#3. Let $h(x) = \begin{cases} -1, & 0<x<1 \\ 1, & 1<x<2 \end{cases}$

Then, from the assumption of the problem

$$h(x) = \sum_{k=1}^{\infty} C_k \sin\left(\frac{k\pi}{2} x\right)$$

This is the Fourier sine series of $h(x)$ on $0<x<2$.

Therefore, $C_k = \frac{2}{2}\int_0^2 h(x)\sin\left(\frac{k\pi}{2}x\right)$

$$= \int_0^1 (-1)\sin\left(\frac{k\pi}{2}x\right)dx + \int_1^2 \sin\left(\frac{k\pi}{2}x\right)dx$$

$$= \frac{2}{k\pi}\left[\cos\left(\frac{k\pi}{2}x\right)\right]_0^1 + \left[-\frac{2}{k\pi}\cos\left(\frac{k\pi}{2}x\right)\right]_1^2$$

$$= \frac{2}{k\pi}\left\{\cos\left(\frac{k\pi}{2}\right) - 1 - \cos(k\pi) + \cos\left(\frac{k\pi}{2}\right)\right\}$$

$$= \frac{2}{k\pi}\left\{-1-(-1)^k + 2\cos\left(\frac{k\pi}{2}\right)\right\}$$ ← used $\cos(k\pi) = (-1)^k$

$$= \begin{cases} \frac{1}{l\pi}\left\{-2+2\cos(l\pi)\right\} & \text{for } k = \text{even} = 2l \\ \frac{2}{(2l+1)\pi}\left\{-1-(-1)+2\cdot 0\right\} & \text{for } k = \text{odd} = 2l+1 \end{cases}$$

here $\cos\left(\frac{k\pi}{2}\right) = 0$ for odd k.

$$\boxed{C_k = \begin{cases} \frac{1}{l\pi}\left\{-2+2(-1)^l\right\} & \text{for } k = \text{even} = 2l \\ 0 & \text{for } k = \text{odd}. \end{cases}}$$

▨

#4. From Fourier sine series we see

$$C_k = \frac{2}{\pi}\int_0^{\pi} e^{-x}\sin\left(\frac{k\pi x}{\pi}\right)dx$$

$$= \frac{2}{\pi}\int_0^{\pi} e^{-x}\sin(kx)\,dx.$$

To compute this integral, apply integration by parts:

$$C_k = \frac{2}{\pi}\int_0^{\pi} \underbrace{e^{-x}}_{f'}\,\underbrace{\sin(kx)}_{g}\,dx = \frac{2}{\pi}\Big[\underbrace{-e^{-x}}_{f}\,\underbrace{\sin(kx)}_{g}\Big]_0^{\pi} - \frac{2}{\pi}\int_0^{\pi}\underbrace{-e^{-x}}_{f}\cdot\underbrace{k\cos(kx)}_{g'}\,dx.$$

$$= \frac{2k}{\pi}\int_0^{\pi} e^{-x}\cos(kx)\,dx$$

Integration by parts again

$$= \frac{2k}{\pi}\Big[-e^{-x}\cos(kx)\Big]_0^{\pi} - \frac{2k}{\pi}\int_0^{\pi}-e^{-x}\cdot(-k)\sin(kx)\,dx$$

$$= \frac{2k}{\pi}\Big[-e^{-\pi}\cos(k\pi)+1\Big] - k^2\cdot\underbrace{\boxed{\frac{2}{\pi}\int_0^{\pi} e^{-x}\sin(kx)\,dx}}_{=C_k}$$

Rearrange the left & right hand sides:

$$(1+k^2)C_k = \frac{2k}{\pi}\Big[-e^{-\pi}\cos(k\pi)+1\Big]$$

$\cos(k\pi) = (-1)^k$

$$\therefore\; C_k = \frac{2k}{\pi(1+k^2)}\Big[1-e^{-\pi}(-1)^k\Big], \quad k = 1, 2, 3, \ldots$$

#5. Plug in $y(t)=\sum_{k=1}^{\infty} c_k \sin\left(\frac{k\pi}{2}t\right)$ to the ODE.

Then get

$$\sum_{k=1}^{\infty} -c_k \cdot \left(\frac{k\pi}{2}\right)^2 \sin\left(\frac{k\pi}{2}t\right) + \sum_{k=1}^{\infty} c_k \sin\left(\frac{k\pi}{2}t\right) = f(t)$$

↑ Be careful.

Thus $$\sum_{k=1}^{\infty} c_k\left[1-\left(\frac{k\pi}{2}\right)^2\right] \sin\left(\frac{k\pi}{2}t\right) = f(t)$$

Now, since this is the Fourier sine series for $f(t)$ on $0<t<2$

We have

$$c_k\left[1-\left(\frac{k\pi}{2}\right)^2\right] = \underset{L=2}{\frac{2}{2}} \int_0^2 f(t) \sin\left(\frac{k\pi}{2}t\right) dt$$

$f(t)=\begin{cases} 0, & 0<t<1 \\ 1, & 1<t<2 \end{cases}$

$$= \int_1^2 \sin\left(\frac{k\pi}{2}t\right) dt$$

$$= \left[-\frac{2}{k\pi}\cos\left(\frac{k\pi}{2}t\right)\right]_1^2$$

$$= -\frac{2}{k\pi}\left[\cos(k\pi) - \cos\left(\frac{k\pi}{2}\right)\right]$$

So, $$c_k = \frac{1}{\left[1-\left(\frac{k\pi}{2}\right)^2\right]} \cdot \frac{-2}{k\pi}\left[\cos(k\pi) - \cos\left(\frac{k\pi}{2}\right)\right], \quad k=1,2,3,\dots$$

$$c_1 = \frac{-2}{\left(1-\frac{\pi^2}{4}\right)\pi} \cdot \left[\underbrace{\cos\pi}_{=-1} - \cancelto{0}{\cos\left(\frac{\pi}{2}\right)}\right] = \frac{2}{\pi\left(1-\frac{\pi^2}{4}\right)}.$$

$$c_2 = \frac{1}{(1-\pi^2)} \cdot \frac{-2}{2\pi}\left[\underbrace{\cos(2\pi)}_{=1} - \underbrace{\cos(\pi)}_{=-1}\right] = \frac{-2}{\pi(1-\pi^2)}.$$

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