Lec 11 · Properties of F.T.

c_1, c_2 constants (can be complex numbers)

last lecture
- linearity $\mathcal{F}[c_1 f(t) + c_2 g(t)](\omega) = c_1 \mathcal{F}[f(t)](\omega) + c_2 \mathcal{F}[g(t)](\omega)$
- time reversal $\mathcal{F}[f(-t)](\omega) = \mathcal{F}[f(t)](-\omega)$
- time-shift $\mathcal{F}[f(t-t_0)](\omega) = e^{-it_0\omega}\mathcal{F}[f(t)](\omega)$

today $\longrightarrow$
- scaling
- differentiation.

Basic Ex · $a>0$ constant $\mathcal{F}[e^{-at}u(t)](\omega) = \dfrac{1}{a+i\omega}$

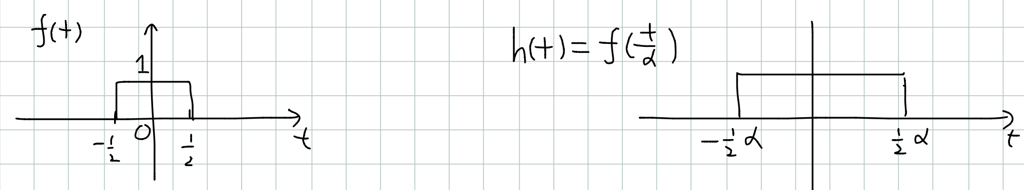
$$\mathcal{F}[\mathrm{rect}(t)](\omega) = \mathrm{sinc}\left(\frac{\omega}{2}\right) = \begin{cases} 1 & \text{for } \omega = 0 \\ \frac{2}{\omega}\sin\left(\frac{\omega}{2}\right) & \text{for } \omega \neq 0 \end{cases}$$

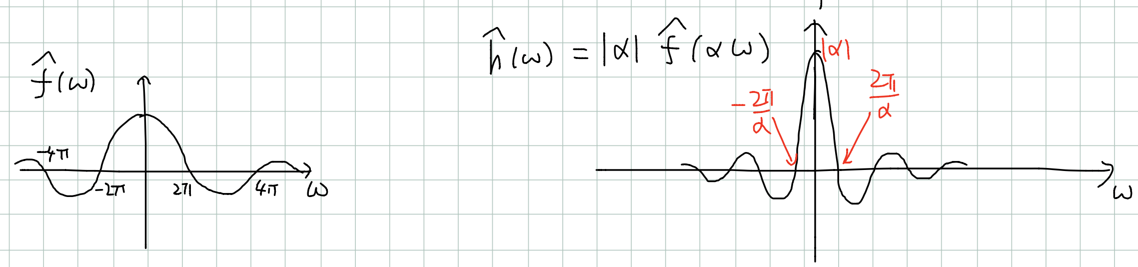
Goal: learn how to use properties of F.T. and basic examples in computing F.T. of signals.

Scaling

for constant $\alpha \neq 0$,

$$\mathcal{F}[f(\tfrac{t}{\alpha})](\omega) = |\alpha|\,\mathcal{F}[f(t)](\alpha\omega)$$

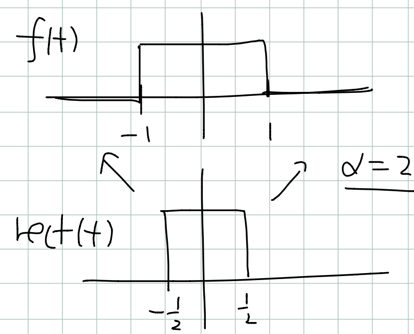




" The wider the signal, the narrower its Fourier transform.

(The wider the Fourier transform, the narrower the original signal.)

EX $f(t) = \begin{cases} 1 & -1 < t < 1 \\ 0 & \text{otherwise} \end{cases}$



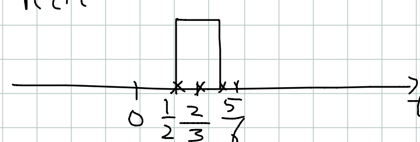
$\hat{f}(\omega) = ?$

<sol> $f(t) = rect\left(\frac{t}{2}\right)$ $\alpha = 2$

$\therefore \hat{f}(\omega) = 2\ \widehat{rect}(2\omega) = 2\ sinc(\omega)$. ▨

EX* $f(t) = rect(3t - 2)$

$rect(3t-2) = rect\left(3\left(t - \frac{2}{3}\right)\right)$



$\hat{f}(\omega) = ?$

<sol>

Method 1 $h(t) = rect(t-2)$

scaling $f(t) = h(3t) \leftarrow \alpha = \frac{1}{3}$

$\therefore \hat{f}(\omega) = \frac{1}{3}\ \hat{h}\left(\frac{1}{3}\omega\right)$

$\hat{h}(\omega) = \mathcal{F}[rect(t-2)]$

time-shifting $= e^{-i2\omega}\ \widehat{rect}(\omega)$

$= e^{-i2\omega}\ sinc\left(\frac{\omega}{2}\right)$

$= \frac{1}{3} e^{-i2\cdot\frac{\omega}{3}} sinc\left(\frac{\omega}{6}\right)$.

Method 2 $rect(3t-2) = rect\left(3\left(t - \frac{2}{3}\right)\right)$

$\hat{f}(\omega) = \mathcal{F}\left[rect\left(3\left(t-\frac{2}{3}\right)\right)\right](\omega)$

$= e^{-i\frac{2}{3}\omega}\ \mathcal{F}[rect(3t)](\omega)$ ← time-shifting

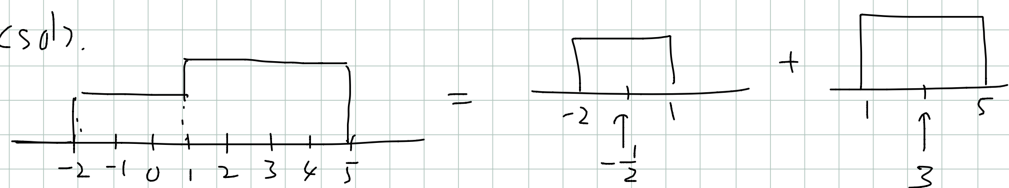
$= e^{-i\frac{2}{3}\omega} \cdot \frac{1}{3} \mathcal{F}[rect(t)]\left(\frac{\omega}{3}\right)$ ← scaling.

$= \frac{1}{3} e^{-i\frac{2}{3}\omega} sinc\left(\frac{\omega}{6}\right)$. ▨

Ex $\quad f(t) = \begin{cases} 2 & -2 < t < 1 \\ 3 & 1 < t < 5 \\ 0 & \text{otherwise} \end{cases}$

Use properties of F.T. & basic examples to compute $\hat{f}(\omega)$

(sol).



$$f(t) = 2\,\mathrm{rect}\left(\tfrac{1}{3}(t+\tfrac{1}{2})\right) + 3\,\mathrm{rect}\left(\tfrac{1}{4}(t-3)\right)$$

$$\therefore \hat{f}(\omega) = 2\,\mathcal{F}\left[\mathrm{rect}\left(\tfrac{1}{3}(t+\tfrac{1}{2})\right)\right](\omega) + 3\,\mathcal{F}\left[\mathrm{rect}\left(\tfrac{1}{4}(t-3)\right)\right](\omega) \qquad \text{"linearity"}$$

$$= 2 \cdot e^{i\frac{\omega}{2}}\,\mathcal{F}\left[\mathrm{rect}\left(\tfrac{1}{3}t\right)\right](\omega) + 3 \cdot e^{-i3\omega}\,\mathcal{F}\left[\mathrm{rect}\left(\tfrac{1}{4}t\right)\right](\omega) \qquad \text{"time-shifting"}$$

$$= 2 \cdot e^{i\frac{\omega}{2}} \cdot 3\,\mathcal{F}[\mathrm{rect}(t)](3\omega) + 3\,e^{-i3\omega} \cdot 4\,\mathcal{F}[\mathrm{rect}(t)](4\omega) \qquad \text{"scaling"}$$

$$= 6e^{i\frac{\omega}{2}}\,\mathrm{sinc}\left(\tfrac{3\omega}{2}\right) + 12\,e^{-i3\omega}\,\mathrm{sinc}(2\omega) \qquad \text{"}\widehat{\mathrm{rect}}(\omega) = \mathrm{sinc}\left(\tfrac{\omega}{2}\right)\text{"}$$

$\blacksquare$

Differentiation

- $h(t) = \frac{d}{dt} f(t) \xrightarrow{\text{F.T.}} \hat{h}(\omega) = i\omega\,\hat{f}(\omega)$

i.e. $\boxed{\mathcal{F}\left[\frac{d}{dt} f(t)\right](\omega) = i\omega\,\mathcal{F}[f(t)](\omega)}$

Can repeat to get $\mathcal{F}\left[\frac{d^k}{dt^k} f(t)\right](\omega) = (i\omega)^k \, \mathcal{F}[f(t)](\omega)$.

k-th derivative

<u>EX</u> $h(t)$

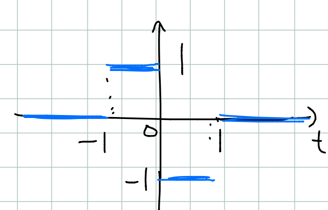
$$h(t) = \begin{cases} t+1 & -1<t<0 \\ -t+1 & 0<t<1 \\ 0 & \text{otherwise} \end{cases}$$

$\hat{h}(\omega) = ?$

<sol>. <u>method 1.</u> Compute directly $\hat{h}(\omega) = \int_{-\infty}^{\infty} h(t) e^{-i\omega t} dt$

<u>method 2</u> Use properties of F.T. & basic examples.

<u>method 2</u>: observe $h'(t)$



$\therefore \; h'(t) = \text{rect}(t+\frac{1}{2}) - \text{rect}(t-\frac{1}{2})$

$\mathcal{F}[h'(t)](\omega) = \mathcal{F}[\text{rect}(t+\frac{1}{2})](\omega) - \mathcal{F}[\text{rect}(t-\frac{1}{2})](\omega)$ "linearity"

time-shift $\longrightarrow$ $= e^{i\frac{1}{2}\omega}\mathcal{F}[\text{rect}(t)](\omega) - e^{-i\frac{1}{2}\omega}\mathcal{F}[\text{rect}(t)](\omega)$

$= \left(e^{i\frac{1}{2}\omega} - e^{-i\frac{1}{2}\omega}\right)\mathcal{F}[\text{rect}(t)](\omega)$

$= 2i\sin(\frac{\omega}{2})\,\text{sinc}(\frac{\omega}{2})$ $\longleftarrow$ recall $\mathcal{F}[\text{rect}(t)](\omega) = \text{sinc}(\frac{\omega}{2})$

Recall $\mathcal{F}[h'(t)](\omega) = i\omega\,\mathcal{F}[h(t)](\omega)$

$\therefore \; i\omega\,\mathcal{F}[h(t)](\omega) = 2i\sin(\frac{\omega}{2})\,\text{sinc}(\frac{\omega}{2})$.

Case $\omega \neq 0$: $\mathcal{F}[h(t)](\omega) = \frac{2}{\omega} \sin\left(\frac{\omega}{2}\right) \operatorname{sinc}\left(\frac{\omega}{2}\right) = \left(\operatorname{sinc}\left(\frac{\omega}{2}\right)\right)^2$

Case $\omega = 0$: $\mathcal{F}[h(t)](0) = \int_{-\infty}^{\infty} h(t)\,dt$ $\leftarrow$ $e^{-i\cdot 0} = e^0 = 1$

$= \frac{1}{2} + \frac{1}{2} = 1.$

$\therefore \mathcal{F}[h(t)](\omega) = \begin{cases} \left(\operatorname{sinc}\left(\frac{\omega}{2}\right)\right)^2 & \omega \neq 0 \\ 1 & \omega = 0 \end{cases}$ $\leftarrow$ note $\operatorname{sinc}(0) = 1$

$= \left(\operatorname{sinc}\left(\frac{\omega}{2}\right)\right)^2$ //

Optional: "rough explanation of the differentiation property"

$$\boxed{\mathcal{F}\left[\frac{d}{dt} f(t)\right](\omega) = i\omega\, \mathcal{F}[f(t)](\omega)}$$

- 2L-periodic case: Suppose $f(t) = \sum_{k=-\infty}^{\infty} c_k e^{i\omega_k t}$ $\omega_k = \frac{k\pi}{L}$

Fourier coeff. of f

Then $\frac{d}{dt} f(t) = \sum_{k=-\infty}^{\infty} \frac{d}{dt}\left(c_k e^{i\omega_k t}\right)$

$$= \sum_{k=-\infty}^{\infty} \underbrace{c_k \cdot i\omega_k}_{\substack{\| \\ \to\ i\omega_k c_k}} e^{i\omega_k t}$$

Therefore,

"k-th Fourier coeff." of $\frac{d}{dt} f(t)$ = $i\omega_k \times$ "k-th Fourier coeff." of $f(t)$.

- general case:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\hat{f}(\omega)}_{= \mathcal{F}[f(t)](\omega)} e^{i\omega t}\, d\omega$$

(← Fourier inversion formula. This can be understood as a representation of $f(t)$.

$$\frac{d}{dt} f(t) = \frac{d}{dt}\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{d}{dt}\left[\hat{f}(\omega) e^{i\omega t}\right] d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\hat{f}(\omega)\cdot i\omega}\, e^{i\omega t}\, d\omega$$

this is $\mathcal{F}\left[\frac{d}{dt} f(t)\right](\omega)$

Thus, can see $\boxed{\mathcal{F}\left[\frac{d}{dt} f(t)\right](\omega) = i\omega\, \mathcal{F}[f(t)](\omega)}$ ▨