Lec 17 · properties of F.T. (see table in p12)
(periodic / finite length discrete-time signals)
· Parseval, periodic convolution

Recall: $x = [x[0], x[1], \cdots, x[N-1]]$ e.g. $x = [0,1,3]$

length $N \xrightarrow[\text{extension}]{N\text{-periodic}}$ period $= N$

$x[N] = x[0]$
$x[-N+1] = x[1]$, $x[-1] = x[N-1]$, etc

· discrete Fourier series (DFS)

period $= N$ case

frequency $\omega_k = \frac{2\pi}{N} k$

$$x[n] \longrightarrow \hat{x}[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n]\, e^{-i \frac{2\pi}{N} k \cdot n}$$

$$x[n] = \sum_{n=0}^{N-1} \hat{x}[k]\, e^{i \frac{2\pi}{N} k \cdot n}$$

· "orthogonality" a, b integer

$$\frac{1}{N} \sum_{n=0}^{N-1} \left(e^{i \frac{2\pi}{N} a \cdot n} \cdot e^{-i \frac{2\pi}{N} b \cdot n} \right) = \begin{cases} 1 & \text{if } \frac{a-b}{N} = \text{integer} \\ 0 & \text{otherwise.} \end{cases}$$

· Parseval (for period / length $= N$)

$$\frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2 = \sum_{k=0}^{N-1} |\hat{x}[k]|^2$$

reason "orthogonality"

$$LHS = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\overline{x[n]}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1}\left[\sum_{k=0}^{N-1}\hat{x}[k]e^{i\frac{2\pi}{N}k\cdot n}\right]\left[\sum_{l=0}^{N-1}\overline{\hat{x}[l]}e^{-i\frac{2\pi}{N}l\cdot n}\right]$$

$$= \sum_{k=0}^{N-1}\sum_{l=0}^{N-1}\hat{x}[k]\overline{\hat{x}[l]}\underbrace{\left(\frac{1}{N}\sum_{n=0}^{N-1}e^{i\frac{2\pi}{N}k\cdot n}e^{-i\frac{2\pi}{N}k\cdot n}\right)}_{\begin{cases}1 & k=l\\ 0 & k\neq l\end{cases}}$$

$$= \sum_{k=0}^{N-1}\hat{x}[k]\overline{\hat{x}[k]}$$

Note $0 \le k, l \le N-1$

$= RHS$

$\boxtimes$

e.g. Let $x = [\overbrace{1, \cdots, 1}^{N}]$. then $\sum_{k=0}^{N-1}(\hat{x}[k])^2 \overset{\text{Parseval}}{=} \frac{1}{N}\sum_{n=0}^{N-1}|x[n]|^2 = 1$.

Note: $\hat{x}[0] = \frac{1}{N}\sum_{n=0}^{N-1}x[n]e^{-i\frac{2\pi}{N}\cdot 0\cdot n} = \frac{1}{N}\sum_{n=0}^{N-1}x[n] = \frac{1}{N}\cdot N = 1$.

Thus $\hat{x}[k] = 0$ for $k = 1, \cdots, N-1$.

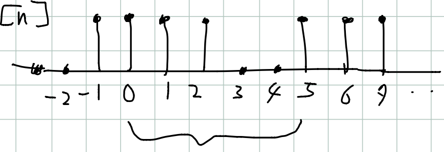
$\therefore \hat{x} = [1, 0, 0, \cdots, 0]$

· <u>time-shifting</u> (period $= N$)

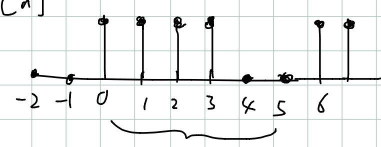
$$y[n] = x[n-n_0] \longleftrightarrow \hat{y}[k] = e^{-i\frac{2\pi}{N}k\cdot n_0}\hat{x}[k]$$

e.g. $N = 6$

$a[n]$



$b[n]$



$$a[n] = b[n+1]$$

$$\hat{b}[k] = \frac{1}{6}\sum_{n=0}^{5} b[n]\, e^{-i\frac{2\pi}{6}k\cdot n}$$

$$b[n] = \begin{cases} 1 & n = 0,1,2,3 \\ 0 & n = 4,5 \end{cases}$$

$$= \frac{1}{6}\sum_{n=0}^{3}\left(e^{-i\frac{\pi}{3}k}\right)^n$$

$$= \frac{1}{6}\sum_{n=0}^{3} r^n \qquad r = e^{-i\frac{\pi}{3}k}$$

$$= \begin{cases} \frac{1}{6}(1+1+1+1) = \frac{2}{3} & k = 0 \\ 6 \cdot \dfrac{1-\left(e^{-i\frac{\pi}{3}k}\right)^4}{1-e^{-i\frac{\pi}{3}k}} & k = 1, \cdots, 5 \end{cases}$$

be careful for the case $r = 1$

$n_0 = 1$

$$\hat{a}[k] = e^{-i\frac{2\pi}{6}k\cdot 1}\,\hat{b}[k] \qquad \text{(time-shift)}$$

$$= \begin{cases} \frac{2}{3} & k = 0 \\ \frac{1}{6} e^{-i\frac{\pi}{3}k} \cdot \dfrac{1-\left(e^{-i\frac{\pi}{3}k}\right)^4}{1-e^{-i\frac{\pi}{3}k}} & k = 1, 2, \cdots, 5. \end{cases}$$

$$\hat{a}[9] = \hat{a}[6+3] = \hat{a}[3] = e^{-i\frac{\pi}{3}}\frac{(1-e^{-i\pi 4})}{1-e^{-i\pi}} = e^{-i\frac{\pi}{3}}\frac{(1-1)}{1+1}$$

$$= 0.$$

∎

Periodic convolution

- Continuous case

$f(t)$, $g(t)$ 2L-periodic

$$\hat{f}[k] = \frac{1}{2L}\int_{-L}^{L} f(t) e^{-i\frac{\pi}{L}kt}\, dt$$

$$(f * g)(t) = \int_{-L}^{L} f(s) g(t-s)\, ds \xrightarrow{\mathcal{F}} 2L\, \hat{f}[k]\, \hat{g}[k]$$

also 2L-periodic

- discrete case

$f[n]$, $g[n]$ period $= N$

$$(f * g)[n] = \sum_{m=0}^{N-1} f[m]\, g[n-m] \xrightarrow{\mathcal{F}} N\, \hat{f}[k]\, \hat{g}[k]$$

also period $= N$

note $f * g = g * f$

Ex $f = [1, 1, 1]$

$g = [1, 1, 0]$

$f * g = ?$

<sol> $(f * g)[n] = \sum_{m=0}^{2} f[m]\, g[n-m]$

$$(f*g)[0] = \sum_{m=0}^{2} f[m]\, g[0-m]$$

$= 1\cdot 1 + 1 \cdot 0 + 1 \cdot 1$

$= 2$

- $g[0] = 1$
- $g[-1] = g[-1+3] = g[2] = 0$
- $g[-2] = g[-2+3] = g[1] = 1$

$(f*g)[1] = f[0]g[1] + f[1]g[0] + f[2]g[-1]$

$= 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 0 = 2$

$(f*g)[2] = f[0]g[2] + f[1]g[1] + f[2]g[0] = 1\cdot 0 + 1 \cdot 1 + 1 \cdot 1 = 2$

$\therefore \underline{f * g = [2, 2, 2]}$ ▨

EX $N=3$.

$a = [1, 0, 1]$

$b = [0, 1, 1]$

Find $\hat{x}$ s.t. $a * x = b$

<sol>. Use F.T.

$$\hat{b} = \widehat{a * x} = \underset{N=3}{3}\, \hat{a}\, \hat{x}$$

$$\hat{a}[0] = \frac{2}{3} \quad \hat{a}[1] = \frac{1}{3}\left(1 + e^{-i\frac{2\pi}{3}}\right), \quad \hat{a}[2] = \frac{1}{3}\left(1 + e^{-i\frac{4\pi}{3}}\right)$$

$$\hat{b}[0] = \frac{2}{3} \quad \hat{b}[1] = \frac{1}{3}\left(e^{-i\frac{\pi}{3}} + e^{-i\frac{2\pi}{3}}\right), \quad \hat{b}[2] = \frac{1}{3}\left(e^{-i\frac{2\pi}{3}} + e^{-i\frac{4\pi}{3}}\right).$$

$$\hat{x}[0] = \frac{1}{3}, \quad \hat{x}[1] = \frac{1}{3}\, \frac{\left(e^{-i\frac{\pi}{3}} + e^{-i\frac{2\pi}{3}}\right)}{\left(1 + e^{-i\frac{2\pi}{3}}\right)}, \quad \hat{x}[2] = \frac{1}{3} \cdot \frac{\left(e^{-i\frac{2\pi}{3}} + e^{-i\frac{4\pi}{3}}\right)}{\left(1 + e^{-i\frac{4\pi}{3}}\right)}$$

Next time: aperiodic (infinite length) signals