Lec 10

Properties of F.T.

- linearity
- time reversal
- time shifting
- scaling

useful for
- computing
- Fourier transform/inversion.

F.T. $\mathcal{F}[f(t)](\omega) = \hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt$

notation

Last time:
- $\mathcal{F}[\mathrm{rect}(x)](\omega) = \mathrm{sinc}\left(\frac{\omega}{2}\right) = \begin{cases} 1 & \omega = 0 \\ \frac{2}{\omega}\sin\left(\frac{\omega}{2}\right) & \omega \neq 0 \end{cases}$
- for constant $a>0$: $\mathcal{F}[e^{-at}u(t)](\omega) = \frac{1}{a+i\omega}$.

Ex 1. $f(t) = e^{-a|t|}$.

$\hat{f}(\omega) = ?$

<sol> method 1

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{at} e^{-i\omega t}\, dt + \int_{0}^{\infty} e^{-at} e^{-i\omega t}\, dt$$

$$= \int_{\infty}^{0} e^{-as} e^{-i(-\omega)s} (-ds) + \int_{0}^{\infty} e^{-at} e^{-i\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-as} e^{-i(-\omega)s}\, ds + \int_{0}^{\infty} e^{-at} e^{-i\omega t}\, dt$$

From previous lecture.

$$= \frac{1}{a+i(-\omega)} + \frac{1}{a+i\omega}$$

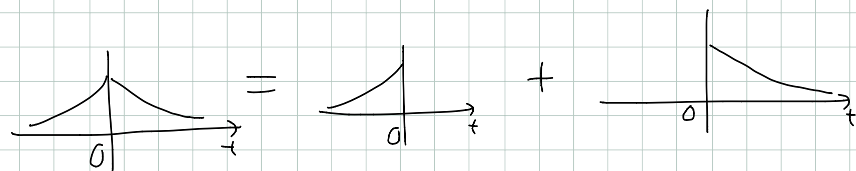
$$= \frac{1}{a-i\omega} + \frac{1}{a+i\omega} = \frac{2a}{a^2+\omega^2}$$

$\boxed{/\!/}$

Method 2 $\begin{cases} \cdot \text{ Express as combinations of more familiar/basic examples} \\ \cdot \text{ Then use properties of Fourier transform.} \end{cases}$

observe

$$e^{-a|t|} = e^{-a(-t)} u(-t) + e^{-at} u(t)$$



Properties of Fourier transform.

linearity

for c_1, c_2 constants

$$\boxed{c_1 f(t) + c_2 g(t) \xrightarrow{\mathcal{F}} c_1 \hat{f}(\omega) + c_2 \hat{g}(\omega)}$$

notation

$$\hat{f}(\omega) = \mathcal{F}[f](\omega)$$

$$\mathcal{F}[c_1 f + c_2 g](\omega) = c_1 \mathcal{F}[f](\omega) + c_2 \mathcal{F}[g](\omega)$$

"time reversal"

$$\boxed{\text{If } g(t) = f(-t) \quad \text{then} \quad \hat{g}(\omega) = \hat{f}(-\omega)}$$

$$\mathcal{F}[f(-t)](\omega) \quad \overset{\text{change of variable}}{=} \quad \mathcal{F}[f](-\omega)$$

$t = -s$

$$\left(\int_{-\infty}^{\infty} f(-t) e^{-i\omega t} dt = -\int_{\infty}^{-\infty} f(s) e^{-i(-\omega)s} ds = \int_{-\infty}^{\infty} f(s) e^{-i(-\omega)s} ds \right)$$

Back to EX 1.

$$\mathcal{F}\left[e^{-a|t|}\right](\omega) = \mathcal{F}\left[e^{-a(-t)}u(-t)\right](\omega) + \mathcal{F}\left[e^{-at}u(t)\right](\omega)$$

"linearity"

time reversal

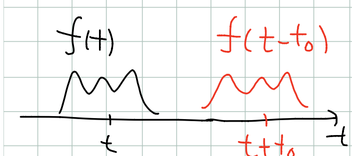
$$= \mathcal{F}\left[e^{-at}u(t)\right](-\omega) + \mathcal{F}\left[e^{-at}u(t)\right](\omega)$$

previous EX

$$\overset{\text{previous EX}}{=} \frac{1}{a-i\omega} + \frac{1}{a+i\omega} = \underline{\frac{2a}{a^2+\omega^2}}$$

▨

· Time shifting



be careful with sign.

$$\boxed{\mathcal{F}\left[f(t-t_0)\right](\omega) = e^{-i\omega t_0}\hat{f}(\omega)}$$

$h(t) = f(t-t_0)$

$\Downarrow$ F.T. $\quad \Uparrow$ Fourier inversion

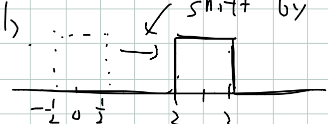
$\hat{h}(\omega) = e^{-i\omega t_0}\hat{f}(\omega)$

<u>note</u> time-shift does <u>not</u> change the amplitude of F.T. but changes the phase of F.T.

<u>EX 2.</u> $h(t) = \begin{cases} 1 & 2<t<3 \\ 0 & \text{otherwise} \end{cases}$

$\hat{h}(\omega) = ?$

<sol>



shift by $2\frac{1}{2} = \frac{5}{2}$ $\quad \therefore \; h(t) = \mathrm{rect}(t-\frac{5}{2})$

$$\hat{h}(\omega) = e^{-i\frac{5}{2}\omega}\,\widehat{\mathrm{rect}}(\omega) = e^{-i\frac{5}{2}\omega}\mathrm{sinc}\left(\frac{\omega}{2}\right)$$

▨

EX3. $h(t) = \begin{cases} 1 & 2<t<3 \\ -3 & 5<t<6 \\ 2 & 10<t<11 \\ 0 & \text{otherwise} \end{cases}$ $\qquad \hat{h}(\omega) = ?$

$5\frac{1}{2} = \frac{11}{2}$ $\qquad 10\frac{1}{2} = \frac{21}{2}$

<sol> $h(t) = \text{rect}(t - \frac{5}{2}) - 3\,\text{rect}(t - \frac{11}{2}) + 2\,\text{rect}(t - \frac{21}{2})$.

$$\hat{h}(\omega) = e^{-i\frac{5}{2}\omega} \text{sinc}\left(\frac{\omega}{2}\right) - 3\, e^{-i\frac{11}{2}\omega} \text{sinc}\left(\frac{\omega}{2}\right) + 2\, e^{-i\frac{21}{2}\omega} \text{sinc}\left(\frac{\omega}{2}\right).$$ ▨

Used · linearity
· time-shift

EX4. $h(t) = \begin{cases} e^{-|t|} & \text{for } |t| > 3 \\ 0 & \text{otherwise} \end{cases}$

<sol>
· sketch graph to get an idea

let this part $h_1(t) = \begin{cases} e^{-t}, & t>3 \\ 0 & \text{otherwise} \end{cases}$

$h(t) = h_1(-t) + h_1(t)$

$h_1(t) = e^{-(t-3)-3} u(t-3)$

$= e^{-3} \underbrace{e^{-(t-3)} u(t-3)}_{f(t-3)}$

for $f(t) = e^{-t} u(t)$

$\therefore \mathcal{F}[h(t)]$

$= \mathcal{F}[h_1(-t)](\omega) \overset{\text{linearity}}{+} \mathcal{F}[h_1(t)](\omega)$

$$= \mathcal{F}[h_1(t)](-\omega) + \mathcal{F}[h_1(t)](\omega)$$

time-reversal

Now. $\mathcal{F}[h_1(t)](\omega) = \mathcal{F}[e^{-3}\, e^{-(t-3)} u(t-3)](\omega)$

$$= e^{-3}\, \mathcal{F}[e^{-(t-3)} u(t-3)](\omega) \qquad \text{for } a>0$$

$$= e^{-3}\, e^{-i3\omega} \frac{1}{1+i\omega}. \longleftarrow \mathcal{F}[e^{-at} u(t)](\omega) = \frac{1}{a+i\omega}.$$

time shifting

So,

$$\mathcal{F}[h(t)](\omega) = e^{-3}\, e^{i3\omega} \frac{1}{1-i\omega} + e^{-3} \cdot e^{-i3\omega} \frac{1}{1+i\omega}$$

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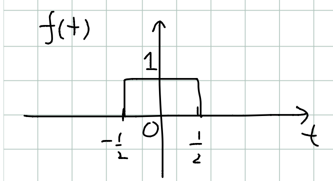
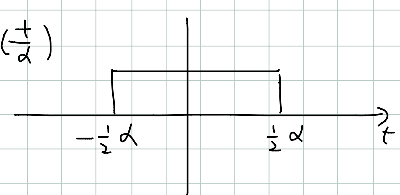
for constant $\alpha \neq 0$,

Scaling $\mathcal{F}[f(\frac{t}{\alpha})](\omega) = |\alpha|\, \mathcal{F}[f(t)](\alpha\omega)$

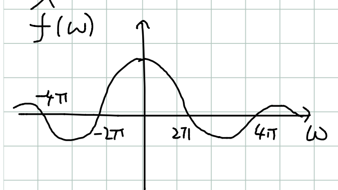
$f(t)$



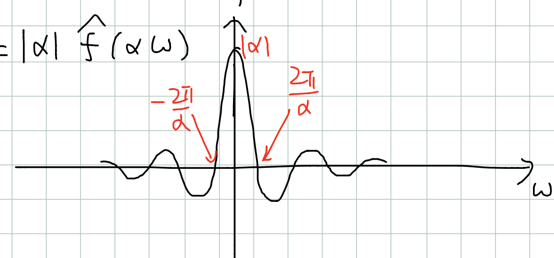
$h(t) = f(\frac{t}{\alpha})$



$\hat{f}(\omega)$



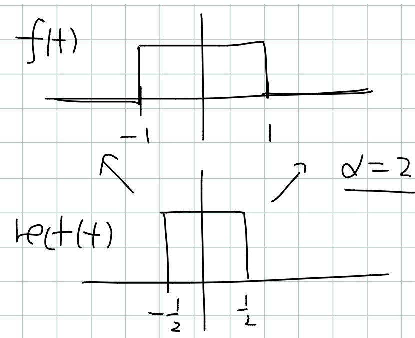
$\hat{h}(\omega) = |\alpha|\, \hat{f}(\alpha\omega)$



" The wider the signal, the narrower its Fourier transform.

(The wider the Fourier transform, the narrower the original signal.)

EX 5. $f(t) = \begin{cases} 1 & -1 < t < 1 \\ 0 & \text{otherwise} \end{cases}$



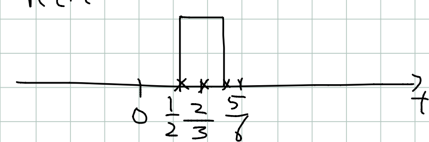
$\hat{f}(\omega) = ?$

<sol> $f(t) = \text{rect}\left(\frac{t}{2}\right) \qquad \alpha = 2$

$\therefore \hat{f}(\omega) = 2\,\widehat{\text{rect}}(2\omega) = 2\,\text{sinc}(\omega)$. $\boxed{}$

EX 6* $f(t) = \text{rect}(3t-2)$

$\text{rect}(3t-2) = \text{rect}\left(3\left(t-\frac{2}{3}\right)\right)$



$\hat{f}(\omega) = ?$

<sol>.

Method 1 $\quad h(t) = \text{rect}(t-2)$

scaling $\quad f(t) = h(3t) \longleftarrow \alpha = \frac{1}{3}$

$\therefore \hat{f}(\omega) = \frac{1}{3}\hat{h}\left(\frac{1}{3}\omega\right)$

$\hat{h}(\omega) = \mathcal{F}[\text{rect}(t-2)]$

time-shifting $\quad = e^{-i2\omega}\,\widehat{\text{rect}}(\omega)$

$= e^{-i2\omega}\,\text{sinc}\left(\frac{\omega}{2}\right)$

$= \frac{1}{3} e^{-i2\cdot\frac{\omega}{3}}\,\text{sinc}\left(\frac{\omega}{6}\right)$.

Method 2

$\text{rect}(3t-2) = \text{rect}\left(3\left(t-\frac{2}{3}\right)\right)$

$\hat{f}(\omega) = \mathcal{F}\left[\text{rect}\left(3\left(t-\frac{2}{3}\right)\right)\right](\omega)$

$= e^{-i\frac{2}{3}\omega}\,\mathcal{F}[\text{rect}(3t)](\omega) \longleftarrow$ time-shifting

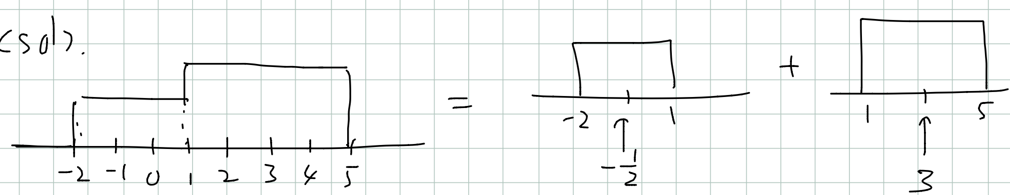
$= e^{-i\frac{2}{3}\omega}\cdot\frac{1}{3}\mathcal{F}[\text{rect}(t)]\left(\frac{\omega}{3}\right) \longleftarrow$ scaling.

$= \frac{1}{3} e^{-i\frac{2}{3}\omega}\,\text{sinc}\left(\frac{\omega}{6}\right)$. $\boxed{}$

Ex 6 $f(t) = \begin{cases} 2 & -2 < t < 1 \\ 3 & 1 < t < 5 \\ 0 & \text{otherwise} \end{cases}$

Use properties of F.T. & basic examples to compute $\hat{f}(\omega)$

(sol).



$$f(t) = 2\,\text{rect}\left(\tfrac{1}{3}\left(t+\tfrac{1}{2}\right)\right) + 3\,\text{rect}\left(\tfrac{1}{4}(t-3)\right)$$

$$\therefore \hat{f}(\omega) = 2\,\mathcal{F}\left[\text{rect}\left(\tfrac{1}{3}\left(t+\tfrac{1}{2}\right)\right)\right](\omega) + 3\,\mathcal{F}\left[\text{rect}\left(\tfrac{1}{4}(t-3)\right)\right](\omega)$$ "linearity"

$$= 2 \cdot e^{i\frac{\omega}{2}}\,\mathcal{F}\left[\text{rect}\left(\tfrac{1}{3}t\right)\right](\omega) + 3 \cdot e^{-i3\omega}\,\mathcal{F}\left[\text{rect}\left(\tfrac{1}{4}t\right)\right](\omega)$$ "time-shifting"

$$= 2 \cdot e^{i\frac{\omega}{2}} \cdot 3\,\mathcal{F}[\text{rect}(t)](3\omega) + 3\,e^{-i3\omega} \cdot 4\,\mathcal{F}[\text{rect}(t)](4\omega)$$ "scaling"

$$= 6\,e^{i\frac{\omega}{2}}\,\text{sinc}\left(\tfrac{3\omega}{2}\right) + 12\,e^{-i3\omega}\,\text{sinc}(2\omega)$$ " $\widehat{\text{rect}}(\omega) = \text{sinc}\left(\frac{\omega}{2}\right)$ "

$\blacksquare$