signal can be viewed as a continuous-time signal of delta functions".

For $x[n]$, construct $x(t) = \sum_{n=-\infty}^{\infty} x[n]\delta_n(t)$

$$\xrightarrow{\mathcal{F}} \hat{x}(\omega) = \underbrace{\int_{-\infty}^{\infty} x(t) e^{-i\omega\cdot t} dt}_{\text{the usual F.T.}}$$

for continuous-time signals

$$= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x[n]\,\delta_n(t)\, e^{-i\omega\cdot t}\, dt$$

$$= \sum_{n=-\infty}^{\infty} x[n] \underbrace{\int_{-\infty}^{\infty} \delta_n(t)\, e^{-i\omega\cdot t}\, dt}_{= e^{-i\omega\cdot n}} = \sum_{n=-\infty}^{\infty} x[n]\, e^{-i\omega n}$$

For 2π-periodic $\hat{x}(\omega)$

Can write $\hat{x}(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-in\omega}$ (Fourier series of $\hat{x}(\omega)$ with $k=-n$)

where $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{x}(\omega)\, e^{i\omega n}\, d\omega$

Now F.I

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{x}(\omega)\, e^{i\omega t}\, d\omega$$ ← this is the usual Fourier inversion for continuous-time signals

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x[n]\, e^{-in\omega}\, e^{i\omega t}\, d\omega$$

$$= \sum_{n=-\infty}^{\infty} x[n]\, \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(t-n)\omega}\, d\omega$$

$$= \sum_{n=-\infty}^{\infty} x[n]\, \delta(t-n)$$

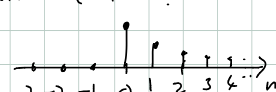
$$= \sum_{n=-\infty}^{\infty} x[n]\, \delta_n(t).$$

// So, We see periodic signals have discrete F.T or F.I //.

EX Fix a constant a, with $|a|<1$.

Consider

$x[n] = a^n u[n]$



Then,

$$\hat{x}(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-i\omega\cdot n} = \sum_{n=-\infty}^{\infty} a^n u[n]\, e^{-i\omega\cdot n} = \sum_{n=0}^{\infty} a^n e^{-i\omega\cdot n}$$

$$= \sum_{n=0}^{\infty} \left(a e^{-i\omega}\right)^n = \frac{1}{1-ae^{-i\omega}}$$

geometric sum $r = ae^{-i\omega}$, $|r| = |a| < 1$

Ex* $N>0$ fixed integer.

$$x[n] = \begin{cases} 1 & -N \le n \le N \\ 0 & \text{otherwise} \end{cases}$$

$\hat{X}(\omega) = ?$

<sol> $\hat{X}(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-i\omega\cdot n} = \sum_{n=-N}^{N} e^{-i\omega\cdot n}$

$e^{-i\omega\cdot n} = e^{-i\omega\cdot(n+N-N)}$

$= e^{-i\omega\cdot(-N)} \cdot e^{-i\omega(n+N)}$

$= e^{-i\omega(-N)} \sum_{k=0}^{2N} e^{-i\omega\cdot k}$

geometric series $\sum_{k=0}^{2N} r^k = \frac{1-r^{2N+1}}{1-r}$

$= e^{i\omega\cdot N} \cdot \frac{1-e^{-i\omega(2N+1)}}{1-e^{-i\omega}} = \frac{e^{i\omega(N+\frac{1}{2})}}{e^{i\frac{\omega}{2}}} \cdot \left(\frac{1 - e^{-i\omega(2N+1)}}{(1-e^{-i\omega})}\right)$

$= \frac{e^{i\omega(N+\frac{1}{2})} - e^{-i\omega(N+\frac{1}{2})}}{e^{i\frac{\omega}{2}} - e^{-i\frac{\omega}{2}}} = \frac{\sin\left((N+\frac{1}{2})\omega\right)}{\sin(\frac{1}{2}\omega)}$ ▨

Ex $\delta[n] = \begin{cases} 1 & n=0 \\ 0 & n\neq 0 \end{cases}$

$\hat{\delta}(\omega) = ?$

<sol>

$\hat{\delta}(\omega) = \sum_{n=-\infty}^{\infty} \delta[n] e^{-i\omega\cdot n} = \underset{n=0}{1} \cdot e^{-i\omega\cdot 0} = \underline{\underline{1}}$ ▨

Ex $\delta_a[n] = \begin{cases} 1 & n=a \\ 0 & n\neq a \end{cases}$. $\quad \hat{\delta}_a(\omega) = ?$

nothing but time shift

<sol>. $\hat{\delta}_a(\omega) = \sum_{n=-\infty}^{\infty} \delta_a[n] e^{-i\omega\cdot n} = 1 \cdot e^{-i\omega\cdot a} = \underline{e^{-ia\omega}}$.

Rmk Similar properties of ordinary F.T. hold for DTFT.

e.g.
- linearity
- time-shift
- frequency-shift, etc ....
- convolution

see Feldman p12. table

For more complete list of properties, see [Hsu, 2nd edition p267, Section 6.4]

Convolution property for DTFT

$x[n], y[n] \quad n \in \mathbb{Z}$

$$(x * y)[n] = \sum_{m=-\infty}^{\infty} x[n-m]y[m] \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftarrows}} \hat{x}(\omega)\hat{y}(\omega)$$

EX $x[n] = \left(\frac{1}{2}\right)^n u[n], \quad y[n] = \left(\frac{1}{3}\right)^n u[n]$

$z[n] = (x * y)[n]$

note $0 < \frac{1}{2}, \frac{1}{3} < 1$

[This condition was used to apply the previous example for $|a|<1$, $a^n u[n] \xrightarrow{\mathcal{F}} \frac{1}{1-ae^{-i\omega}}$]

$\hat{z}(\omega) = ?$

<sol> $\hat{z}(\omega) = \hat{x}(\omega)\hat{y}(\omega)$

$$= \frac{1}{1-\frac{1}{2}e^{-i\omega}} \cdot \frac{1}{1-\frac{1}{3}e^{-i\omega}}$$

$$\begin{cases} \hat{x}(\omega) = \dfrac{1}{1-\frac{1}{2}e^{-i\omega}} \\ \hat{y}(\omega) = \dfrac{1}{1-\frac{1}{3}e^{-i\omega}} \end{cases}$$

Sometimes, can use $\mathcal{F}, \mathcal{F}^{-1}$ to compute convolution.

EX $x[n] = \left(\frac{1}{2}\right)^n u[n], \quad y[n] = \left(\frac{1}{3}\right)^n u[n]$

$(x * y)[n] = ?$

<sd>

method 1 Directly compute $\displaystyle\sum_{m=-\infty}^{\infty} x[n-m]\, y[m]$

$$= \sum_{m=-\infty}^{\infty} \left(\tfrac{1}{2}\right)^{n-m} u[n-m] \left(\tfrac{1}{3}\right)^{m} u[m]$$

$$= \left(\tfrac{1}{2}\right)^{n} \underbrace{\sum_{m=-\infty}^{\infty} \left(\tfrac{1}{2}\right)^{-m} \cdot \left(\tfrac{1}{3}\right)^{m} \cdot u[n-m]\, u[m]}$$

to compute this, follow similar procedure as in Lect 18, computation of $(u * u)[n]$.

method 2 Use Fourier transform.

Note $\hat{x}(\omega) = \dfrac{1}{1-\frac{1}{2}e^{-i\omega}}$, $\hat{y}(\omega) = \dfrac{1}{1-\frac{1}{3}e^{-i\omega}}$

$$\widehat{x*y}(\omega) = \hat{x}(\omega)\,\hat{y}(\omega) = \frac{1}{1-\frac{1}{2}e^{-i\omega}} \cdot \frac{1}{1-\frac{1}{3}e^{-i\omega}}$$

$$\overset{\text{partial fraction}}{=} \frac{A}{1-\frac{1}{2}e^{-i\omega}} + \frac{B}{1-\frac{1}{3}e^{-i\omega}}$$

since $\begin{cases} A+B=1, \\ -\frac{1}{3}A - \frac{1}{2}B = 0 \end{cases} \Rightarrow A=3,\ B=-2$

Here we used the basic example $a^n u[n] \xrightarrow{\mathcal{F}} \dfrac{1}{1-ae^{-i\omega}}$ with $|a|<1$

$$\therefore\ \widehat{x*y}(\omega) = \frac{3}{1-\frac{1}{2}e^{-i\omega}} + \frac{2}{1-\frac{1}{3}e^{-i\omega}}$$

$$\xrightarrow{\mathcal{F}^{-1}} 3 \cdot \left(\tfrac{1}{2}\right)^n u[n] + 2\left(\tfrac{1}{3}\right)^n u[n] \quad \blacksquare$$

Ex (time-shift)

$$x[n-n_0] = (\delta_{n_0} * x)[n] \xrightarrow{\mathcal{F}} \hat{\delta}_{n_0}(\omega)\, \hat{x}(\omega)$$
$$= e^{-in_0\omega}\, \hat{x}(\omega).$$

Ex [n-difference] ← similar to differentiation property for continuous-time signals.

$$y[n] = x[n] - x[n-1] \xrightarrow{\mathcal{F}} \hat{y}(\omega) = \hat{x}(\omega) - e^{-i\omega}\hat{x}(\omega)$$
$$= (1-e^{-i\omega})\,\hat{x}(\omega).$$

Ex $u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$ $n \in \mathbb{Z}$.

$\hat{u}(\omega) = ?$ for $0 < \omega < 2\pi$.

<sol>. $u[n] - u[n-1] = \delta[n]$.

$$\xrightarrow{\mathcal{F}} \underbrace{(1-e^{-i\omega})}_{\neq 0 \text{ for } 0<\omega<2\pi}\, \hat{u}(\omega) = 1.$$

$\therefore$ For $0 < \omega < 2\pi$,

$$\hat{u}(\omega) = \frac{1}{1-e^{-i\omega}}. \quad \boxtimes$$

In fact,

$$\boxed{u[n] \xrightarrow{\mathcal{F}} \frac{1}{1-e^{-i\omega}} + \pi\,\delta(\omega) \quad \text{for } -\pi \leq \omega \leq \pi.}$$

<u>Ex</u> Find $y[n]$ when

$$y[n+1] - 2y[n] = x[n] \quad \text{for } n \in \mathbb{Z}.$$

with $x[n] = \delta[n-1]$.

<sol> Can use F.T.

$$y[n+1] - 2y[n] = x[n]$$

$$\xrightarrow{\mathcal{F}} \quad e^{i\omega}\hat{y}(\omega) - 2\hat{y}(\omega) = \hat{x}(\omega)$$

$$= e^{-i\omega} \cdot 1$$

$$\therefore \ (e^{i\omega} - 2)\hat{y}(\omega) = e^{-i\omega}$$

$$\therefore \ \hat{y}(\omega) = \frac{e^{-i\omega}}{e^{i\omega} - 2} = \frac{e^{-i\omega}}{-2} \cdot \frac{1}{1 - \frac{1}{2}e^{i\omega}}$$

note $0 < \frac{1}{2} < 1$

thus $\frac{1}{1-\frac{1}{2}e^{-i\omega}} \xrightarrow{\mathcal{F}^{-1}} \left(\frac{1}{2}\right)^n u[n]$

$$\therefore \ y[n] = \mathcal{F}^{-1}[\hat{y}(\omega)](n)$$

time-shift

$$= -\frac{1}{2}\mathcal{F}^{-1}\left[\frac{1}{1-\frac{1}{2}e^{i\omega}}\right][n-1]$$

$$\mathcal{F}^{-1}\left[\frac{1}{1-\frac{1}{2}e^{i\omega}}\right][n] = \left(\frac{1}{2}\right)^{-n} u[-n]$$

time-reversal

$$= -\frac{1}{2}\left(\frac{1}{2}\right)^{-(n-1)} u[-(n-1)]$$

$$= -\left(\frac{1}{2}\right)^{-n+2} u[1-n]$$