Lec 14

* F.T. of $u(t)$
* Convolution.
  - Calculation
  - F.T. & convolution

EX For $\omega \neq 0$, $\mathcal{F}[u(t)](\omega) = ?$

<sol>.

$$\begin{array}{ccc} \frac{d}{dt}u(t) & \xrightarrow{\mathcal{F}} & i\omega\, \mathcal{F}[u(t)](\omega) \\ \| & & \| \\ \delta(t) & & 1 \end{array}$$

$\therefore \quad i\omega\, \mathcal{F}[u(t)](\omega) = 1$

For $\omega \neq 0$, $\mathcal{F}[u(t)](\omega) = \frac{1}{i\omega}$. ▨

In fact $\boxed{\mathcal{F}[u(t)](\omega) = \frac{1}{i\omega} + \pi\,\delta(\omega)}$

e.g. $\mathcal{F}[u(2t)](\omega) \underset{\text{scaling}}{=} \frac{1}{2}\mathcal{F}[u(t)](\tfrac{1}{2}\omega) = \frac{1}{2}\left[\frac{1}{i\frac{\omega}{2}} + \pi\,\delta(\tfrac{\omega}{2})\right]$

note $u(2t) = u(t)$.

$= \frac{1}{i\omega} + \pi\,\frac{1}{2}\delta(\frac{\omega}{2}) = \frac{1}{i\omega} + \pi\,\delta(\omega)$.

EX $\mathcal{F}[u(2t+1)](\omega) = ?$

<sol> $\mathcal{F}[u(2t+1)](\omega) = \mathcal{F}[u(2(t+\frac{1}{2}))](\omega)$

$= e^{i\frac{1}{2}\omega}\,\mathcal{F}[u(2t)](\omega)$ ← time shifting

$= e^{i\frac{1}{2}\omega}\,\mathcal{F}[u(t)](\omega)$ since $u(2t) = u(t)$

$= e^{i\frac{1}{2}\omega}\left[\frac{1}{i\omega} + \pi\,\delta(\omega)\right]$

▨

Convolution

Q. Given $\hat{g}(\omega) = \hat{H}(\omega)\hat{f}(\omega)$,

how to express/compute $g(t)$ using $H(t)$ & $f(t)$?

(In a circuit,

$H(t)$ impulse response

$\hat{H}(\omega)$ frequency response

$f(t)$ input signal

$g(t)$ output signal)

$$f(t) \longrightarrow \overset{H(t)}{\boxed{\text{circuit}}} \longrightarrow g(t)$$

e.g. (filtering)

To modify a given signal at frequency level (i.e. $\hat{f}(\omega)$)

what actually has to be done for the physical signal (i.e. $f(t)$) ?

(- equalizer
- denoising)

Fact $\hat{g}(\omega) = \hat{H}(\omega)\hat{f}(\omega) \underset{\mathcal{F}}{\overset{\mathcal{F}^{-1}}{\rightleftarrows}} g(t) = (H * f)(t)$

$\quad$ ↳ convolution

Def

$$(H * f)(t) = \int_{-\infty}^{\infty} H(s) f(t-s)\, ds$$

← the result is a function of t.

$$\overset{\text{also}}{=} \int_{-\infty}^{\infty} H(t-s) f(s)\, ds$$

(Property: $H * f = f * H$)

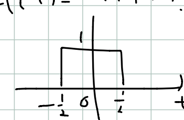
EX filter: $H(t) = \text{rect}(t)$
signal: $u(t)$.

Compute $(H * u)(t)$.

<sol>.

$$(H*u)(t) = \int_{-\infty}^{\infty} H(s)\, u(t-s)\, ds$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} u(t-s)\, ds \qquad \leftarrow H(t) = \text{rect}(t).$$

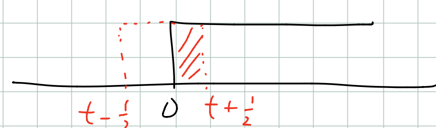


$$= \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} u(z)\, dz \qquad z = t-s$$

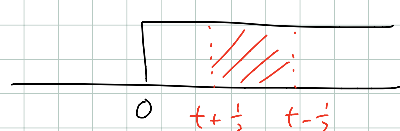
Cases

① no overlap

② partial overlap



③ full overlap

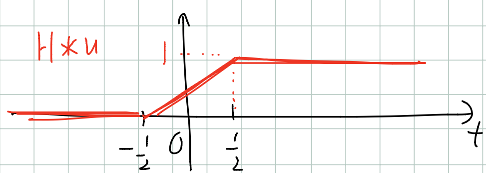


① : $t + \frac{1}{2} \le 0$ i.e. $\underline{t \le -\frac{1}{2}}$ $\Rightarrow$ $(H*u)(t) = 0$

② $t - \frac{1}{2} < 0 < t + \frac{1}{2}$ i.e. $\underline{-\frac{1}{2} < t < \frac{1}{2}}$ $\Rightarrow$ $(H*u)(t) = t + \frac{1}{2}$

③ $0 \leq t - \frac{1}{2}$ i.e. $\underline{\frac{1}{2} \leq t}$ $\Longrightarrow$ $(H * u)(t) = 1$.

$$\therefore \ (H * u)(t) = \begin{cases} 0 & t \leq -\frac{1}{2} \\ t + \frac{1}{2} & -\frac{1}{2} < t < \frac{1}{2} \\ 1 & t \geq \frac{1}{2} \end{cases}$$



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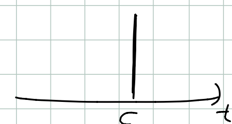
<u>EX</u> Let $\underline{\delta_c(t) \overset{\text{notation}}{=} \delta(t-c)}$



$$(\delta_c * f)(t) = f(t-c)$$

(pf)

$$(\delta_c * f)(t) = (f * \delta_c)(t) = \int_{-\infty}^{\infty} f(t-s)\, \delta_c(s)\, ds$$

$$= \int_{-\infty}^{\infty} f(t-s)\, \delta(s-c)\, ds$$

$$= f(t-c) \longleftarrow \text{the value of } f(t-s) \text{ when } s = c.$$

$\boxtimes$

$$f \xrightarrow[* \delta_c]{} f(t-c)$$

shifting.

e.g. $(f * \delta)(t) = (\delta * f)(t) = f(t)$

Convolution & F.T.

- $\mathcal{F}[f * g](\omega) = \mathcal{F}[f] \cdot \mathcal{F}[g]$
- $\mathcal{F}[f(t)\, g(t)](\omega) = \frac{1}{2\pi}\left(\mathcal{F}[f] * \mathcal{F}[g]\right)(\omega)$

EX. $\mathcal{F}[\,\mathrm{rect}(t) * u(t)\,](\omega)$

$$= \mathcal{F}[\mathrm{rect}(t)](\omega) \cdot \mathcal{F}[\,u(t)\,](\omega)$$

$$= \mathrm{sinc}\left(\frac{\omega}{2}\right) \cdot \left[\frac{1}{i\omega} + \pi\,\delta(\omega)\right]$$

$$= \frac{1}{i\omega}\,\mathrm{sinc}\left(\frac{\omega}{2}\right) + \pi \cdot \underbrace{\mathrm{sinc}\left(\frac{\omega}{2}\right)\delta(\omega)}_{=\,\mathrm{sinc}(0)\,\delta(\omega)}$$

(see HW 7 #3)

$$= \frac{1}{i\omega}\,\mathrm{sinc}\left(\frac{\omega}{2}\right) + \pi\,\delta(\omega).$$

EX $\hat{y}(\omega) = \hat{H}(\omega)\,\hat{x}(\omega)$

$\hat{H}(\omega) = \mathrm{sinc}(\omega)$

$\hat{x}(\omega) = \mathrm{sinc}(\omega)$

Find $y(t)$.

<sol).

Rmk Trying to find $\mathcal{F}^{-1}[\,\mathrm{sinc}(\omega)\,\mathrm{sinc}(\omega)]$ using basic examples may not be easy.

$$\mathrm{sinc}(\omega)\,\mathrm{sinc}(\omega) \xrightarrow{\mathcal{F}^{-1}} \mathcal{F}^{-1}[\mathrm{sinc}(\omega)] * \mathcal{F}^{-1}[\mathrm{sinc}(\omega)]$$

$$\mathrm{rect}(t) \xrightarrow{\mathcal{F}} \mathrm{sinc}\left(\frac{\omega}{2}\right)$$

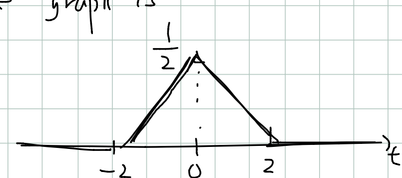
$$\frac{1}{2}\mathrm{rect}\left(\frac{t}{2}\right) \longrightarrow \mathrm{sinc}(\omega) \qquad \text{"Scaling"}$$

$$\therefore\ \mathcal{F}^{-1}\left[\mathrm{sinc}(\omega)\,\mathrm{sinc}(\omega)\right](t) = \frac{1}{2}\mathrm{rect}\left(\frac{t}{2}\right) * \frac{1}{2}\mathrm{rect}\left(\frac{t}{2}\right)$$

$$= \frac{1}{4}\int_{-\infty}^{\infty} \mathrm{rect}\left(\frac{t-s}{2}\right)\mathrm{rect}\left(\frac{s}{2}\right)\,ds$$

$$\overset{\text{exercise}}{=} \begin{cases} 0 & t < -2 \\ \frac{1}{4}(t+2) & -2 < t < 0 \\ \frac{1}{4}(2-t) & 0 < t < 2 \\ 0 & t > 2 \end{cases}$$

whose graph is



$\square$