Lec 13

* Parseval's relation.
* Delta function ("unit impulse")

· Parseval's relation

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega \qquad \left(\text{Similar to } \frac{1}{2L}\int_{-L}^{L} |f(t)|^2 dt = \sum_{k=-\infty}^{\infty} |c_k|^2\right)$$

Ex $\displaystyle\int_{-\infty}^{\infty} |\operatorname{sinc}(\omega)|^2 d\omega = \int_{-\infty}^{\infty} \left|\frac{\sin(\omega)}{\omega}\right|^2 d\omega = ?$

<sol> Recall $\operatorname{rect}(t) \xrightarrow{\mathcal{F}} \operatorname{sinc}\left(\frac{\omega}{2}\right)$

$$\frac{1}{2}\operatorname{rect}\left(\frac{t}{2}\right) \xrightarrow{\mathcal{F}} \operatorname{sinc}(\omega)$$

Scaling

· $f(\alpha t) \xrightarrow{\mathcal{F}} \frac{1}{\alpha}\hat{f}\left(\frac{\omega}{\alpha}\right)$

· $\hat{f}(\alpha\omega) \xrightarrow{\mathcal{F}^{-1}} \frac{1}{\alpha} f\left(\frac{t}{\alpha}\right)$

$$\therefore \int_{-\infty}^{\infty} |\operatorname{sinc}(\omega)|^2 d\omega$$

$$= 2\pi \cdot \frac{1}{2\pi}\int_{-\infty}^{\infty} |\operatorname{sinc}(\omega)|^2 d\omega$$

$$= 2\pi \cdot \int_{-\infty}^{\infty} \left|\frac{1}{2}\operatorname{rect}\left(\frac{t}{2}\right)\right|^2 dt$$

note $\operatorname{rect}\left(\frac{t}{2}\right) = \begin{cases} 1 & -1<t<1 \\ 0 & \text{otherwise} \end{cases}$

$$= 2\pi \cdot \frac{1}{4}\int_{-1}^{1} dt = 2\pi \cdot \frac{2}{4} = \underline{\underline{\pi}} \qquad \square$$

Rmk $\displaystyle f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} dt$

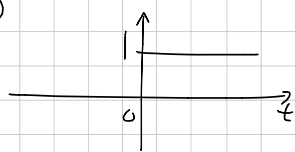
This identity works for <u>most</u> of t

but <u>NOT</u> every t UNLESS $\begin{cases} \cdot\ f(t) \text{ is a continuous function.} \\ \cdot\ \text{both } \int_{-\infty}^{\infty} |f(t)|dt,\ \int_{-\infty}^{\infty} |\hat{f}(\omega)|\, d\omega < \infty \end{cases}$

Delta function

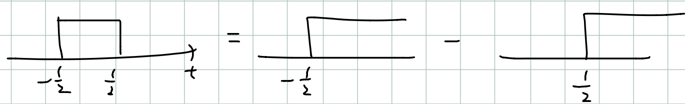
unit step function

$u(t)$



$$u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

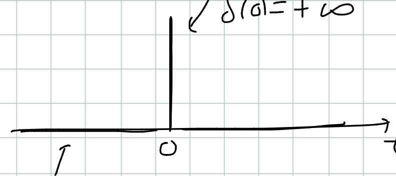
ex. $\text{rect}(t) = u(t+\frac{1}{2}) - u(t-\frac{1}{2})$



Q. $\frac{d}{dt}u(t) = ?$

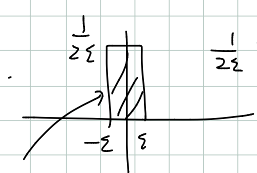
Ans: $\delta(t) = \frac{d}{dt}u(t)$

$\delta(0) = +\infty$



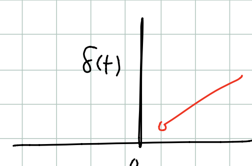
$\delta(t) = 0$ for $t \neq 0$.

an "impulse" at $t = 0$



area $=1$

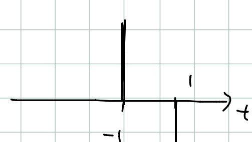
$\frac{1}{2\varepsilon}\text{rect}\left(\frac{t}{2\varepsilon}\right)$ $\xrightarrow[\text{as } \varepsilon \to 0]{}$



a unit mass is "concentrated" at at $t = 0$

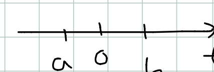
e.g. $\delta(t+1) + \delta(t-1)$

$2\delta(t+1) - \delta(t-1)$



Properties of $\delta(t)$

- $\delta(t) = \delta(-t)$
- $\int_{-\infty}^{\infty} \delta(t)\,dt = 1$ (In fact for any $a < 0 < b$ $\int_a^b \delta(t)\,dt = 1$



So, $\int_{-\infty}^{t} \delta(t)\,dt = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases} = u(t)$

· For any $f(t)$ continuous at $t=0$

$$\boxed{\int_{-\infty}^{\infty} \delta(t)\, f(t)\, dt = f(0)}$$

$$\overset{\shortparallel}{\int_{-\varepsilon}^{\varepsilon} \delta(t) f(t)\, dt} \overset{\text{approximation}}{\approx} f(0) \overbrace{\int_{-\varepsilon}^{\varepsilon} \delta(t)\, dt}^{=1} = f(0)$$

f is continuous at $t=0$,
so near $t=0$, $f(t) \approx f(0)$. ☐

e.g. $\displaystyle\int_{-\infty}^{\infty} \delta(t)\, e^{i\omega t}\, dt = e^{i\omega\cdot 0} = 1.$

$$\boxed{\therefore\ \mathcal{F}[\delta(t)](\omega) = 1}$$

e.g. $\displaystyle\int_{-\infty}^{\infty} \delta(t)\, u(t)\, dt \neq 1$ or 0

but $= \dfrac{1}{2}$.

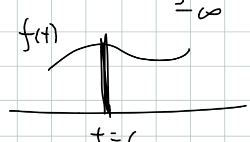
<u>Reason</u>:

$$\int_{-\infty}^{\infty} \frac{1}{2\varepsilon} \operatorname{rect}\left(\frac{t}{2\varepsilon}\right) u(t)\, dt$$

$$= \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} u(t)\, dt = \frac{1}{2}$$ ☐

· If $f(t)$ is continuous at $t=c$

then $\displaystyle\int_{-\infty}^{\infty} \delta(t-c)\, f(t)\, dt = f(c)$

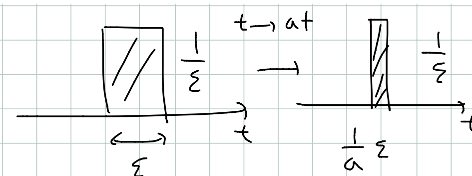


$$\shortparallel$$

$$\int_{-\infty}^{\infty} \delta(c-t)\, f(t)\, dt$$

· scaling of $\delta(t)$.

$a > 0$ const.



$$\delta(at) = \frac{1}{a}\delta(t).$$

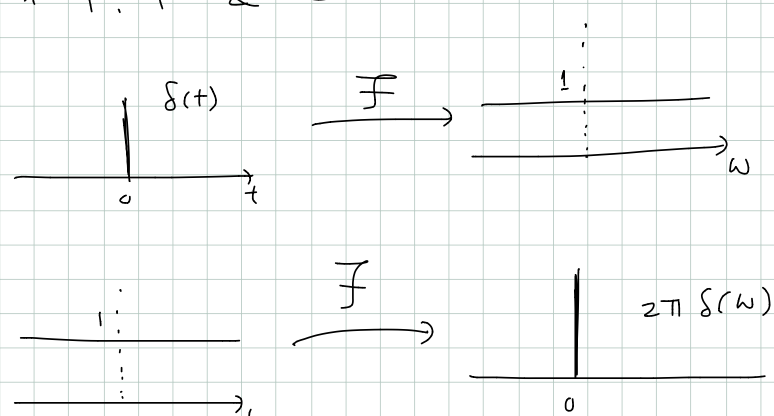
Reason $\int_{-\infty}^{\infty} \delta(at)\,dt = \int_{-\infty}^{\infty} \delta(u)\frac{du}{a} = \frac{1}{a}.$ ☒

e.g. $\int_{-\infty}^{\infty} \delta(3t)\, e^{it}\, dt = \int_{-\infty}^{\infty} \frac{1}{3}\delta(t)\, e^{it}\, dt = \underline{\frac{1}{3}}.$

e.g. $\int_{-\infty}^{\infty} \delta(2t-1)\, e^{it}\, dt$

$$= \int_{-\infty}^{\infty} \delta\left(2\left(t-\tfrac{1}{2}\right)\right) e^{it}\, dt = \frac{1}{2}\int_{-\infty}^{\infty} \delta\left(t-\tfrac{1}{2}\right) e^{it}\, dt = \underline{\frac{1}{2}e^{i\frac{1}{2}}}.$$

* F.T & delta function



used "duality" & $\delta(\omega) = \delta(-\omega)$

$$\boxed{\begin{aligned} \mathcal{F}[\delta(t)](\omega) &= 1 \\ \mathcal{F}[1](\omega) &= 2\pi\,\delta(\omega) \end{aligned}}$$

$\underline{Ex}$. $c \in \mathbb{R}$. $\mathcal{F}[\delta(t-c)](\omega) = e^{-i\omega c}$

$a \in \mathbb{R}$. $\mathcal{F}[e^{iat}](\omega) = 2\pi\,\delta(\omega - a)$.

$\underline{\text{reason}}$. $\mathcal{F}[\delta(t-c)](\omega) = e^{-i\omega c}\,\mathcal{F}[\delta(t)](\omega)$ time-shift

$= e^{-i\omega c}$

$\mathcal{F}[e^{iat}](\omega) = \mathcal{F}[1](\omega - a)$ frequency shift

$= 2\pi\,\delta(\omega - a)$.

(or. $\mathcal{F}[e^{iat}](\omega) = \mathcal{F}[e^{-i(-a)t}](\omega)$

$= 2\pi\,\delta(-\omega - (-a))$ duality

$= 2\pi\,\delta(-\omega + a)$ $\delta(\omega) = \delta(-\omega)$.

$= 2\pi\,\delta(\omega - a)$).

$\underline{Ex}$ $e^{it} + e^{i2t} \xrightarrow{\mathcal{F}} 2\pi\,\delta(\omega - 1) + 2\pi\,\delta(\omega - 2)$.

$\underline{Ex}$ $\mathcal{F}[\cos t](\omega)$

<sol> $\cos t = \frac{1}{2}(e^{it} + e^{-it}) \xrightarrow{\mathcal{F}} \frac{1}{2}[2\pi\,\delta(\omega - 1) + 2\pi\,\delta(\omega + 1)]$ ☑

$\underline{Ex}$ $\mathcal{F}^{-1}[\cos\omega](t)$

<sol> $\cos\omega = \frac{1}{2}(e^{i\omega} + e^{-i\omega}) \xrightarrow{\mathcal{F}^{-1}} \frac{1}{2}\left(\delta(t+1) + \delta(t-1)\right)$ ☑