Lec 15

A non comprehensive review for F.T.

* ODE & F.T.
(e.g. RLC circuit)

$$\underset{\text{input}}{x(t)} \longrightarrow \boxed{\text{circuit}} \longrightarrow \underset{\text{output}}{y(t)}$$

EX.

$$3\,y''(t) + 4\,y'(t) + y(t) = x(t)$$

Find $y(t)$ when

(a) $x(t) = \delta(t)$

(b) $x(t) = e^{-t}u$

(c) $x(t) = u(t)$

<sol>.

$$3\,y''(t) + 4\,y'(t) + y(t) = x(t)$$

$$\Big\downarrow \text{F.T.}$$

$$\mathcal{F}\left[\frac{d}{dt}f(t)\right](\omega) = i\omega\,\mathcal{F}[f(t)](\omega)$$

$$\mathcal{F}\left[\frac{d^2}{dt^2}f(t)\right](\omega) = (i\omega)^2\,\mathcal{F}[f(t)](\omega) = -\omega^2\,\mathcal{F}[f(t)](\omega)$$

$$3(i\omega)^2\,\hat{y}(\omega) + 4i\omega\,\hat{y}(\omega) + \hat{y}(\omega) = \hat{x}(\omega)$$

$$\left[3(i\omega)^2 + 4i\omega + 1\right]\hat{y}(\omega) = \hat{x}(\omega)$$

$$\therefore\ \hat{y}(\omega) = \left(\boxed{\frac{1}{3(i\omega)^2 + 4i\omega + 1}}\right)\hat{x}(\omega)$$

frequency response.

(a) $x(t) = \delta(t) \Rightarrow \hat{x}(\omega) = 1$. $\therefore\ \hat{y}(\omega) = \dfrac{1}{3(i\omega)^2 + 4i\omega + 1}$

$$\therefore\ \hat{y}(\omega) = \frac{1}{(3i\omega+1)(i\omega+1)}$$

$$= \frac{1}{3}\ \frac{1}{i\omega+\frac{1}{3}}\ \frac{1}{i\omega+1}$$

$$= \frac{1}{3}\cdot\frac{3}{2}\left[\frac{1}{i\omega+\frac{1}{3}} - \frac{1}{i\omega+1}\right]$$

$$= \frac{1}{2}\left[\frac{1}{i\omega+\frac{1}{3}} - \frac{1}{i\omega+1}\right]$$

$$\Big\downarrow \mathcal{F}^{-1}$$

$$\underline{y(t) = \frac{1}{2}\cdot\left[e^{-\frac{1}{3}t}u(t) - e^{-t}u(t)\right]}$$

(e.g. $a = 3-i$)

- $\underline{\mathrm{Re}\,a > 0}$: $\frac{1}{i\omega+a} \xrightarrow{\mathcal{F}^{-1}} e^{-at}u(t)$
- $\underline{\mathrm{Re}\,a < 0}$: $\frac{1}{i\omega+a} \xrightarrow{\mathcal{F}^{-1}} -e^{-at}u(-t)$

(e.g. $a = -3+i$)

This output $y(t)$ for input $\underline{x(t) = \delta(t)}$ is called <u>impulse response</u>.

($\mathcal{F}$[impulse response] = frequency response.

(b) $x(t) = e^{-t}u(t) \Longrightarrow \hat{x}(\omega) = \frac{1}{i\omega+1}$

$$\therefore\ \hat{y}(\omega) = \frac{1}{2}\left[\frac{1}{i\omega+\frac{1}{3}} - \frac{1}{i\omega+1}\right]\frac{1}{i\omega+1}$$

$$= \frac{1}{2}\left(\frac{1}{i\omega+\frac{1}{3}}\ \frac{1}{i\omega+1}\right) - \frac{1}{2}\ \frac{1}{(i\omega+1)^2}$$

$$= \frac{1}{2}\cdot\frac{3}{2}\left[\frac{1}{i\omega+\frac{1}{3}} - \frac{1}{i\omega+1}\right] - \frac{1}{2}\ \frac{1}{(i\omega+1)^2}$$

$$= \frac{3}{4}\ \frac{1}{i\omega+\frac{1}{3}} - \frac{3}{4}\ \frac{1}{i\omega+1} - \frac{1}{2}\ \frac{1}{(i\omega+1)^2}$$

$$\Big\downarrow \mathcal{F}^{-1}$$

$$y(t) = \frac{3}{4}\cdot e^{-\frac{1}{3}t}u(t) - \frac{3}{4}\cdot e^{-t}u(t) - \frac{1}{2}\cdot\mathcal{F}^{-1}\left[\frac{1}{(i\omega+1)^2}\right](t)$$

$$\mathcal{F}^{-1}\left[\frac{1}{(i\omega+1)^2}\right](\omega) = ?$$

<u>differentiation & F.T.</u>

- $\frac{d}{dt}x(t) \xrightarrow{\mathcal{F}} i\omega\,\hat{x}(\omega)$
- $t\,x(t) \xrightarrow{\mathcal{F}} i\frac{d}{d\omega}\hat{x}(\omega)$

$$\frac{1}{(i\omega+1)^2} = i\frac{d}{d\omega}\frac{1}{(i\omega+1)} \qquad \left(\because \frac{d}{d\omega}\frac{1}{i\omega+1} = -i\frac{1}{(i\omega+1)^2}\right)$$

$$\downarrow$$

$$\mathcal{F}^{-1}\left[\frac{1}{(i\omega+1)^2}\right](t) = t\,\mathcal{F}^{-1}\left[\frac{1}{i\omega+1}\right](t) = \underline{te^{-t}u(t)}$$

Finally, $y(t) = \frac{3}{4}e^{-\frac{1}{3}t}u(t) - \frac{3}{4}e^{-t}u(t) - \frac{1}{2}te^{-t}u(t)$

(c) $x(t) = u(t) \Longrightarrow \hat{x}(\omega) = \hat{u}(\omega) = \frac{1}{i\omega} + \pi\delta(\omega)$

$$\hat{y}(\omega) = \frac{1}{2}\left[\frac{1}{i\omega+\frac{1}{3}} - \frac{1}{i\omega+1}\right]\hat{x}(\omega)$$

$$= \frac{1}{2}\left[\frac{1}{i\omega+\frac{1}{3}} - \frac{1}{i\omega+1}\right]\left[\frac{1}{i\omega} + \pi\delta(\omega)\right]$$

$$= \frac{1}{2}\frac{1}{i\omega+\frac{1}{3}}\cdot\frac{1}{i\omega} - \frac{1}{2}\frac{1}{i\omega+1}\cdot\frac{1}{i\omega} + \underbrace{\frac{1}{2}\left(\frac{1}{i\omega+\frac{1}{3}} - \frac{1}{i\omega+1}\right)\pi\delta(\omega)}$$

$$= \frac{1}{2}\cdot 3\left[\frac{1}{i\omega} - \frac{1}{i\omega+\frac{1}{3}}\right] - \frac{1}{2}\left[\frac{1}{i\omega} - \frac{1}{i\omega+1}\right] + \pi\delta(\omega) \qquad \underbrace{\frac{\pi}{2}\cdot\left(\frac{1}{i0+\frac{1}{3}} - \frac{1}{i0+1}\right)\delta(\omega)}_{\pi\delta(\omega).}$$

$$= -\frac{3}{2}\frac{1}{i\omega+\frac{1}{3}} + \frac{1}{2}\frac{1}{i\omega+1} + \frac{1}{i\omega} + \pi\delta(\omega)$$

$\downarrow \mathcal{F}^{-1}$

$\mathcal{F}^{-1}$ $\frac{1}{i\omega} + \pi\delta(\omega)$

$$y(t) = -\frac{3}{2} e^{-\frac{t}{3}} u(t) + \frac{1}{2} e^{-t} u(t) + u(t)$$

F. T. & convolution

$$(f * g)(t) \xrightarrow{\mathcal{F}} \hat{f}(\omega)\,\hat{g}(\omega)$$

$$\xleftarrow[\mathcal{F}^{-1}]{}$$

$$(f*g)(t) = \int_{-\infty}^{\infty} f(s)\, g(t-s)\, ds = \int_{-\infty}^{\infty} f(t-s)\, g(s)\, ds = (g * f)(t)$$

Ex $\mathcal{F}^{-1}\left[\frac{1}{(i\omega+1)^2}\right](t)$

Note $e^{-t} u(t) \xrightarrow{\mathcal{F}} \frac{1}{i\omega+1}$

$$\therefore \mathcal{F}^{-1}\left[\frac{1}{(i\omega+1)^2}\right](t) = \int_{-\infty}^{\infty} e^{-s} u(s)\, e^{-(t-s)} u(t-s)\, ds$$

$$= e^{-t} \int_{-\infty}^{\infty} u(s)\, u(t-s)\, ds$$

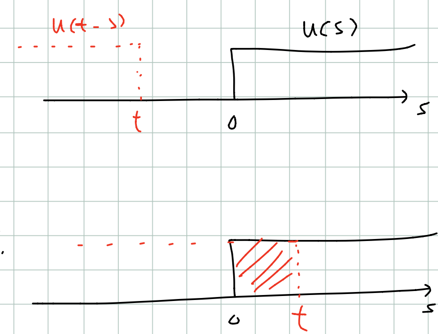
$(u * u)(t)$

$$= \int_{-\infty}^{\infty} u(s)\, u(t-s)\, ds$$

$$= \begin{cases} 0 & t < 0 \\ t & t > 0 \end{cases}$$

$\underline{t<0:}$

$\underline{t>0:}$



$\therefore \quad (u * u)(t) = t\, u(t).$

$\therefore \quad \mathcal{F}^{-1}\left[\dfrac{1}{(i\omega+1)^2}\right](t) = \underline{t\, e^{-t}\, u(t)}$ $\boxed{/}$

$\underline{\text{EX}}$ $\quad \mathcal{F}^{-1}\left[\dfrac{1}{i\omega+2}\, \hat{u}(\omega)\right](t)$

<sol> $\quad$ · one can use $\quad \hat{u}(\omega) = \dfrac{1}{i\omega} + \pi(\omega)$

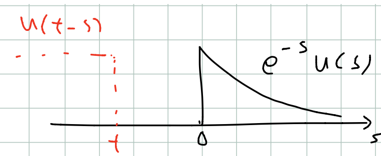
OR . alternatively,

$$\mathcal{F}^{-1}\left[\frac{1}{i\omega+2}\hat{u}(\omega)\right](t)$$

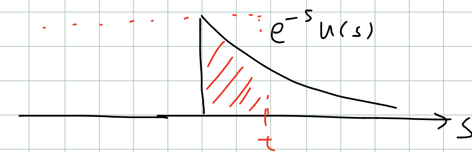
$$= (f * u)(t) \qquad \text{for } f(t) = \mathcal{F}^{-1}\left[\frac{1}{i\omega+2}\right](t)$$

$$\qquad\qquad\qquad\qquad\qquad = e^{-2t} u(t)$$

$$= \int_{-\infty}^{\infty} e^{-2s} u(s)\, u(t-s)\, ds$$



$$= \begin{cases} 0 & t<0 \\ \int_0^t e^{-2s} u(s)\,ds & t>0 \end{cases}$$



$$= \begin{cases} 0 & t<0 \\ \int_0^t e^{-2s}\,ds & t>0 \end{cases}$$

$$= \begin{cases} 0 & t<0 \\ \frac{1}{2}\left(1-e^{-2t}\right) & t>0 \end{cases}$$

$$= \frac{1}{2}\left(1-e^{-2t}\right)u(t)$$

Use other method

$$\frac{1}{i\omega+1}\hat{u}(\omega) = \frac{1}{i\omega+2}\left(\frac{1}{i\omega}+\pi\delta(\omega)\right)$$

$$= \frac{1}{i\omega+2}\frac{1}{i\omega} + \frac{\pi}{i\omega+2}\delta(\omega)$$

If g is continuous at 0, then $g(\omega)\delta(\omega) = g(0)\delta(\omega)$

$$= \frac{1}{i\omega+2}\frac{1}{i\omega} + \frac{\pi}{i0+2}\delta(\omega)$$

$$= \frac{1}{2}\left(\frac{1}{i\omega} - \frac{1}{i\omega+2}\right) + \frac{1}{2}\pi\delta(\omega)$$

$$= \frac{1}{2}\left[-\frac{1}{i\omega+2} + \frac{1}{i\omega} + \pi\delta(\omega)\right]$$

$$\mathcal{F}^{-1}\downarrow$$

$$\mathcal{F}^{-1}\left[\frac{1}{i\omega+1}\hat{u}(\omega)\right](t) = \frac{1}{2}\left[-e^{-2t}u(t) + u(t)\right] = \frac{1}{2}\left(1-e^{-2t}\right)u(t) \quad \square$$