Lec 3 . Separation of Variables method – Continue

— how to solve eigenvalue problem for ODE

— Matching IC: introduction to Fourier Series.

* Eigenvalue Problem

Find all $\gamma \in \mathbb{R}$ and functions $X(x) \not\equiv 0$ solving
$$\begin{cases} X''(x) + \gamma X(x) = 0 \\ X(0) = 0 = X(L) \end{cases}$$

<sol>.

To solve the ODE try $X(x) = e^{rx}$:
$$(e^{rx})'' + \gamma (e^{rx}) = 0$$
$$\Rightarrow e^{rx}(r^2 + \gamma) = 0$$

get characteristic eqn : $\underline{r^2 + \gamma = 0}$

$\Rightarrow r = \pm i\sqrt{\gamma}$.

<u>Case</u> $\gamma = 0$: $X'' = 0 \Rightarrow X(x) = A + Bx$, A, B constants.

But <u>BC</u> $\left.\begin{array}{l} 0 = X(0) = A \\ 0 = X(L) = B \cdot L \end{array}\right\} \Rightarrow A = 0 = B$

$\therefore X(x) = 0$ trivial solution (NOT what we want!)

Thus, discard this case. Therefore, $\gamma \neq 0$.

<u>Case</u> $\gamma \neq 0$: $X(x) = C_1 e^{i\sqrt{\gamma}x} + C_2 e^{-i\sqrt{\gamma}x}$. C_1, C_2 constants

But, $\underline{\underline{BC}}$ $\quad 0 = X(0) = C_1 + C_2 \Rightarrow C_2 = -C_1$

$$0 = X(L) = C_1 e^{i\sqrt{\gamma} L} + C_2 e^{-i\sqrt{\gamma} L}$$

$$= C_1 \left[e^{i\sqrt{\gamma} L} - e^{-i\sqrt{\gamma} L} \right]$$

$C_2 = -C_1$

Note: $C_1 \neq 0$ (otherwise $X \equiv 0$ which is not what we want)

Thus $\quad e^{i\sqrt{\gamma} L} - e^{-i\sqrt{\gamma} L} = 0$

$$\Rightarrow \quad e^{2i\sqrt{\gamma} L} = 1.$$

$e^{i\theta} = 1$

$\theta = 2\sqrt{\gamma} L$

$$\Rightarrow \quad 2\sqrt{\gamma} L = 2\pi k, \quad k = \text{integer}.$$

$$\therefore \quad \boxed{\gamma = \frac{k^2 \pi^2}{L^2}, \quad k = 1, 2, 3, \ldots}$$

can also let $k = \pm 1, \pm 2, \pm 3, \ldots$ but it is redundant because of the expression k^2

(here $k \neq 0$ since $\gamma \neq 0$)

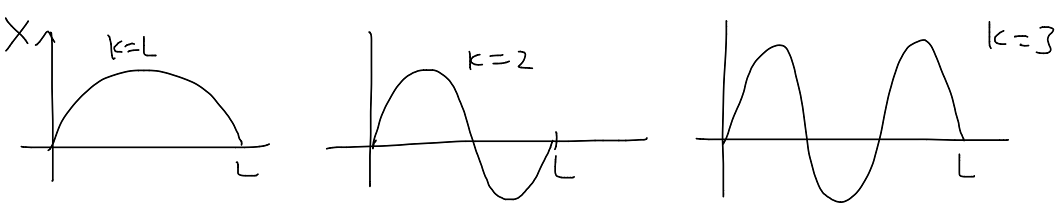
$\sqrt{\gamma} = \frac{k\pi}{L}$

$$\therefore \quad \boxed{X(x) = C_1 \left[e^{i \frac{k\pi}{L} x} - e^{-i \frac{k\pi}{L} x} \right]}$$

$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$

here also can write

$$\therefore \quad \boxed{X(x) = D_1 \sin\left(\frac{k\pi}{L} x\right), \quad k = 1, 2, 3, \ldots \quad D_1 = \text{arbitrary constant.}}$$



* How to match IC.

Recall (WE) $u_{tt} = c^2 u_{xx}$

(BC) $u(0,t) = 0 = u(L,t)$, $t > 0$

(IC) $\begin{cases} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$ for $0 < x < L$

From previous lecture $\begin{cases} \text{I. separation of variables solutions} \\ \text{II. superposition of solutions} \end{cases}$

We get a general solution:

$$u(x,t) = \sum_{k=1}^{\infty} \sin\left(\frac{k\pi}{L}x\right)\left[\alpha_k \cos\left(\frac{k\pi}{L}ct\right) + \beta_k \sin\left(\frac{k\pi}{L}ct\right)\right]$$

satisfying (WE) & (BC).

To match (IC), need to find α_k, β_k (constants) such that

- $f(x) = u(x,0) = \sum_{k=1}^{\infty} \alpha_k \sin\left(\frac{k\pi}{L}x\right)$ for $0 < x < L$

for $t=0$ $\cos\left(\frac{k\pi}{L}c\cdot 0\right) = \cos(0) = 1$ & $\sin\left(\frac{k\pi}{L}c\cdot 0\right) = \sin 0 = 0$

- $g(x) = u_t(x,0) = \sum_{k=1}^{\infty} \beta_k \frac{k\pi c}{L} \sin\left(\frac{k\pi}{L}x\right)$ for $0 < x < L$.

note $\left.\frac{\partial}{\partial t}\right|_{t=0} \left(\alpha_k \cos\left(\frac{k\pi}{L}ct\right) + \beta_k \sin\left(\frac{k\pi}{L}ct\right)\right) = \frac{k\pi c}{L}\beta_k$

- Is it possible?

YES: "Fourier sine series"

Any "reasonable" function $h(x)$ on $0 < x < L$

can be expressed as

$$h(x) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi}{L}x\right)$$

for some b_k = Fourier coeff. of $h(x)$.

* How to find b_k "Fourier coeff."

Ans: $\boxed{b_k = \frac{2}{L}\int_0^L h(x)\sin\left(\frac{k\pi}{L}x\right)dx}$ $\quad k = 1, 2, 3, \cdots$

How do we derive this formula for b_k?

Important exercise "orthogonality"

For n, k natural numbers

$$\boxed{\int_0^L \sin\left(\frac{k\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx = \begin{cases} 0 & \text{if } n \neq k \\ \frac{L}{2} & \text{if } n = k \end{cases}}$$

This is very useful:

Let $h(x) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi}{L}x\right)$.

for each $n = 1, 2, 3, \cdots$

Then $\int_0^L h(x)\sin\left(\frac{n\pi}{L}x\right)dx = \int_0^L \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)$

$$= \sum_{k=1}^{\infty} b_k \int_0^L \sin\left(\frac{k\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx$$

$$= 0 + 0 + \cdots + 0 + \underbrace{b_n \frac{L}{2}}_{\text{when } k=n} + 0 + 0 + \cdots\cdots$$

Comparing each sides we get

$\therefore$ $\underbrace{b_n = \frac{2}{L}\int_0^L h(x)\sin\left(\frac{n\pi}{L}x\right)dx}$. for $n = 1, 2, 3, \cdots$

$\square$

EX Solve
$$\begin{cases} u_{tt} = 4u_{xx} & \\ u(0,t) = 0 = u(1,t) & t > 0 \\ u(x,0) = 1 & 0 < x < 1 \\ u_t(x,0) = \sin(2\pi x) - 2\sin(4\pi x) & \end{cases}$$

← This is your exercise before the next lecture.

We will quickly cover it in the next lecture.