MATH 121:201 Honours Integral Calculus

Written Home Work (WHW) assignment 2

Due: **Jan. 23, (Fri), 2015, by 2pm.**

Material: [Adams& Essex] Sections 5.1 - 5.6 and Appendix IV.

In the following problems, you can use the definitions and results contained in the textbook 5.1-5.6, Appendix IV (but, not the exercises).

You have to give **clear** justifications.

Problem 1: Solve the integral equation, namely, find the function f (assuming f is continuous) satisfying the equation:

$$f(x^3) = 1 + \int_{x^3}^{1} f(t)dt$$

Problem 2: Evaluate the following sums.

A.

$$\lim_{n\to\infty} \sum_{k=1}^{2n} k\frac{e^{k^2/n^2}}{n^2}$$

Give justification.

B.*

$$\lim_{n\to\infty} \sum_{k=1}^{n} \tan(k/n^2)$$

Give a justification. This problem is subtile and you would need to apply an ϵ argument for a rigorous treatment of the limit.

Problem 3: You need to recall the rigorous definitions of continuity and differentiability of a function.

Suppose f is a Riemann integrable function on the interval $[0,1]$. (Notice that we do not assume that f is continuous.) Suppose that $0 \leq f(x) \leq 1$ for all $x \in [0,1]$. Define the function $F(x) = \int_0^x f(t)dt$ for $0 \leq x \leq 1$.

A. Is $F(x)$ continuous for $0 \leq x \leq 1$? Justify rigorously.

B. Is $F(x)$ differentiable for $0 < x < 1$? Justify rigorously.

Problem 4:

In this problem, you have to give rigorous justification (sometimes using ϵ argument).

For each $x_0 \in \mathbf{R}$ define a function h_{x_0} by

$$h_{x_0} = \begin{cases} 1 & \text{if } x = x_0 \ , \\ 0 & \text{if } x \neq x_0 . \end{cases}$$

A. Prove that h_{x_0} is integable on any $[a, b]$ and that $\int_a^b h_{x_0}(x)dx = 0$.

B. Assume f is integrable on $[a, b]$ and $g : [a, b] \to \mathbf{R}$ satisfies $g(x) = f(x)$ for all $x \neq x_0$, for some $x_0 \in [a, b]$. Prove that g is also integrable on $[a, b]$ and $\int_a^b g(x)dx = \int_a^b f(x)dx$.

   Hint: Show that $g = f + ch_{x_0}$ for some c.

C. Show there is no solution F to the differential equation $F'(x) = h_0(x)$ for $x \in [-1, 1]$.