Lec 7

- Two more properties of the definite integral. } §5.4
- mean value thm for integral.
- The fundamental thm of calculus. §5.5

Thm (Properties of the definite integral) [Thm 3. §5.4]

Let f, g be (bounded) & integrable on an interval $[a_0, b_0]$

(Note in this case f, g are integrable on any subinterval, say, $[\alpha, \beta] \subset [a_0, b_0]$)

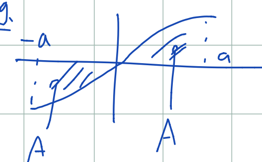
Let $a, b, c \in [a_0, b_0]$.

Then.

(g) odd function f. $-f(x) = f(-x)$. $\forall x$.

Then, $\int_{-a}^{a} f(x)\,dx = 0$

e.g.

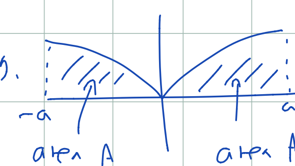


$\int_{-a}^{a} f(x)\,dx = -A + A = 0$.

(h) even function f $f(x) = f(-x)$

Then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

e.g.



$\int_{-a}^{a} f(x)\,dx = A + A$

$= 2\int_{0}^{a} f(x)\,dx$.

e.g.
- $\int_{-1}^{1} x e^{x^2}\,dx = 0$ since $x e^{x^2}$ is an odd function.
- $\int_{-1}^{1} e^{x^2}\,dx = 2\int_{0}^{1} e^{x^2}\,dx$. since e^{x^2} is an even function

Mean Value thm.

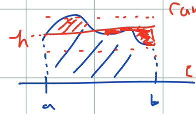
$a<b$.

f continuous on $[a,b]$

$\Rightarrow \exists\ c \in [a,b]$ such that $\int_a^b f(x)\,dx = (b-a) f(c)$.

"There exists"

"reason" Case $f \geq 0$:



can find height h

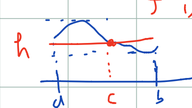
$\min_{[a,b]} f \leq h \leq \max_{[a,b]} f$

such that the area above h is the same as the area of the region below h and above $y=f(x)$

Then $\int_a^b f(x)\,dx = h\cdot(b-a)$.

And since $\min_{[a,b]} f(x) \leq h \leq \max_{[a,b]} f(x)$

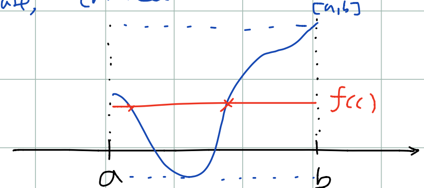
f is continuous on $[a,b]$



there exists $c \in [a,b]$

s.t. $f(c) = h$.

For general case, consider $f(x) - \min_{[a,b]} f(x) \geq 0$.



More rigorous proof

Let $m = \min_{[a,b]} f(x)$.

$M = \max_{[a,b]} f(x)$

By continuity of f on $[a,b]$, such max & min exist, and there are points $\alpha, \beta \in [a,b]$ such that $f(\alpha) = M$, $f(\beta) = m$.

Now, since $m \leq f(x) \leq M$ on $[a,b]$ $a<b$,

$\int_a^b m\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b M\,dx \Rightarrow m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$

$\Rightarrow m \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq M$.

Due to continuity of f & $M = f(\alpha)$, $m = f(\beta)$, we can apply intermediate value theorem and see there exists $c \in [\alpha, \beta]$ if $\alpha \le \beta$ (or $c \in [\beta, \alpha]$ if $\alpha \geq \beta$)

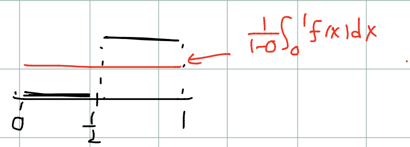
such that $f(c) = \frac{1}{b-a} \int_a^b f(x)\,dx$. $\boxed{//}$

In the mean value theorem,

<u>note</u> Continuity of f is essential.

<u>Example</u> $f(x) = \begin{cases} 1 & x \le \frac{1}{2} \\ 0 & x > \frac{1}{2} \end{cases}$ on $[0,1]$.

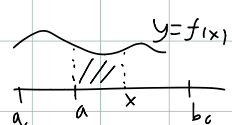
$$\int_0^1 f(x)\,dx = \frac{1}{2} \neq (1-0) f(c) \quad \text{for any } c \in [0,1].$$



The Fundamental Theorem of Calculus. §5.5

<u>Thm</u> (F.T.C). 1. Suppose f continuous on $[a_0, b_0]$. $a \in [a_0, b_0]$.

Let $F(x) = \int_a^x f(t)\,dt$.



Then F is differentiable for $x \in (a_0, b_0)$

& $F'(x) = f(x)$

i.e. $\frac{d}{dx} \int_a^x f(t)\,dt = f(x)$.

2. Suppose G is a differentiable function on $[a_0, b_0]$

& $G'(x)$ is bounded Riemann integrable on $[a_0, b_0]$.

Then $\int_a^b G'(x)\,dx = G(b) - G(a)$ $\forall\, a, b \in [a_0, b_0]$.

Proof of part 1.

$$F(x+h) - F(h)$$

$$= \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt$$

$$= \int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt - \int_a^x f(t)\,dt$$

$$= \int_x^{x+h} f(t)\,dt.$$

$$\therefore \frac{d}{dx} F(x) = \lim_{h \to 0} \frac{1}{h}\left[F(x+h) - F(x)\right]$$

$$= \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt$$

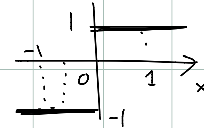
$$= \lim_{h \to 0} f(c)$$ for some c between x & $x+h$ by the mean value thm. (possible since f is continuous.)

$$= f(x)$$ because as $h \to 0$, $c \to x$ & f is continuous.

$\square$ part 1.

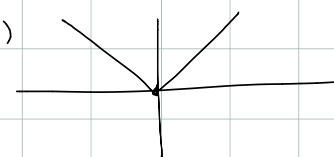
Q. Is continuity assumption on f essential in part 1. ?

Yes e.g. $f(x) = \begin{cases} 1 & x < 0 \\ -1 & x \geq 0 \end{cases}$



Let $F(x) = \int_0^x f(t)\,dt$:

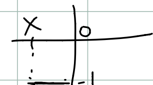
F(X)



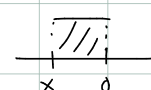
$F(0) = 0$

$x > 0 \Rightarrow \quad F(x) = \int_0^x 1\,dt = x$

$x < 0 \Rightarrow \quad F(x) = \int_0^x -1\,dt = -\int_0^x dt = -\left(-\int_x^0 dt\right)$



$= \int_x^0 dt = |x| = -x.$



$$F(x) = \begin{cases} x & x \geq 0 \\ -x & x \leq 0 \end{cases}$$

$$\therefore \quad \frac{F(x) - F(0)}{x - 0} = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

Thus $\lim_{x \to 0} \frac{F(x) - F(0)}{x - 0}$ does NOT exist.

i.e. $F(x) = \int_0^x f(t)\,dt$ is not differentiable at $x = 0$

By the same reason $F_2(x) = \int_{-1}^x f(t)\,dt$

is NOT differentiable at $x = 0$. //

Proof of Part 2.

We skip the full proof, but

We prove only the special case where G' is continuous.

Assume G' is continuous on $[a_0, b_0]$.

Fix $a, b \in [a_0, b_0]$

Then define $F(x) = \int_a^x G'(t)\,dt$.

From part 1, $F'(x) = G'(x)$ for $x \in [a_0, b_0]$.

Then from this & using Thm 13, Section 2.8.

$$F(x) = G(x) + C \quad \text{for some constant.}$$

Thus

$$\left.\begin{aligned} F(a) &= G(a) + C \\ F(b) &= G(b) + C \\ \text{but} \quad F(a) &= \int_a^a G'(t)\,dt = 0 \\ F(b) &= \int_a^b G'(t)\,dt \end{aligned}\right\} \Longrightarrow \underline{\int_a^b G'(t)\,dt = G(b) - G(a).}$$

$\blacksquare$

Hueristic explanation for part 2. (not a rigorous proof.)

$$\int_a^b G'(t)\,dt \approx \sum_{k=1}^{N} G'(x_k)\,\Delta x_k$$

$$G'(x_k) \approx \frac{G(x_k) - G(x_{k-1})}{x_k - x_{k-1}}$$

$$\approx \sum_{k=1}^{N} \frac{G(x_k) - G(x_{k-1})}{x_k - x_{k-1}} \overbrace{(x_k - x_{k-1})}^{\Delta x_k}$$

$$= \sum_{k=1}^{N} \left[G(x_k) - G(x_{k-1}) \right] \longleftarrow \text{telescopic sum}$$

$$= G(x_N) - G(x_0) = G(b) - G(a)$$

//