Math 267 Sec 202
HW3 solutions

#1. This problem is about "orthogonality".

(a) $\int_0^3 f(x)^2\,dx = \int_0^3 \left(3\sin(3\pi x) + 5\sin(7\pi x)\right)^2 dx$

$$= 3^2\int_0^3 \sin(3\pi x)\sin(3\pi x)\,dx + 30\underbrace{\cancel{\int_0^3 \sin(3\pi x)\sin(7\pi x)\,dx}}_{\text{"orthogonality"}} \to 0$$

$$+ 5^2\int_0^3 \sin(7\pi x)\sin(7\pi x)\,dx$$

$$= 3^2 \times \frac{3}{2} + 5^2\cdot\frac{3}{2}$$

$$= \frac{3}{2}\left(3^2+5^2\right)$$

$$= \frac{3}{2}\cdot 34 = 3\times 17$$

$$= \underline{51}.$$ ▨

for $k, l = 1, 2, 3, \dots$

$$\boxed{\int_0^L \sin\left(\frac{k\pi}{L}x\right)\sin\left(\frac{l\pi}{L}x\right)dx = \begin{cases} 0 & k\neq l \\ \frac{L}{2} & k = l \end{cases}}$$

(b) $\int_0^3 f(x)\,p(x)\,dx = \int_0^3 \left(3\sin(3\pi x) + 5\sin(7\pi x)\right)\sum_{k=100}^{200} k^2\sin\left(\frac{k\pi x}{3}\right)dx$

$$= \sum_{k=100}^{200} 3\cdot\int_0^3 \sin(\underbrace{3\pi x}_{\frac{9\pi}{3}x})\sin\left(\frac{k\pi}{3}x\right)dx$$

$$+ \sum_{k=100}^{200} 5\cdot\int_0^3 \sin(\underbrace{7\pi x}_{\frac{21}{3}\pi x})\sin\left(\frac{k\pi}{3}x\right)dx$$

$= 0 + 0$ (since for $100\le k\le 200$, $k\neq 9$, $k\neq 21$ "orthogonality" applies.) ▨

(c) $\int_0^3 g(x)h(x)\,dx = \int_0^3 \sum_{k=1}^{100} \sin(k\pi x/3) \cdot \sum_{\ell=1}^{50} \sin(\ell\pi x/3)\ dx$

(we changed the index, not to be confused with the other sum)

$$= \sum_{k=1}^{100}\sum_{\ell=1}^{50} \int_0^3 \sin\left(\frac{k\pi x}{3}\right) \sin\left(\frac{\ell\pi x}{3}\right) dx$$

Here apply "orthogonality"

$$= \sum_{\substack{k=\ell \\ 1\le k\le 100 \\ 1\le \ell\le 50}} \frac{3}{2} = \sum_{\ell=1}^{50} \frac{3}{2} = \frac{3}{2}\cdot 50 = \underline{\underline{75}}$$

$\boxed{//}$

(d) $\int_0^3 h(x)p(x)\,dx$

$$= \int_0^3 \sum_{k=1}^{50} \sin\left(\frac{k\pi x}{3}\right) \sum_{\ell=100}^{200} \ell^2 \sin\left(\frac{\ell\pi x}{3}\right) dx$$

$$= \sum_{k=1}^{50}\sum_{\ell=100}^{200} \ell^2 \underbrace{\int_0^3 \sin\left(\frac{k\pi x}{3}\right) \sin\left(\frac{\ell\pi x}{3}\right) dx}_{=0 \text{ since } k\ne\ell \text{ for } 1\le k\le 50,\ 100\le \ell\le 200}$$

$$= \underline{0}$$

$\boxed{//}$

(e) $\int_0^3 g(x)\,p(x)\,dx = \int_0^3 \sum_{k=1}^{100} \sin\left(\frac{k\pi x}{3}\right) \sum_{\ell=100}^{200} \ell^2 \sin(\ell\pi x/3)\ dx$

$$= \sum_{k=1}^{100}\sum_{\ell=100}^{200} \ell^2 \underbrace{\int_0^3 \sin\left(\frac{k\pi x}{3}\right) \sin\left(\frac{\ell\pi x}{3}\right) dx}_{= \begin{cases} \frac{3}{2} & \text{for } k=\ell \\ 0 & \text{for } k\ne\ell \end{cases} = \begin{cases} \frac{3}{2} & \text{for } k=\ell=100 \\ 0 & \text{otherwise} \end{cases}}$$

$$= \underbrace{100^2\cdot\frac{3}{2}}_{\ell=100}$$

$\boxed{//}$

note for $1\le k\le 100$, $100\le \ell\le 200$, $k=\ell=100$ is the only case where $k=\ell$.

#2.

(a) Let $u(x,t) = X(x)T(t)$

$u_t = 9u_{xx} \Rightarrow XT' = 9X''T$

$$\Rightarrow \frac{T'}{9T} = \frac{X''}{X} = \text{constant} \overset{\text{set}}{=} -\gamma.$$

$u(0,t) = 0 = u(3,t) \Rightarrow \begin{matrix} X(0)T(t) = 0 \\ X(3)T(t) = 0 \end{matrix} \Rightarrow X(0) = 0 = X(3).$

Therefore $\boxed{1}$ $\begin{cases} X'' + \gamma X = 0 \\ X(0) = 0 = X(3) \leftarrow BC. \end{cases}$

$\boxed{2}$ $T' + 9\gamma T = 0$

$\boxed{1}$: chr. eqn. $r^2 + \gamma r = 0 \Rightarrow r = \pm\sqrt{\gamma}\, i$

<u>Case $\gamma = 0$</u>: $X''(x) = 0 \Rightarrow X(x) = Ax + B.$

BC: $X(0) = 0 \Rightarrow B = 0$

$X(3) = 0 \Rightarrow 0 = 3A \Rightarrow A = 0$

$\therefore X \equiv 0.$ trivial solution, so we exclude this case.

<u>Case $\gamma \neq 0$</u> $X(x) = C_1 e^{i\sqrt{\gamma}x} + C_2 e^{-i\sqrt{\gamma}x}$

BC: $0 = X(0) = C_1 + C_2 \Rightarrow C_1 = -C_2$

$0 = X(3) = C_1\left[e^{i\sqrt{\gamma}\cdot 3} - e^{-i\sqrt{\gamma}\cdot 3}\right]$

$C_1 \neq 0$ since otherwise $X \equiv 0$ trivial sol.

Thus. $e^{i\sqrt{\gamma}3} - e^{-i\sqrt{\gamma}3} = 0$

i.e. $e^{i\sqrt{\gamma}\cdot 3} = e^{-i\sqrt{\gamma}3}$

$\Rightarrow e^{i6\sqrt{\gamma}} = 1 = e^{i2k\pi}$ k integer

$\Rightarrow 6\sqrt{\gamma} = 2k\pi$

$\Rightarrow \sqrt{\gamma} = \frac{k\pi}{3}$ ✓ note that this automatically says $k > 0$ since $\gamma \neq 0$ & $\sqrt{\gamma}$ is <u>real</u>. (from $\sqrt{\gamma} = \frac{k\pi}{3}$)

$\therefore \sqrt{\gamma} = \frac{k\pi}{3}$, $k = 1, 2, 3, \ldots$

$\gamma = \left(\frac{k\pi}{3}\right)^2$.

& $X(x) = C_1\left[e^{i\frac{k\pi}{3}x} - e^{-i\frac{k\pi}{3}x}\right]$

$= D_1 \sin\left(\frac{k\pi}{3}x\right)$, $k = 1, 2, 3, \ldots$

Using the same γ & $\boxed{2}$, we get $T(t) = e^{-\left(\frac{k\pi}{3}\right)^2 \cdot 9t}$

$$\boxed{\text{Thus. } X(x)T(t) = D_1 \sin\left(\frac{k\pi}{3}x\right) e^{-9\left(\frac{k\pi}{3}\right)^2 t}, \quad k = 1, 2, 3 \quad \text{where } D_1 \text{ arbitrary constant.}}$$

$\boxed{///}$

(b) general solution is obtained from superposition

$$u(x,t) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi}{3}x\right) e^{-9\left(\frac{k\pi}{3}\right)^2 t}$$

where b_k arbitrary constant.

note that for each k, we have different arbitrary constant.

$\blacksquare$

(c) Match IC

$$u(x,0) = 1 = \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi}{3}x\right).$$

$$b_k = \frac{2}{3}\int_0^3 1 \cdot \sin\left(\frac{k\pi}{3}x\right) dx = \frac{2}{3}\left[\frac{3}{k\pi}\left(-\cos\left(\frac{k\pi x}{3}\right)\right)\right]_0^3$$

$L=3$

$k=1,2,3,\ldots$ (i.e. $k\neq 0$, k can be in the denominator.)

$$= \frac{2}{k\pi}\left[-\underbrace{\cos(k\pi)}_{(-1)^k} + 1\right]$$

$$= \frac{2}{k\pi}\left(1-(-1)^k\right)$$

$$\therefore\ u(x,t) = \sum_{k=1}^{\infty} \frac{2}{k\pi}\left(1-(-1)^k\right) \sin\left(\frac{k\pi}{3}x\right) e^{-9\left(\frac{k\pi}{3}\right)^2 t}.$$

$\blacksquare$

#3 From the formula for general solution,

$$u(x,t) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi}{2}x\right) e^{-4\cdot\left(\frac{k\pi}{2}\right)^2 t}$$

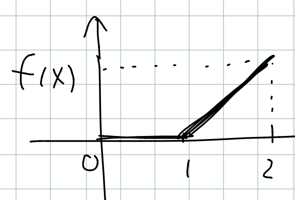
Match IC:

$$f(x) = u(x,0) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi}{2}x\right)$$

Fourier sine series

$$b_k = \frac{2}{2}\int_0^2 f(x)\sin\left(\frac{k\pi}{2}x\right)dx$$

$L=2$



$$= \int_0^2 f(x)\sin\left(\frac{k\pi}{2}x\right)dx$$

$$= \int_0^1 0\sin\left(\frac{k\pi}{2}x\right)dx + \int_1^2 (x-1)\sin\left(\frac{k\pi}{2}x\right)dx$$

$$\therefore\ b_k = \int_1^2 x\sin\left(\frac{k\pi}{2}x\right)dx - \int_1^2 \sin\left(\frac{k\pi}{2}x\right)dx$$

for $k=1,2,3,\cdots$

To compute the integrals,

① $$\int_1^2 x\sin\left(\frac{k\pi}{2}x\right)dx = \left[x\cdot\frac{-2}{k\pi}\cos\left(\frac{k\pi}{2}x\right)\right]_1^2 - \int_1^2 1\cdot\frac{-2}{k\pi}\cos\left(\frac{k\pi}{2}x\right)dx$$

integration by parts

$$= \frac{-2}{k\pi}\left[x\cos\left(\frac{k\pi}{2}x\right)\right]_1^2 + \frac{2}{k\pi}\int_1^2 \cos\left(\frac{k\pi}{2}x\right)dx$$

$$= \frac{-2}{k\pi}\left[x\cos\left(\frac{k\pi}{2}x\right)\right]_1^2 + \left(\frac{2}{k\pi}\right)\left[\left(\frac{2}{k\pi}\right)\sin\left(\frac{k\pi}{2}x\right)\right]_1^2$$

$$= \frac{-2}{k\pi}\left[x\cos\left(\frac{k\pi}{2}x\right)\right]_1^2 + \left(\frac{2}{k\pi}\right)^2\left[\sin\left(\frac{k\pi}{2}x\right)\right]_1^2$$

$$= \frac{-2}{k\pi}\left[2\underbrace{\cos(k\pi)}_{=(-1)^k} - \underbrace{\cos\left(\frac{k\pi}{2}\right)}_{=0 \text{ if } k \text{ odd}}\right] + \left(\frac{2}{k\pi}\right)^2\left[\underbrace{\cancel{\sin(k\pi)}}_{0} - \underbrace{\sin\left(\frac{k\pi}{2}\right)}_{=0 \text{ if } k \text{ even}}\right]$$

$$= \begin{cases} \frac{-2}{k\pi}\left[2(-1)^k - \cos\left(\frac{k\pi}{2}\right)\right] & \text{if } k \text{ even} \\ -\frac{2}{k\pi}\cdot 2(-1)^k - \left(\frac{2}{k\pi}\right)^2 \sin\left(\frac{k\pi}{2}\right) & \text{if } k \text{ odd} \end{cases}$$

$$= \begin{cases} \frac{+2}{k\pi}\left[-2 + \cos\left(\frac{k\pi}{2}\right)\right] & \text{for } k \text{ even} \\ \frac{2}{k\pi}\left[2 - \left(\frac{2}{k\pi}\right)\sin\left(\frac{k\pi}{2}\right)\right] & \text{for } k \text{ odd} \end{cases}$$

② $$\int_1^2 \sin\left(\frac{k\pi}{2}x\right)dx \underset{k\neq 0}{=} \frac{2}{k\pi}\left[-\cos\left(\frac{k\pi}{2}x\right)\right]_1^2$$

$$= \frac{2}{k\pi}\left[-\cos(k\pi) + \cos\left(\frac{k\pi}{2}\right)\right]$$

$$= \begin{cases} \frac{2}{k\pi}\left[-1 + \cos\left(\frac{k\pi}{2}\right)\right], & k \text{ even} \\ \frac{2}{k\pi}, & k \text{ odd.} \end{cases}$$

$$\therefore \quad b_k = ① - ②$$

$$= \begin{cases} \frac{+2}{k\pi}\left[-2 + \cos\left(\frac{k\pi}{2}\right)\right] & \text{for } k \text{ even} \\ \frac{2}{k\pi}\left[2 - \left(\frac{2}{k\pi}\right)\sin\left(\frac{k\pi}{2}\right)\right] & \text{for } k \text{ odd} \end{cases}$$

$$- \begin{cases} \frac{2}{k\pi}\left[-1 + \cos\left(\frac{k\pi}{2}\right)\right], & k \text{ even} \\ \frac{2}{k\pi}, & k \text{ odd.} \end{cases}$$

$$= \begin{cases} \frac{-2}{k\pi} & k \text{ even} \\ \frac{2}{k\pi} - \left(\frac{2}{k\pi}\right)^2 \sin\left(\frac{k\pi}{2}\right) & k \text{ odd} \end{cases} = \frac{2}{k\pi}(-1)^k - \left(\frac{2}{k\pi}\right)^2 \sin\left(\frac{k\pi}{2}\right).$$

$$\boxed{u(x,t) = \sum_{\substack{k>0 \\ k \text{ odd}}} \left[\frac{2}{k\pi}(-1)^k - \left(\frac{2}{k\pi}\right)^2 \sin\left(\frac{k\pi}{2}\right)\right] \sin\left(\frac{k\pi}{2}x\right) e^{-4\left(\frac{k\pi}{2}\right)^2 t}}$$

$\leftarrow$ This answer should be enough.

#4.

(a)

Step1. steady state. $u_p(x)$. (for the PDE & BC)

$$u_t = 4u_{xx} \Rightarrow 0 = 4u_p''(x) \Rightarrow u_p''(x) = 0$$

$$\Rightarrow u_p(x) = Ax + B \qquad A, B \text{ constant.}$$

$$\begin{matrix} u(0,t)=1 \\ u(2,t)=9 \end{matrix} \Rightarrow \begin{matrix} u_p(0) = 1 = B \\ u_p(2) = 9 = 2A + B \end{matrix} \qquad \therefore B = 1, \ A = 4$$

$$\therefore \underline{u_p(x) = 4x + 1}$$

(b)

Step2. general sol. to homogeneous problem. $\begin{cases} u_t = 4u_{xx} \\ u(0,t) = 0 \\ u(2,t) = 0 \end{cases}$

$0 < x < 2$

$t > 0$

$$u_h(x,t) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi}{2}x\right) e^{-4\left(\frac{k\pi}{2}\right)^2 t}$$

Step3 general sol. to the PDE & BC. inhomogeneous

$$u(x,t) = u_p(x) + u_h(x,t)$$

$$= 4x + 1 + \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi}{2}x\right) e^{-4\left(\frac{k\pi}{2}\right)^2 t}$$

Step 4 Matching IC.

$$g(x) = u(x,0) = 4x+1 + \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi}{2}x\right).$$

$$g(x) - (4x+1) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi}{2}x\right)$$

$$\therefore \quad b_k = \frac{2}{2}\int_0^2 \left[g(x) - (4x+1)\right] \sin\left(\frac{k\pi}{2}x\right) dx$$

$$= \int_0^2 \left[g(x) - (4x+1)\right] \sin\left(\frac{k\pi}{2}x\right) dx$$

$$= \int_0^1 \left[-1 - (4x+1)\right] \sin\left(\frac{k\pi}{2}x\right) dx$$

$$+ \int_1^2 \left[1 - (4x+1)\right] \sin\left(\frac{k\pi}{2}x\right) dx$$

$$= \int_0^1 (-4x-2) \sin\left(\frac{k\pi}{2}x\right) dx$$

$$+ \int_1^2 -4x \sin\left(\frac{k\pi}{2}x\right) dx$$

$$= -4\int_0^1 x \sin\left(\frac{k\pi}{2}x\right) dx - 2\int_0^1 \sin\left(\frac{k\pi}{2}x\right) dx$$

$$-4\int_1^2 x \sin\left(\frac{k\pi}{2}x\right) dx$$

$$= -4\int_0^2 x \sin\left(\frac{k\pi}{2}x\right) dx - 2\int_0^1 \sin\left(\frac{k\pi}{2}x\right) dx$$

(used $\int_a^b f + \int_b^c f = \int_a^c f$)

Now
$$\int_a^b x\sin\left(\frac{k\pi}{2}x\right)dx \underset{k\neq 0}{=} -\frac{2}{k\pi}\,x\cos\left(\frac{k\pi}{2}x\right)\Bigg]_a^b + \frac{2}{k\pi}\int_a^b \cos\left(\frac{k\pi}{2}x\right)dx$$

$$= \left[-\frac{2}{k\pi}\,x\cos\left(\frac{k\pi}{2}x\right) + \left(\frac{2}{k\pi}\right)^2 \sin\left(\frac{k\pi}{2}x\right)\right]_a^b$$

$$\therefore \int_0^2 x\sin\left(\frac{k\pi}{2}x\right)dx = -\frac{2}{k\pi}\,2\cdot\cos(k\pi) + \left(\frac{2}{k\pi}\right)^2 \cancelto{0}{\sin(k\pi)} = -\frac{4}{k\pi}(-1)^k$$

Also
$$\int_0^1 \sin\left(\frac{k\pi}{2}x\right)dx = -\frac{2}{k\pi}\cos\left(\frac{k\pi}{2}x\right)\Bigg]_0^1$$

$$= -\frac{2}{k\pi}\left[\cos\left(\frac{k\pi}{2}\right) - 1\right]$$

All things together,

$$b_k = \frac{16}{k\pi}(-1)^k + \frac{4}{k\pi}\left[\cos\left(\frac{k\pi}{2}\right)-1\right] = \frac{4}{k\pi}\left[4(-1)^k + \cos\left(\frac{k\pi}{2}\right) - 1\right]$$

$$\therefore u(x,t) = \sum_{k=1}^{\infty} \frac{4}{k\pi}\left[4(-1)^k + \cos\left(\frac{k\pi}{2}\right) - 1\right]\sin\left(\frac{k\pi}{2}x\right)e^{-4\left(\frac{k\pi}{2}\right)^2 t}$$

QED

#5.

(a) steady state $u_p(x)$

from PDE

$u_{tt} = 4u_{xx} \Rightarrow \underset{0}{\overset{}{\underset{\|}{\partial^2_{tt} u_p(x)}}} = 4 \underbrace{\partial^2_{xx} u_p}_{u_p''}$

$\therefore\ 0 = 4u_p''$ $\qquad \therefore\ u_p'' = 0$ $\qquad$ So, $u_p(x) = Ax + B$

from BC

$u_x(0,t) = 1 \Rightarrow u_p'(0) = 1 \Rightarrow \underline{A = 1}$

$u(2,t) = -3 \Rightarrow u_p(2) = -3 \Rightarrow 2A + B = -3 \qquad \therefore\ \underline{B = -5}$

So, $\underline{u_p(x) = x - 5.}$ ☒

(b) The general solution will be $u_p(x) + u_h(x,t)$

where

u_h : general sol. to $u_{tt} = 4u_{xx}$ with homogeneous BC

!!! $\longrightarrow \begin{cases} u_x(0,t) = 0 \\ u(2,t) = 0 \end{cases}$

To find u_h :

Separation of variables

try $X(x)T(t)$

$u_{tt} = 4u_{xx} \Rightarrow XT'' = 4X''T$

$\Rightarrow \dfrac{T''}{4T} = \dfrac{X''}{X} = \text{const.} \overset{\text{say}}{=} -\gamma$

get . $X'' + \gamma X = 0$

. $T'' + 4\gamma T = 0$

Use BC : $u_x(0,t) = 0 \Rightarrow X'(0)\,T(t) = 0$

$u(2,t) = 0 \Rightarrow X(2)\,T(t) = 0$

$\Rightarrow$ $X'(0) = 0$, $X(2) = 0$

$\therefore$ two problems

$\boxed{1}$ $\begin{cases} X'' + \gamma X = 0 \\ X'(0) = 0 = X(2) \end{cases}$

$\boxed{2}$ $T'' + \gamma T = 0$.

Solve $\boxed{1}$: chr. eqn $r^2 + \gamma = 0$ $r = \pm\sqrt{\gamma}$;

Case $\gamma = 0$: $X'' = 0 \Rightarrow X(x) = Cx + D$

BC $\Rightarrow$ $X'(0) = 0 \Rightarrow C = 0$

$X(2) = 0 \Rightarrow 2C + D = 0$

$\therefore C = 0$, $D = 0$ $\Rightarrow X \equiv 0$

got trivial sol.

Case $\gamma \neq 0$ $X(x) = C_1 e^{i\sqrt{\gamma}x} + C_2 e^{-i\sqrt{\gamma}x}$ (in this case $\gamma \neq 0$)

BC : $X'(0) = 0 \Rightarrow C_1 i\sqrt{\gamma} - C_2 i\sqrt{\gamma} = 0 \Rightarrow C_1 = C_2$

$X(2)=0 \Rightarrow C_1 e^{2i\sqrt{\gamma}} + C_2 e^{-2i\sqrt{\gamma}} = 0$

$\Rightarrow C_1 \left[e^{2i\sqrt{\gamma}} + e^{-2i\sqrt{\gamma}} \right] = 0$

$C_1 \neq 0$ (otherwise $0 = C_1 = C_2 \Rightarrow X \equiv 0$ trivial sol.)

$\therefore \quad e^{2i\sqrt{\gamma}} + e^{-2i\sqrt{\gamma}} = 0$

$\therefore \quad \underline{e^{4i\sqrt{\gamma}} = -1} = e^{i(\pi + 2k\pi)}$ $\quad k$ integer.

$\therefore \quad 4\sqrt{\gamma} = \pi + 2k\pi = (2k+1)\pi$

$\underline{\sqrt{\gamma} = \frac{(2k+1)\pi}{4}}, \quad k = 0, 1, 2, 3, \cdots$

(also notice that $\sqrt{\gamma} = \frac{2k+1}{4}\pi$

real

implies $\sqrt{\gamma} > 0$

$\therefore \quad \left(\frac{2k+1}{4}\right)\pi > 0$

This is why $k = 0, 1, 2, 3 \cdots$

(not, $-1, -2, -3 \cdots$)

$\therefore \quad \gamma = \left[\left(\frac{2k+1}{4}\right)\pi\right]^2, \quad \underline{k = 0, 1, 2, 3, \cdots}$

$\therefore \quad X(x) = C_1 e^{i\sqrt{\gamma}x} + C_2 e^{-i\sqrt{\gamma}x} \quad \swarrow (C_1 = C_2)$

$= C_1 \left[e^{i\sqrt{\gamma}x} + e^{-i\sqrt{\gamma}x} \right]$

$= 2C_1 \cos(\sqrt{\gamma}x)$

$X(x) = D_1 \cos\left(\frac{2k+1}{4}\pi x\right) \quad k = 0, 1, 2, 3, 4 \cdots$

For each k,
corresponding $T(t)$ is

$$T'' + 4\gamma T = 0$$

$$\Rightarrow \quad T(t) = \alpha_k \cos(2\sqrt{\gamma}x) + \beta_k \sin(2\sqrt{\gamma}x)$$

$$= \alpha_k \cos\left(2\cdot\frac{2k+1}{4}\pi t\right) + \beta_k \sin\left(2\cdot\frac{2k+1}{4}\pi t\right)$$

$$\therefore \; T(t) = \alpha_k \cos\left(\frac{2k+1}{2}\pi t\right) + \beta_k \sin\left(\frac{2k+1}{2}\pi t\right)$$

Combining (for each $k = 0, 1, 2, 3, \cdots$)

$$X(x)T(t) = a_k \cos\left(\frac{2k+1}{4}\pi x\right)\left[\alpha_k \cos\left(\frac{2k+1}{2}\pi t\right) + \beta_k \sin\left(\frac{2k+1}{2}\pi t\right)\right]$$

arbitrary const.

- Superposition

$$\boxed{u_h(x,t) = \sum_{k=0}^{\infty} a_k \cos\left(\frac{2k+1}{4}\pi x\right)\left[\alpha_k \cos\left(\frac{2k+1}{2}\pi t\right) + \beta_k \sin\left(\frac{2k+1}{2}\pi t\right)\right]}$$

Finally,

$$\boxed{\begin{aligned} u(x,t) &= u_p(x) + u_h(x) \\ &= x - 5 \\ &+ \sum_{k=0}^{\infty} a_k \cos\left(\frac{2k+1}{4}\pi x\right)\left[\alpha_k \cos\left(\frac{2k+1}{2}\pi t\right) + \beta_k \sin\left(\frac{2k+1}{2}\pi t\right)\right] \end{aligned}}$$

the final answer