Math 267. 2013 Jan – Apr.

Announcement : Midterm 1. Jan 31. Thur. 7 pm (1 hr)
Midterm 2. Mar 7. Thur. 7 pm (1 hr)

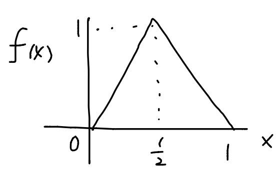
Lecture 1

- Idea of Fourier series
- complex numbers
- complex exponential

• Fourier series.

Main idea A "reasonable" function can be expressed as the (infinite) sum of "basic" functions.

e.g.



$$f(x) = \frac{1}{4} - 2 \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \frac{1}{\pi^2 n^2} \cos(2\pi n x)$$

$$\overset{\text{i.e.}}{=} \frac{1}{4} - 2\left(\frac{1}{\pi^2} \cos(2\pi x) + \frac{1}{\pi^2 3^2} \cos(6\pi x) + \frac{1}{\pi^2 5^2} \cos(10\pi x) + \cdots \right)$$

More generally,
for $f(x)$ on $[0, L]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(2\pi \frac{n}{L} x\right) + b_n \sin\left(2\pi \frac{n}{L} x\right) \right\}$$

another form

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi i k x} = \cdots + c_{-1} e^{-2\pi i x} + c_0 + c_1 e^{2\pi i x} + c_2 e^{4\pi i x} + \cdots$$

where $i = \sqrt{-1}$, $e^{2\pi i k x} = \cos(2\pi k x) + i \sin(2\pi k x)$

Why useful?

$$\sum c_k e^{i2\pi kx}$$

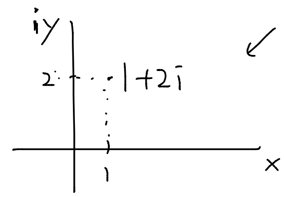
information (e.g. talking on a cell phone $f(x)$) $\xrightarrow{\text{transform}}$ signals as sums of basic signals (functions) by giving different coefficients. c_k

- can make systematic operations on such sums. (e.g. transmission, denoising, encryption, compression, etc)

We have to know

Complex Numbers ($\mathbb{C}$) (real numbers $= \mathbb{R}$)

$i = \sqrt{-1}, \quad i^2 = -1.$



complex plane

e.g. $1 + 2i$

real part ↗ ↖ imaginary part

$Re(1+2i) = 1$

$Im(1+2i) = 2$

- addition: $(3+5i) + (7-3i)$
$= 3+7 + (5-3)i$

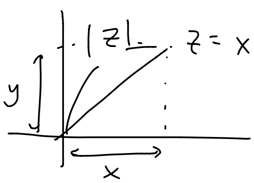
- multiplication: $(3+5i) \times (7-3i)$
$= 3\times 7 + 3\times(-3i) + 5i\times 7 + 5i\times(-3i)$
$= 21 - 9i + 35i \underbrace{-15i^2}_{15}$ ← $i^2 = -1$
$= 36 + 26i$

Complex conjugate

$z = x + iy \longrightarrow \bar{z} = x - iy$ ← replace i with $-i$

$$z\bar{z} = (x+iy)(x-iy) = x^2 - (iy)^2 = x^2 + y^2$$

$(A+B)(A-B) = A^2 - B^2$



$z = x + iy$ $\quad |z| = \sqrt{x^2+y^2}$ $\quad \therefore \boxed{z\bar{z} = |z|^2}$

"modulus" of z

(or magnitude)

Division

$\frac{1}{i} = ?$

$i \cdot (-i) = -i^2 = -(-1) = 1 \qquad \therefore \boxed{\frac{1}{i} = -i}$

Also,

$$\frac{1}{z} = \frac{1}{z} \cdot \frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{|z|^2}$$

Ex Find real & imaginary part of $\frac{2+i}{1-i}$

<sol> $\frac{2+i}{1-i} = \frac{(2+i)(1+i)}{(1-i)(1+i)} = \frac{2+2i+i+(-1)}{1^2+1^2} = \frac{1+3i}{2} = \frac{1}{2} + \frac{3}{2}i$

$\mathrm{Re}\left(\frac{2+i}{1-i}\right) = \frac{1}{2}$

$\mathrm{Im}\left(\frac{2+i}{1-i}\right) = \frac{3}{2}$. ▨

Ex $z \in \mathbb{C}$. Express $\mathrm{Re}\, z$, $\mathrm{Im}\, z$ using z & $\bar{z}$.

<sol> Let $z = x + iy$. Then $\bar{z} = x - iy$

$\therefore z + \bar{z} = 2x \qquad z - \bar{z} = 2iy$

$\mathrm{Re}\, z = x = \frac{z+\bar{z}}{2}$, $\qquad \mathrm{Im}\, z = y = \frac{z-\bar{z}}{2i}$ ▨

Ex modulus of $(2+i)^{12}$

(sol) Note $\boxed{|z_1 z_2| = |z_1||z_2|}$

$$|(2+i)^{12}| = |2+i|^{12} = \left(\sqrt{2^2+1^2}\right)^{12} = 5^{\frac{12}{2}} = 5^6 \quad \blacksquare$$

Complex exponential

$t \in \mathbb{R}$. $\quad e^{it} = \cos t + i \sin t$

$x, y \in \mathbb{R}$
$z = x+iy$ $\quad e^z = e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i \sin y)$

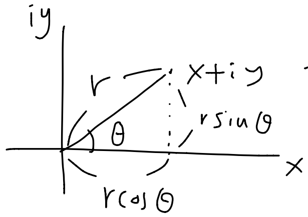
e.g. $\overline{e^z} = \overline{e^x \cos y + i e^x \sin y} = e^x \cos y - i e^x \sin y$

$$= e^x(\cos y - i \sin y) = e^x(\cos(-y) + i \sin(-y))$$

$$= e^{x-iy} = e^{\bar{z}}. \qquad \boxed{\overline{e^z} = e^{\bar{z}}}$$

Note : $z_1, z_2 \in \mathbb{C}$. $\quad \underline{e^{z_1} e^{z_2} = e^{z_1+z_2}}$

Polar form



here $r = |z|$.

$$x+iy = r\cos\theta + i r \sin\theta = r(\cos\theta + i\sin\theta)$$

$$= r e^{i\theta}$$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

note $|e^{i\theta}| = 1$.

e.g.

$i = e^{i\frac{\pi}{2}} = e^{i(\frac{\pi}{2}+2\pi)} = e^{i(\frac{\pi}{2}+4\pi)}$

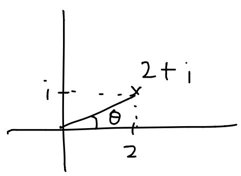
$-1 \qquad 1 = e^{i0} = e^{i2\pi} = e^{-i2\pi}$

$\parallel$

$e^{i\pi} = e^{-i\pi} \qquad\qquad 1 = e^{i2n\pi} \quad n = 0, \pm 1, \pm 2, \cdots$

· Ex Express $(2+i)^{12}$ in polar form.

<sol>



$\theta = \tan^{-1}(\frac{1}{2})$

$r = \sqrt{2^2+1^2} = \sqrt{5}$

$\therefore (2+i)^{12} = \left[\sqrt{5}\, e^{i\tan^{-1}(\frac{1}{2})}\right]^{12} = 5^{\frac{12}{2}} e^{i12\tan^{-1}(\frac{1}{2})}$

$= 5^6 e^{i12\tan^{-1}(\frac{1}{2})}$ ▨

· Ex solve $z^5 = 18i$.

<sol> Use polar form.

· $z = re^{i\theta} \qquad z^5 = (re^{i\theta})^5 = r^5 e^{i5\theta}$

· $18i = 18e^{i(\frac{\pi}{2}+2n\pi)}$, $n = 0, \pm 1, \pm 2, \cdots$

$z^5 = 18i \iff r^5 e^{i5\theta} = 18e^{i(\frac{\pi}{2}+2n\pi)}$

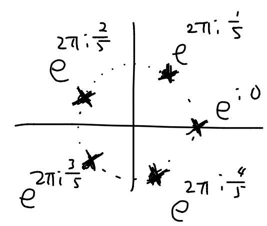
$\therefore r^5 = 18, \quad 5\theta = \frac{\pi}{2} + 2n\pi$

$r = 18^{\frac{1}{5}}, \quad \theta = \frac{\pi}{10} + \frac{2}{5}n\pi, \quad n = 0, \pm 1, \pm 2, \cdots$

$\therefore z = 18^{\frac{1}{5}} e^{i(\frac{\pi}{10}+\frac{2}{5}n\pi)}$, $n = 0, \pm 1, \pm 2, \cdots$

$= \boxed{18^{\frac{1}{5}} e^{i\frac{\pi}{10}} \cdot e^{2\pi i \frac{n}{5}}, \quad n = 0, 1, 2, 3, 4}$ ▨

$e^{\frac{i2n\pi}{5}}$



Ex Write $\sum_{k=-\infty}^{\infty} 2^{-|k|} \cos(2kx)$ as a sum of complex exponentials.

(like $\sum c_k e^{i2\pi kx}$)

<sol>. Note $\boxed{\cos\theta = Re(e^{i\theta}) = \frac{1}{2}(e^{i\theta} + e^{-i\theta})}$

$$\therefore \quad \cos(2kx) = \frac{1}{2}\left[e^{i2kx} + e^{-i2kx}\right]$$

$$\therefore \quad \sum_{k=-\infty}^{\infty} 2^{-|k|} \cos(2kx) = \sum_{k=-\infty}^{\infty} 2^{-|k|} \frac{1}{2}\left[e^{i2kx} + e^{-i2kx}\right]$$

$$= \sum_{k=-\infty}^{\infty} 2^{-|k|-1}\left[e^{i2kx} + e^{-i2kx}\right]$$

$$= \sum_{k=-\infty}^{\infty} 2^{-|k|-1} e^{i2kx} + \sum_{k=-\infty}^{\infty} 2^{-|k|-1} e^{-i2kx}$$

$$= \sum_{k=-\infty}^{\infty} 2^{-|k|-1} e^{i2kx} + \sum_{k=-\infty}^{\infty} 2^{-|k|-1} e^{i2kx}$$

replace k with -k.

$$= \sum_{k=-\infty}^{\infty} 2^{-|k|} e^{i2kx}$$

$\blacksquare$