Lec 9.

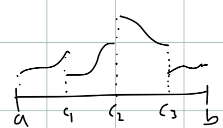
- piece-wise continuous functions.
- computation of integrals. substitution technique. §5.6.

piecewise continuous functions on $[a,b]$

are the functions that are continuous

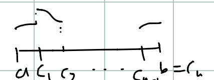
except at finitely many points.

e.g.



Thm Suppose f is a bounded piecewise continuous function on $[a,b]$

Then, f is integrable



Suppose further that f is given by for $a=c_0 < c_1 < c_2 \cdots < c_n = b$

$$f(x) = g_i(x) \quad \text{for } c_{i-1} \le x < c_i$$

g_i continuous on $[c_{i-1}, c_i]$.

$$\text{Then } \int_a^b f(x)\,dx = \sum_{i=1}^{n} \int_{c_{i-1}}^{c_i} g_i(x)\,dx$$

e.g.

$$f(x) = \begin{cases} e^x & 0 \le x \le 1 \\ -\sin x & 1 < x \le 3 \\ \frac{1}{x} & 3 < x \le 4 \end{cases}$$

$$\int_0^4 f(x)\,dx = \int_0^1 e^x\,dx + \int_1^3 -\sin x\,dx + \int_3^4 \frac{1}{x}\,dx$$

$$= e^x\Big]_0^1 + \Big[\cos x\Big]_1^3 + \ln x\Big]_3^4$$

$$= e - 1 + \cos 2 - \cos 1 + \ln 4 - \ln 3.$$

Comments about proof

The proof is optional.

You use integrability of continuous functions.

But, there are finitely many discontinuous points.

When you control $U(f,P) - L(f,P) = \sum_{i=1}^{N} (M_i - m_i)\Delta x_i$

Make the partition in such a way the contribution from discontinuity points are arbitrary small.

This proof has similar ideas (but more complicated) as WHW 1 Problem 3. (a).

$\square$

Comments on WHW 1.

3.A.

Fix $\varepsilon > 0$.

$\frac{\varepsilon}{4}$ — f is not continuous on $[0, \frac{\varepsilon}{4}]$.

$f(x)$ is continuous on $[\frac{\varepsilon}{4}, 1]$, thus integrable on $[\frac{\varepsilon}{4}, 1]$.

: $\exists$ partition P_1 of $[\frac{\varepsilon}{4}, 1]$ s.t. $U(f, P_1) - L(f, P_1) < \frac{\varepsilon}{2}$

But $\sup_{[0,\frac{\varepsilon}{4}]} f - \inf_{[0,\frac{\varepsilon}{4}]} f = 1 - (-1) = 2.$

& $2 \cdot \frac{\varepsilon}{4} = \frac{\varepsilon}{2}$.

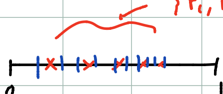
Thus for partition $P = \{0, \frac{\varepsilon}{4}\} \cup P_1$: $x_0 = 0 < x_1 = \frac{\varepsilon}{4} < \underbrace{x_2 < \cdots < x_N}_{\text{from } P_1} = 1$

$$U(f,P) - L(f,P) \leq \sum_{i=1}^{N} (M_i - m_i)\Delta x_i = (M_1 - m_1)\frac{\varepsilon}{4} + \sum_{i=2}^{N} (M_i - m_i)\Delta x_i$$

$$\leq \frac{\varepsilon}{2} + U(f, P_1) - L(f, P_1) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad \blacksquare$$

3.B. $f(r_k) = 2^{-k}$ $k = 1, 2, 3, \cdots$

Fix $\varepsilon > 0$. $\{r_1, r_2, \ldots, r_{N_1}\}$ s.t. $2^{-N_1} < \frac{\varepsilon}{2}$.



$\exists$ partition P s.t. subintervals containing $r_1, \cdots, r_{N_1}$ have total length $< \frac{\varepsilon}{2}$.

$$\text{Then } U(f,P) - L(f,P) = \sum_{\substack{r_k \in I_i \\ k=1,2,\cdots,N_1}} \underbrace{(\sup_{I_i} f - \inf_{I_i} f)}_{\leq 1} |I_i| + \sum_{\substack{\text{other intervals } I_j \\ \text{i.e. } \{r_1,\cdots,r_{N_1}\} \cap I_j = \emptyset}} \overbrace{(\sup_{I_j} f - \inf_{I_j} f)}^{\leq 2^{-N_1} < \frac{\varepsilon}{2}} |I_j|$$

$$\leq \sum_{\substack{r_k \in I_i \\ k=1,2,\cdots,N_1}} |I_i| + \sum_{\substack{\text{other intervals } I_j \\ \text{i.e. } \{r_1,\cdots,r_{N_1}\} \cap I_j = \emptyset}} \frac{\varepsilon}{2} |I_j|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \sum |I_j| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad \blacksquare$$

Computation of definite integrals: Substitution technique.
(change of variables)

e.g. $\int_2^3 x e^{x^2} dx$:

Let $u = x^2$

$\frac{d}{dx} e^u = \frac{d e^u}{du} \cdot \frac{du}{dx}$ chain rule

$= e^u \cdot 2x$

$= e^{x^2} \cdot 2x = 2x e^{x^2}$

$\therefore$ see $\frac{d}{dx}\left[\frac{1}{2} e^{x^2}\right] = x e^{x^2}$

$$\therefore \int_2^3 x e^{x^2} dx = \left.\frac{1}{2} e^{x^2}\right]_2^3$$

$$= \frac{1}{2}\left(e^{3^2} - e^{2^2}\right) = \frac{1}{2}\left(e^9 - e^4\right)$$

More systematically,

$u = x^2$. $\quad du = 2x\,dx$

$$\therefore \quad x e^{x^2} dx = e^u \frac{du}{2}$$

$$\therefore \int_2^3 x e^{x^2} dx = \int_{x=2}^{x=3} \frac{e^u}{2} du$$

$$= \int_{u=4}^{u=9} \frac{e^u}{2} du$$

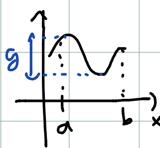
$$= \left.\frac{1}{2} e^u\right]_4^9 = \frac{1}{2}\left(e^9 - e^4\right).$$

Anti-derivative: $\displaystyle\int f(g(x))\, g'(x)\, dx = \int f(u)\, du \quad \leftarrow u = g(x)$

definite integral $\displaystyle\int_a^b f(g(x))\, g'(x)\, dx = \int_{u=g(a)}^{u=g(b)} f(u)\, du$

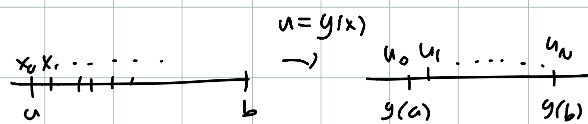
for g differentiable & continuous on $[a,b]$

f is continuous on the range of g



<u>Rmk</u> More generally, it holds if g' is bounded integrable & f is bounded integrable. (e.g. piecewise continuous)

<u>Idea</u> Let $u = g(x)$.



$\Delta u_k = u_k - u_{k-1} = g(x_k) - g(x_{k-1})$

(It may happen $g(b) < g(a)$. . .)

$= g'(c_k)(x_k - x_{k-1})$ for some $c_k \in [x_{k-1}, x_k]$.

$= g'(c_k)\, \Delta x_k$ due to mean value theorem

$$\int_{u=g(a)}^{u=g(b)} f(u)\, du \approx \sum_k f(u_k^*)\, \Delta u_k \quad \leftarrow \text{let } u_k^* = g(c_k).$$

$$= \sum_k f(g(c_k))\, (g(x_k) - g(x_{k-1}))$$

$$= \sum_k f(g(c_k))\, g'(c_k)\, \Delta x_k$$

$$\approx \int_a^b f(g(x))\, g'(x)\, dx$$

Here, the key point: $\Delta u_k = g'(c_k)\, \Delta x_k$.

In infinitesimal notation $\boxed{du = g'(x)\, dx}$ //.

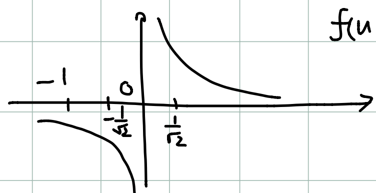
$\underline{\text{Ex}}$ $\displaystyle\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$ $\qquad \cos x = u$, $du = -\sin x\,dx$

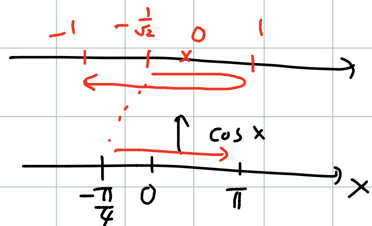
$$= \int \frac{-du}{u} = -\ln|u| + C = -\ln|\cos x| + C$$

Be CAREFUL when you compute definite integrals using substitution.

- $\displaystyle\int_{-\frac{\pi}{4}}^{\pi} \tan x\,dx \;\cancel{=}\; \Big[-\ln|\cos x|\Big]_{-\frac{\pi}{4}}^{\pi} = \cancel{-\ln|\cos \pi|}^{\,0} + \ln\left|\cos\left(\tfrac{\pi}{4}\right)\right| = \ln\frac{1}{\sqrt{2}}$

wrong.

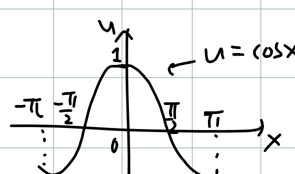


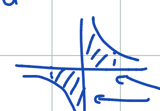


range of $u = \cos x$ for $-\frac{\pi}{4} \le x \le \pi$, is $-1 \le u \le 1$.

$\frac{1}{u}$ is not continuous on $[-1, 1]$. (nor integrable)

In fact $\displaystyle\int_{-1}^{0} \frac{1}{u}\,du = -\infty$, $\displaystyle\int_{0}^{1} \frac{1}{u}\,du = +\infty$.

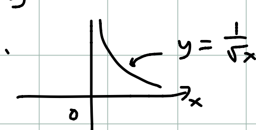




both infinite area.

$\underline{\text{Rmk}}$ In § 6.5 "Improper integrals" we will learn more about integrals of unbounded functions

e.g.



Substitution technique requires practice! Do MANY exercises.

Ex $\displaystyle\int \frac{1}{\cos x}\,dx$ ← Anti derivative of $\frac{1}{\cos x}$

$$= \int \frac{\cos x}{\cos x \cos x}\,dx$$

$$= \int \frac{\cos x}{\cos^2 x}\,dx$$

← $\cos^2 x = 1 - \sin^2 x$

$$= \int \frac{\cos x}{1-\sin^2 x}\,dx$$

$u = \sin x \qquad du = \cos x\,dx$

$$= \int \frac{du}{1-u^2}$$

$$= \int \frac{1}{2}\left(\frac{1}{1-u} + \frac{1}{1+u}\right)du$$

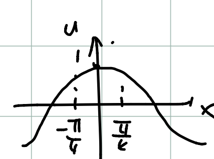
$$= \frac{1}{2}\left[\int \frac{1}{1-u}\,du + \int \frac{1}{1+u}\,du\right]$$

- $\left(\ln|1+u|\right)' = \frac{1}{1+u}$
- $\left(\ln|1-u|\right)' = -\frac{1}{1-u}$.

$$= \frac{1}{2}\left(-\ln|1-u| + \ln|1+u|\right) + C$$

$$= \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right| + C. = \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + C.$$

$u = \cos x \qquad -\frac{\pi}{4} \le x \le \frac{\pi}{4}$



e.g. $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{\cos x}\,dx$

$\frac{1}{u}$ is <u>continuous</u> on $\cos\left(\frac{\pi}{4}\right) \le u \le 1$.

$$= \frac{1}{2}\ln\frac{1+\sin x}{1-\sin x}\Bigg]_{x=-\frac{\pi}{4}}^{x=\frac{\pi}{4}}$$

← can apply substitution

$$= \frac{1}{2}\left(\ln\frac{1+\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}} - \ln\frac{1-\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}}\right) = \ln\frac{1+\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}} = \underline{\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)}$$ ▨