Lec 18 Discrete-time signals

* aperiodic (infinite length) case
  - important examples
  - convolution
  - Fourier transform $\rightarrow$ Next week

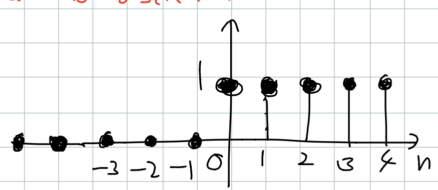
* aperiodic signals (length $= +\infty$) $x[n]$ $\quad -\infty < n < \infty$
(non-periodic) $\quad n = 0, \pm 1, \pm 2, \pm 3, \cdots$

"This theory is relavant to handle signals with different lengths together
e.g. telephone

[Discrete-time unit step function]

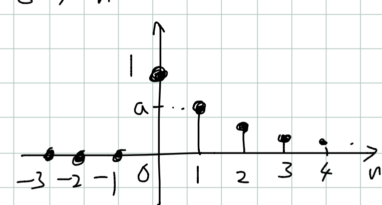
$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \qquad n = \text{integer}.$$

notation: bracket for discrete case



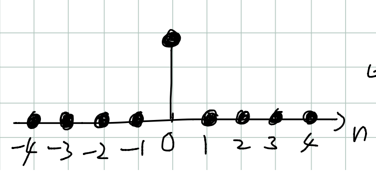
e.g. $0 < a < 1$

$$x[n] = a^n u[n]$$



[Discrete-time impulse] "discrete-time delta function"

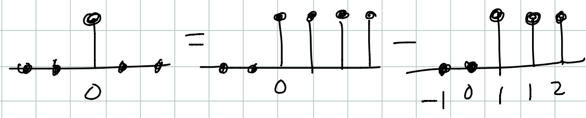
$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$



"unit mass/energy is concentrated at $n=0$"

- $\delta[n] = u[n] - u[n-1]$ ← similar to continuous case $\delta(t) = \frac{d}{dt} u(t)$

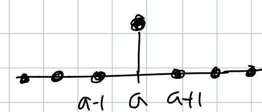
shift by 1.



- $\sum_{n=-\infty}^{\infty} x[n]\,\delta[n] = x[0]$ ← $\int_{-\infty}^{\infty} \delta(x) f(x)\,dx = f(0)$

<u>Notation</u> a : integer

$$\delta_a[n] = \delta[n-a] = \begin{cases} 1 & n = a \\ 0 & \text{otherwise} \end{cases}$$



e.g. $\sum_{n=-\infty}^{\infty} \delta_a[n]\, x[n] = x[a]$

* Convolution (non-periodic / infinite length)

$x[n]$, $y[n]$ n : integer

$$(x * y)[n] = \sum_{m=-\infty}^{\infty} x[m]\, y[n-m]$$

$$= \sum_{m'=-\infty}^{\infty} x[n-m']\, y[m'] \quad \leftarrow m' = n - m$$

$$= (y * x)[n]$$

<u>EX</u> $(\delta_a * x)[n] = x[n-a]$ ← time shift

<u>Continuous case</u>

$(\delta_a * x)(t) = x(t-a)$

<reason> $\text{LHS} = \sum_{m=-\infty}^{\infty} \delta_a[m]\, x[n-m]$ ← nonzero only for $m = a$.

$= x[n-a]$

<u>EX</u> $(\delta_a * \delta_b)[n] = ?$

<sol> $\text{LHS} = \delta_b[n-a] = \delta[n-a-b] = \delta_{a+b}[n]$

EX $(\delta_a * \delta_b * x)[n] = ((\delta_a * \delta_b) * x)[n] = (\delta_{a+b} * x)[n] = \underline{x[n-a-b]}$

EX $(u * u)[n]$

<sol> $LHS = \sum_{m=-\infty}^{\infty} u[m]\, u[n-m]$

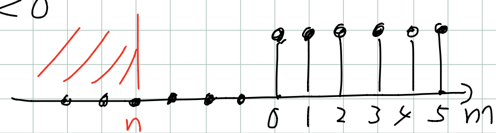
$$u[n-m] = \begin{cases} 1 & n-m \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \sum_{n-m \geq 0} u[m]$$

Here, n fixed, m varies.

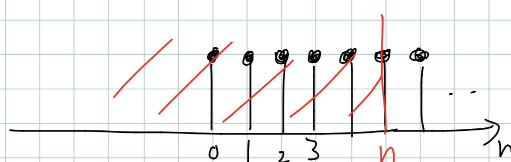
$$\begin{pmatrix} n-m \geq 0 \\ \Longleftrightarrow \\ m \leq n \end{pmatrix} \qquad = \sum_{m=-\infty}^{n} u[m]$$

Case $n<0$



the sum $= 0$

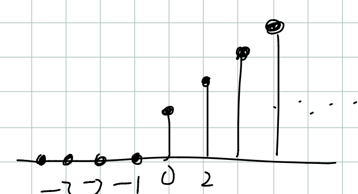
Case $n \geq 0$



$$\text{the sum} = \underset{m=0}{1} + 1 + 1 + \cdots + \underset{m=n}{1} = n+1$$

$$\therefore \ (u * u)[n] = \begin{cases} 0 & \text{if } n<0 \\ n+1 & \text{if } n \geq 0 \end{cases}$$

$$\boxed{(u * u)[n] = (n+1)\, u[n]}$$



Properties of convolution

- $f * g = g * f$
- $(f * g) * h = f * (g * h)$
-

· linearity: c: const.

$$f*(cg) = c(f*g).$$

$$(f+g)*h = f*h + g*h$$

$$f*(g+h) = f*g + f*h$$

<u>EX</u> $\left((3\delta_a + 2u)*u\right)[n] = ?$

<sol> $LHS = (3\delta_a * u)[n] + 2(u*u)[n]$

$$= 3u[n-a] + 2(n+1)u[n]$$

<u>EX</u> (compare with EX 6, page 9 in Feldman's course notes)

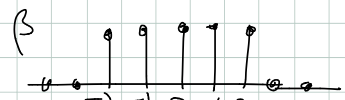
$$\beta[n] = \begin{cases} 1 & |n| \le 2 \\ 0 & |n| > 2 \end{cases}$$

$$\beta * \beta = ?$$

<sol> <u>Method 1</u>. see EX 6, page 9.

<u>Method 2</u> Use properties of convolution

β



$$\beta[n] = u[n+2] - u[n-3]$$

$$= (\delta_{-2} * u)[n] - (\delta_3 * u)[n]$$

i.e. $\beta = \delta_{-2} * u - \delta_3 * u$

$$= (\delta_{-2} - \delta_3) * u$$

$$\beta * \beta = \left((\delta_{-2} - \delta_3) * u\right) * \left((\delta_{-2} - \delta_3) * u\right)$$

reordering

$$= \left((\delta_{-2} - \delta_3) * (\delta_{-2} - \delta_3)\right) * u * u$$

$$= (\delta_{-2} * \delta_{-2} - \delta_3 * \delta_{-2} - \delta_{-2} * \delta_3 + \delta_3 * \delta_3) * (u * u)$$

$$= (\delta_{-4} - \delta_1 - \delta_1 + \delta_6) * (u * u)$$

$\delta_a * \delta_b$
$= \delta_{a+b}$

$$= (\delta_{-4} - 2\delta_1 + \delta_6) * (u * u)$$

$$= \delta_{-4} * (u * u) - 2\,\delta_1 * (u * u) + \delta_6 * (u * u)$$

shift.

Thus, $(\beta * \beta)[n] = ((n+4)+1)\,u(n+4) - 2\,((n-1)+1)\,u[n-1] + ((n-6)+1)\,u[n-6]$

$$= (n+5)\,u[n+4] - 2n\,u[n-1] + (n-5)\,u[n-6]$$

$\boxed{/\!/}$

$(u * u)[n]$
$= (n+1)\,u[n]$