Lec 16

- Discrete time signals
- periodic / finite length discrete-time signals
- Discrete Fourier series

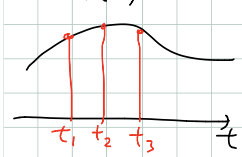
* Discrete time signals

e.g. binary code (computer) $\cdots 0110101 \cdots$

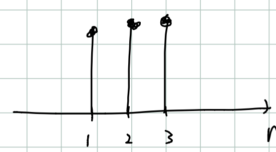
e.g. sampling (useful in scientific/engineering computing)

continuous data $\longrightarrow$ discrete data

$x(t)$



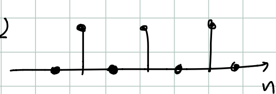
$x[n] = x(t_n)$ $n = 0, \pm 1, \pm 2, \cdots$



- periodic or finite length discrete signals

period $= N$: $x[n+N] = x[n]$ $\forall n \in \mathbb{Z}$.

e.g. $N=2$



period $= N \Rightarrow$ $x[0], x[1], \cdots, x[N-1]$ determine the whole signal

- period$=N$ signal $\Rightarrow$ length$=N$ signal
- length$=N$ signal $\Rightarrow$ period$=N$ signal

N-periodic extension

$$\boxed{\begin{array}{l} \because k \text{ integer.} \\ \therefore N = \text{period / length} \\ x[n+kN] = x[n] \end{array}}$$

<u>notation</u> $x = [x[0], x[1], \cdots, x[N-1]]$

e.g. $x = [-1, 0, 1, 2]$ $\leftarrow$ $x[3] = 2$,

$x[0] = -1$ $x[1] = 0$

period $= 4$.

$x[4] = x[0] = -1$.

$x[5] = x[1] = 0$,

$x[-1] = x[-1+4] = x[3]$
$= 0$

* discrete Fourier series (D.F.S.)

$x[n]$ period $= N$ $\xrightarrow{\mathcal{F}}$ $\hat{x}[k] = \underbrace{\frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-i\frac{2\pi}{N}k\cdot n}}$

$(n = 0, 1, \cdots, N-1)$

period $= N$

$(k = 0, 1, \ldots, N-1)$

similar to the continuous case

$\hat{x}(k) = \frac{1}{N}\int_0^N x(t)\, e^{-i(\frac{2\pi}{N})kt}\, dt$

$\underbrace{x[n] = \sum_{k=0}^{N-1} \hat{x}[k]\, e^{i\frac{2\pi}{N}k n}}$ $\xleftarrow{\mathcal{F}^{-1}}$ $\hat{x}[k]$

So any periodic/finite length signal can be written as a superposition of complex exponentials.

Ex $x = [3, 1, 0, 1]$ $\hat{x} = ?$

<sol> length $N = 4$.

$$\hat{x}[k] = \frac{1}{4}\sum_{n=0}^{4-1} x[n]\, e^{-i\frac{2\pi}{4}\cdot k\cdot n}$$

$$= \frac{1}{4}\left(3\, e^{-i\frac{2\pi}{4}\cdot k\cdot 0} + 1\cdot e^{-i\frac{2\pi}{4}\cdot k\cdot 1} + 0 + 1\cdot e^{-i\frac{2\pi}{4}\cdot k\cdot 3}\right)$$

$$= \frac{1}{4}\left(3 + e^{-i\frac{\pi}{2}k} + e^{-i\frac{3\pi}{2}k}\right)$$

$$= \frac{1}{4}\begin{cases} 3+1+1 & k=0 \\ 3-i+i & k=1 \\ 3-1-1 & k=2 \\ 3+i-i & k=3 \end{cases}$$

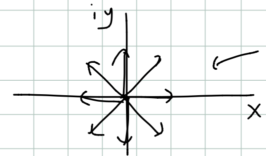
$$\therefore \hat{x} = \left[\frac{5}{4}, \frac{3}{4}, \frac{1}{4}, \frac{3}{4}\right]$$ ▨

* Calculation with discrete complex exponentials.

geometric sum: $\displaystyle\sum_{n=0}^{N-1} r^n = \begin{cases} N & r=1 \\ \dfrac{1-r^N}{1-r} & \text{otherwise} \end{cases}$

<u>EX</u> $\displaystyle\sum_{n=0}^{N-1} e^{i\frac{2\pi}{N}n} = 0$ for $N \geq 2$.

<sol> · geometric:



← cancelation of complex numbers.

· algebraic: $\displaystyle\sum_{n=0}^{N-1} e^{i\frac{2\pi}{N}n} \underset{N\geq 2}{=} \frac{1-\left(e^{i\frac{2\pi}{N}}\right)^N}{1-e^{i\frac{2\pi}{N}}} = \frac{1-1}{1-e^{i\frac{2\pi}{N}}} = 0$ ▨

<u>EX</u> $x = \left[1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}\right]$

$\hat{x} = ?$

<sol>. $N=4$. $x[n] = \left(\frac{1}{2}\right)^n$ $n = 0, 1, 2, 3$.

$$\hat{x}[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-i\frac{2\pi}{N}k\cdot n}$$

$$= \frac{1}{4}\sum_{n=0}^{3} \left(\frac{1}{2}\right)^n e^{-i\frac{2\pi}{4}k\cdot n}$$

$$= \frac{1}{4}\sum_{n=0}^{3} \left[\frac{1}{2} e^{-i\frac{2\pi}{4}k}\right]^n$$

$$= \frac{1}{4}\cdot\frac{1-\left(\frac{1}{2}e^{-i\frac{2\pi}{4}\cdot k}\right)^4}{1-\frac{1}{2}e^{-i\frac{2\pi}{4}k}} = \frac{1}{4}\cdot\frac{1-\frac{1}{2^4}}{1-\frac{1}{2}e^{-i\frac{\pi}{2}k}}$$

$$\therefore\ \underline{\hat{x}[k] = \frac{15}{16}\cdot\frac{1}{2-e^{-i\frac{\pi}{2}k}}}$$

plugin $k = 0, 1, 2, 3$

$$x = \left[\frac{15}{16}\cdot\frac{1}{2},\ \frac{15}{16}\cdot\frac{1}{2+i},\ \frac{15}{16}\cdot\frac{1}{3},\ \frac{15}{16}\cdot\frac{1}{2-i}\right]$$

Ex "orthogonality" of discrete complex exponential.

a, b integer

$$\frac{1}{N}\sum_{n=0}^{N-1}\Big(\underbrace{e^{i\frac{2\pi}{N}a\cdot n}}_{\substack{a \text{ fixed} \\ n \text{ variable}}}\cdot e^{-i\frac{2\pi}{N}b\cdot n}\Big) = \begin{cases} 1 & \text{if } \frac{a-b}{N} = \text{integer} \\ 0 & \text{otherwise} \end{cases}$$

Reason. Summand $= \left(e^{i\frac{2\pi}{N}(a-b)}\right)^n$

$\therefore$ The sum $= \frac{1}{N}\sum_{n=0}^{N-1} r^n$ with $r = e^{i2\pi\frac{a-b}{N}}$

$$= \begin{cases} 1 & r=1 \;\left(\Leftrightarrow \frac{a-b}{N} = \text{integer}\right) \\ \frac{1}{N}\frac{1-r^N}{1-r} & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1 & \frac{a-b}{N} = \text{integer} \\ 0 & \text{otherwise} \end{cases} \longleftarrow \text{since } r^N = \left(e^{i\frac{2\pi}{N}(a-b)}\right)^N = 1$$

Ex $x[n] = e^{i\frac{2\pi}{N}a\cdot n}$

$$\Rightarrow \hat{x}[k] = \begin{cases} 1 & \frac{a-k}{N} = \text{integer} \\ 0 & \text{otherwise} \end{cases}$$

e.g. $x[n] = e^{i\frac{2\pi}{4}\cdot 2n}$ $\xrightarrow{\mathcal{F}}$ $\hat{x} = [0, 0, 1, 0]$

$n = 0, 1, \cdots, 3$

at $k=0$

at $k=2$

e.g. $x = [1, 1, \cdots, 1] \Rightarrow \hat{x} = [1, 0, 0, \cdots, 0]$

(since $x[n] = e^{i\frac{2\pi}{N}\cdot 0\cdot n}$)