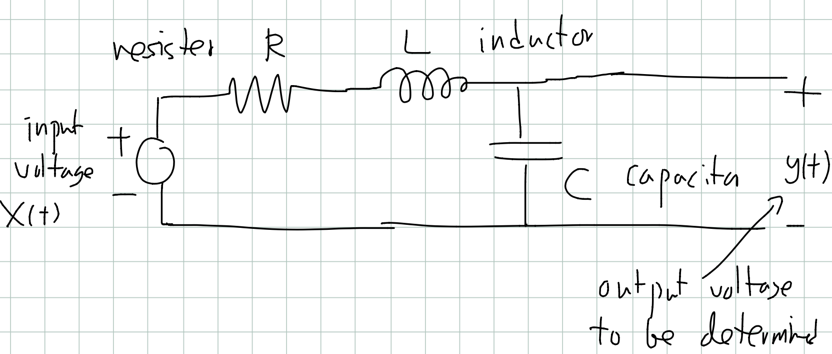
Lec 12

- Fourier Inversion (Inverse Fourier Transform)
  - motivating example: RLC circuit
  - computations: Use properties of F.T.
- Duality between F.T. & F.I.

RLC Circuit & F.T.



given

ODE:

$$LCy''(t) + RCy'(t) + y(t) = X(t)$$

"We will see similar systems in discrete setting!" (later in the course)

$y(t) = ?$

$$\Big\Downarrow \text{ F.T.} \quad \left(\text{Used } \mathcal{F}\left[\tfrac{d}{dt}y(t)\right](\omega) = i\omega\,\hat{y}(\omega) \right)$$

$$LC(i\omega)^2\hat{y}(\omega) + RC\,i\omega\,\hat{y}(\omega) + \hat{y}(\omega) = \hat{X}(\omega)$$

Rearrange $\Rightarrow$

$$\underbrace{\hat{y}(\omega)}_{\text{output frequency}} = \underbrace{\frac{1}{-LC\omega^2 + RC\omega + 1}}_{\hat{H}(\omega)\text{, called "transfer function" of the circuit}} \underbrace{\hat{X}(\omega)}_{\text{input frequency}}$$

e.g. Audio equalizer: change $\hat{H}(\omega)$ to change output frequencies.

* To recover $y(t)$ from $\hat{y}(\omega)$: Do Fourier inversion of $\hat{H}(\omega)\hat{X}(\omega)$.

* Fourier Inversion. (F. I.) "get $y(t)$ from $\hat{y}(t)$"

$$y(t) \xrightarrow{\mathcal{F}} \hat{y}(\omega) = \int_{-\infty}^{\infty} y(t) e^{-i\omega t} dt \qquad \xrightarrow{\mathcal{F}^{-1}} y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{y}(\omega) e^{i\omega t} d\omega$$

this integral can be very hard to compute directly in many cases.

strategy: try to reduce to already known cases for $\mathcal{F}^{-1}$ & use properties of F.T / F.I.

(of course, sometimes, you may have to compute the integral directly.)

EX. Suppose $\hat{X}(\omega) = 1$ (we will learn such a function "impulse" $\delta(t)$ with $\mathcal{F}[\delta(t)](\omega) = 1$)

. $\hat{H}(\omega) = \dfrac{1}{-2\omega^2 + i3\omega + 1}$ ← $R=3$, $L=2$, $C=1$

. $\hat{y}(\omega) = \hat{H}(\omega)\hat{X}(\omega)$

Find $y(t)$.

<sol> Rewrite $\hat{Y}(\omega)$: $\hat{Y}(\omega) = \hat{H}(\omega)\hat{X}(\omega) = \dfrac{1}{(2i\omega+1)(i\omega+1)}$

partial fraction

$$= \frac{a}{(2i\omega+1)} + \frac{b}{(i\omega+1)}$$

$1 = a(i\omega+1) + b(2i\omega+1)$

$= (a+b) + i(a+2b)\omega$

$a+b=1$, $a+2b=0$

$\therefore a=2$, $b=-1$

$$\to \; = \frac{2}{2i\omega+1} - \frac{1}{i\omega+1}$$

each of these is F.T. of basic examples

For constant $a>0$,

Recall $\mathcal{F}[e^{-at}u(t)](\omega) = \dfrac{1}{a+i\omega}$ i.e. $\dfrac{1}{a+i\omega} \xrightarrow{\mathcal{F}^{-1}} e^{-at}u(t)$.

$$\frac{2}{2i\omega+1} = \frac{1}{i\omega+\frac{1}{2}} \xrightarrow{\mathcal{F}^{-1}} e^{-\frac{1}{2}t}u(t)$$

$$\hat{Y}(\omega) = \underbrace{\frac{1}{i\omega+\frac{1}{2}}}_{\mathcal{F} \nearrow\, e^{-\frac{1}{2}t}u(t)} - \underbrace{\frac{1}{i\omega+1}}_{\mathcal{F} \nearrow\, e^{-t}u(t)}$$

$$\frac{1}{i\omega+1} \xrightarrow{\mathcal{F}^{-1}} e^{-t}u(t).$$

$$\therefore \quad \hat{Y}(\omega) = \mathcal{F}\left[e^{-\frac{1}{2}t}u(t) - e^{-t}u(t)\right](\omega)$$

$$\mathcal{F}^{-1} \Downarrow$$

Thus $Y(t) = e^{-\frac{1}{2}t}u(t) - e^{-t}u(t)$. ▨

$$\boxed{f(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftarrows}} \hat{f}(\omega)}$$

Note $\mathcal{F}^{-1}\left[c_1\hat{f}_1(\omega) + c_2\hat{f}_2(\omega)\right](t) = c_1 f_1(t) + c_2 f_2(t)$.

both F.T. (Fourier transform) & F.I. (Fourier inversion) are linear!

EX. $\mathcal{F}^{-1}\left[\mathrm{sinc}(\omega)\right](t) = ?$

(sol) $\mathrm{rect}(t) \xrightarrow{\mathcal{F}} \mathrm{sinc}\left(\frac{\omega}{2}\right)$

scaling $f\left(\frac{t}{\alpha}\right) \xrightarrow{\mathcal{F}} |\alpha|\,\hat{f}(\alpha\omega)$

$$\mathrm{rect}\left(\frac{t}{2}\right) \xrightarrow{\mathcal{F}} 2\,\mathrm{sinc}\left(\frac{2\omega}{2}\right)$$

$$\frac{1}{2}\mathrm{rect}\left(\frac{t}{2}\right) \xleftarrow{\mathcal{F}^{-1}} \frac{1}{2}\cdot 2\,\mathrm{sinc}(\omega)$$

$$\therefore \quad \mathcal{F}[\mathrm{sinc}(\omega)] = \frac{1}{2}\mathrm{rect}\left(\frac{t}{2}\right) \quad ▨$$

Ex $\mathcal{F}^{-1}[e^{i\omega}\,\mathrm{sinc}(\omega)](t) = ?$

<sol> $f(t-t_0) \xrightarrow{\mathcal{F}} e^{-i\omega t_0}\hat{f}(\omega)$ ← time-shifting

So, we have $\boxed{\mathcal{F}^{-1}[e^{-i\omega t_0}\, g(\omega)](t) = \mathcal{F}^{-1}[g(\omega)](t-t_0)}$

Thus, $\mathcal{F}^{-1}[e^{i\omega}\,\mathrm{sinc}(\omega)](t)$

$= \mathcal{F}^{-1}[\mathrm{sinc}(\omega)](t+1)$ ← $t_0=-1$

previous example

$= \underbrace{\frac{1}{2}\mathrm{rect}\left(\frac{t+1}{2}\right)}$ ← replaced t with $t+1$.

∎

Ex $\mathcal{F}^{-1}\left[\frac{1}{1+\omega^2}\right](t) = ?$

<sol> $\frac{1}{1+\omega^2} = \frac{1}{2}\frac{1}{1+i\omega} + \frac{1}{2}\frac{1}{1-i\omega}$

time reversal

$= \frac{1}{2}\mathcal{F}[e^{-t}u(t)](\omega) + \frac{1}{2}\mathcal{F}[e^{-t}u(t)](-\omega)$

$= \frac{1}{2}\mathcal{F}[e^{-t}u(t)](\omega) + \frac{1}{2}\mathcal{F}[e^{t}u(-t)](\omega)$

$= \mathcal{F}\left[\frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{t}u(-t)\right](\omega)$

$\therefore \mathcal{F}^{-1}\left[\frac{1}{1+\omega^2}\right](t) = \frac{1}{2}\left[e^{-t}u(t) + e^{t}u(-t)\right]$

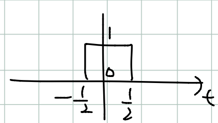
$= \underline{\underline{e^{-|t|}}}$ ∎

* Duality

Idea: F.T. & F.I are similar (not the same)

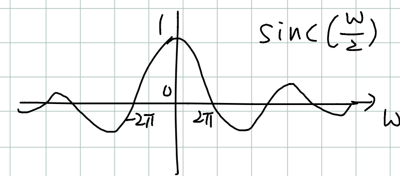
EX

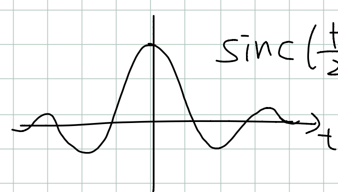
rect(t)



$\xrightarrow{\mathcal{F}}$

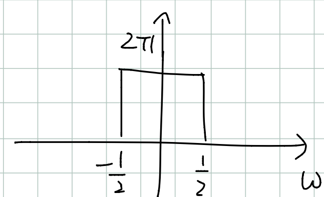
$\xleftarrow[\mathcal{F}^{-1}]{}$





$\xrightarrow{\mathcal{F}}$

$\xleftarrow[\mathcal{F}^{-1}]{}$



duality: suppose $f(t) \xrightarrow{\mathcal{F}} g(\omega)$

then $g(t) \xrightarrow{\mathcal{F}} 2\pi f(-\omega)$.

Proof

$$\mathcal{F}[g(t)](\omega) = \int_{-\infty}^{\infty} g(t)\, e^{-i\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} g(t)\, e^{i(-\omega)t}\, dt$$

note $f(t) \xrightarrow{\mathcal{F}} g(\omega)$

$\therefore f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\omega)\, e^{i\omega t}\, d\omega$

$$= 2\pi f(-\omega)$$

change the role of ω & t.

□

Ex $\mathcal{F}\left[\frac{1}{1+it}\right](\omega) = ?$

<sol> $e^{-t}u(t) \xrightarrow{\mathcal{F}} \frac{1}{1+i\omega}$

$\therefore$ By duality $\frac{1}{1+it} \xrightarrow{\mathcal{F}} 2\pi e^{-(-\omega)} u(-\omega) = \underline{2\pi e^{\omega} u(-\omega)}$ ▨

Ex $\mathcal{F}[\mathrm{sinc}(t)](\omega) = ?$

<sol>. Recall $\mathrm{sinc}(\omega) \xrightarrow{\mathcal{F}^{-1}} \frac{1}{2}\,\mathrm{rect}\left(\frac{t}{2}\right)$

$\therefore$ By duality

$$\mathrm{sinc}(t) \xrightarrow{\mathcal{F}} 2\pi\,\frac{1}{2}\,\mathrm{rect}\left(-\frac{\omega}{2}\right)$$

$$= \underline{\pi\,\mathrm{rect}\left(-\frac{\omega}{2}\right)}$$ ▨

Ex. $\mathcal{F}\left[\frac{1}{1+t^2}\right](\omega) = ?$ ← For this example, calculating directly the corresponding integral will be a mess.

<sol>. Consider duality

$$\frac{1}{1+\omega^2} = \frac{1}{(1+i\omega)(1-i\omega)} = \frac{1}{2}\,\frac{1}{1+i\omega} + \frac{1}{2}\,\frac{1}{1-i\omega}$$

$$= \frac{1}{2}\mathcal{F}[e^{-t}u(t)](\omega) + \frac{1}{2}\mathcal{F}[e^{-t}u(t)](-\omega)$$

$$= \frac{1}{2}\mathcal{F}[e^{-t}u(t)](\omega) + \frac{1}{2}\mathcal{F}[e^{t}u(-t)](\omega)$$

time-reversal.

Thus, $\frac{1}{1+\omega^2} \xrightarrow{\mathcal{F}^{-1}} \frac{1}{2} e^{-t} u(t) + \frac{1}{2} e^{t} u(-t)$.

duality:

$$\frac{1}{1+t^2} \xrightarrow{\mathcal{F}} 2\pi \cdot \left(\frac{1}{2} e^{-(-\omega)} u(-\omega) + \frac{1}{2} e^{-\omega} u(-(-\omega)) \right)$$

$$= \pi \left[e^{\omega} u(-\omega) + e^{-\omega} u(\omega) \right]$$

$$= \pi e^{-|\omega|}. \quad \square$$