Lec39.

- Ratio / Root test examples
- Alternating series test.

Ratio test

For a given $\sum_{n=1}^{\infty} a_n$,

- IF $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$ THEN $\sum_{n=1}^{\infty} a_n$ converges.

strict inequality

- IF $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$ THEN $\sum_{n=1}^{\infty} a_n$ diverges.

(including the case $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$.) ← IMPORTANT!

Root test

For a given $\sum_{n=1}^{\infty} a_n$,

- IF $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$, THEN $\sum_{n=1}^{\infty} a_n$ converges.

strict inequality

- IF $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$ THEN $\sum_{n=1}^{\infty} a_n$ diverges.

(including the case $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$.) ← IMPORTANT!

The idea for Ratio/Root tests

Let $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ (if the limit exists).

(or $\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ (if the limit exists.))

Then, FOR VERY LARGE n (For convergence/divergence what matters is the behavior for larger n.)

(i.e. $n \geq K$ for a very large K)

$|a_{n+1}| \approx \rho |a_n|$, So, $\sum_{n=K}^{\infty} |a_n| \approx a_K \sum_{j=0}^{\infty} \rho^j$

i.e. the tail of $\sum_{n=1}^{\infty} |a_n|$

looks like a geometric series $\sum_{j=0}^{\infty} \rho^j$

note $\rho \geq 0$

since $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ (≥ 0), or $\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|}$

- $\sum_{j=0}^{\infty} \rho^j$ converges if $0 \leq \rho < 1$
  diverges if $\rho > 1$.
- So, $\sum_{n=1}^{\infty} |a_n|$ converges if $\rho < 1$
  diverges if $\rho > 1$

the same for $\sum_{n=0}^{\infty} a_n$.

Ex $\sum_{n=1}^{\infty} \frac{n!}{2^n}$.

: $a_n = \frac{n!}{2^n}$.

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{2^{n+1}} \Big/ \frac{n!}{2^n}$$
$$= \frac{(n+1)!}{n!} \Big/ \frac{2^{n+1}}{2^n}$$
$$= \frac{(n+1)}{2}$$

$\therefore \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n+1}{2} = \infty > 1$.

So, by Ratio test, $\sum_{n=1}^{\infty} \frac{n!}{2^n}$ diverges. ▨ .

Ex $\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^{n^2}$

(sol) $a_n = \left(1-\frac{1}{n}\right)^{n^2}$, $\sqrt[n]{a_n} = \left(\left(1-\frac{1}{n}\right)^{n^2}\right)^{\frac{1}{n}} = \left(1-\frac{1}{n}\right)^n$

$$\ln (1-x)^{\frac{1}{x}} = \frac{1}{x} \ln(1-x)$$
$$\lim_{x\to 0} \ln(1-x)^{\frac{1}{x}} = \lim_{x\to 0} \frac{-\frac{1}{1-x}}{1} \leftarrow \text{L'Hopital}$$
$$= -1.$$
$$\therefore \lim_{x\to 0} (1-x)^{\frac{1}{x}} = e^{\lim_{x\to 0} \ln(1-x)^{\frac{1}{x}}} = e^{-1} = \frac{1}{e}.$$

$\therefore \sqrt[n]{a_n} \xrightarrow[n\to\infty]{} \frac{1}{e} < 1$.

Thus, the series converges. ▨

Note For $\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^n$ the root test does not work, but we easily see it diverges since $\left(1-\frac{1}{n}\right)^n \to \frac{1}{e} \neq 0$.

MORE EXAMPLES.

Ex Define a function $f(x)$ by

$$f(x) = \sum_{n=1}^{\infty} \frac{1+n}{n^3\, 2^n} x^n .$$

typo corrected →

Find all x where $f(x)$ is well-defined

i.e. those x where the series converges.

<sol>. • $a_n = \dfrac{1+n}{n^3\, 2^n} x^n$ ← involves exponent changing in n.

natural to try ratio test.

• $$\frac{a_{n+1}}{a_n} = \left[\frac{1+n+1}{(n+1)^3\, 2^{n+1}} \cdot x^{n+1}\right] \Big/ \left[\frac{1+n}{n^3\, 2^n} x^n\right]$$

$$= \frac{1+n+1}{1+n} \cdot \frac{n^3}{(n+1)^3} \cdot \frac{x}{2}$$

• $$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1+n+1}{1+n} \cdot \frac{n^3}{(n+1)^3} \cdot \frac{|x|}{2}.$$

• Consider $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

$$= \frac{|x|}{2} \cdot \lim_{n\to\infty} \left\{\left[\frac{1+n+1}{1+n}\right]\left[\frac{n^3}{(n+1)^3}\right]\right\}$$

$$= \frac{|x|}{2} \cdot \underbrace{\left(\lim_{n\to\infty} \frac{1+n+1}{1+n}\right)}_{=1} \cdot \underbrace{\left(\lim_{n\to\infty} \frac{n^3}{(n+1)^3}\right)}_{=1}$$

$$= \frac{|x|}{2}$$

• $\displaystyle\lim_{n\to\infty} \frac{1+n+1}{1+n} = \lim_{n\to\infty} \frac{n(\frac{1}{n}+1+\frac{1}{n})}{n(1+\frac{1}{n})} = 1$

• $\displaystyle\lim_{n\to\infty} \frac{n^3}{(n+1)^3} = \lim_{n\to\infty} \frac{n^3}{n^3(1+\frac{1}{n})^3} = 1$

• So, by Ratio test, $\sum_{n=1}^{\infty} \frac{1+n}{n^3 2^n} x^n$

- converges for $\frac{|x|}{2} < 1$ i.e. for $|x| < 2$
- diverges for $\frac{|x|}{2} > 1$ i.e. for $|x| > 2$.

• For $x = 2$, $x = -2$?

• $x = 2$: $\sum_{n=1}^{\infty} \frac{1+n}{n^3 2^n} 2^n = \sum_{n=1}^{\infty} \frac{1+n}{n^3}$

It behaves like $\sum_{n=1}^{\infty} \frac{n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^2}$. guess it converges.

In fact, • $\frac{1+n}{n^3} < \frac{2n}{n^3} = \frac{2}{n^2}$ for $n \geq 1$

• $\sum_{n=1}^{\infty} \frac{2}{n^2}$ converges by integral test.

(We knew this from Lec 32).

So, by Comparison test,

$\sum_{n=1}^{\infty} \frac{1+n}{n^3}$ converges. So $x = 2$ is chosen.

• $x = -2$: $\sum_{n=1}^{\infty} \frac{1+n}{n^3 2^n} (-2)^n = \sum_{n=1}^{\infty} \frac{1+n}{n^3} (-1)^n$.

Note. $\left| \frac{1+n}{n^3} (-1)^n \right| = \frac{1+n}{n^3} < \frac{2n}{n^3}$ for $n \geq 1$.

• $\sum_{n=1}^{\infty} \frac{2}{n^2}$ converges.

So, by the comparison test $\sum_{n=1}^{\infty} \frac{1+n}{n^3 2^n} (-2)^n$ converges.
so. choose $x = -2$.

• The series converges for $-2 \leq x \leq 2$ only. ▨

including $x = 2, -2$.

Convergence / divergence for $\sum_{n=0}^{\infty} a_n$

"strategy".

try

⓪ see "rough" behavior, make guesses.
(e.g. Identify dominating terms).

① divergence test: · see whether $a_n \to 0$
- If not, $\sum_{n=0}^{\infty} a_n$ diverges

② If a_n looks complicated
: Find a simpler, but related series $\sum_{n=0}^{\infty} b_n$

try comparison test (Including · limit comparison · ratio test · root test)
(usually, want · $|a_n| \leq b_n$ for all $n \geq M$
or · $a_n \geq b_n$ for all $n \geq M$

③ For not so complicated a_n
(e.g. $\sum_{n=0}^{\infty} b_n$ found in ②)

try - integral test: for some monotonically decreasing a_n.
involving n^p, $\ln n$, etc.

- ratio test: for a_n involving r^n, $n!$, etc.

Ex. Check convergence: $\sum_{n=1}^{\infty} \frac{n^2+\sin(n)}{n!+e^{-n}}$

<sol>.

- By considering dominating terms n^2, $n!$

  we see:

  The sequence $a_n = \frac{n^2+\sin(n)}{n!+e^{-n}}$ behaves like $\frac{n^2}{n!}$

- $n!$ grows much faster than n^2.

  $n! = 1\cdot 2\cdot 3 \cdots (n-3)(n-2)(n-1)\,n$ ← many factors growing like n

  $n^2 = n \cdot n$ ← only two factors growing like n.

  So, can guess convergence of $\sum_{n=1}^{\infty} \frac{n^2}{n!}$,

  so, also for $\sum_{n=1}^{\infty} \frac{n^2+\sin(n)}{n!+e^{-n}}$.

- Find a simpler but related sequence:

$$0 \leq \frac{n^2+\sin(n)}{n!+e^{-n}} \leq \frac{n^2+1}{n!} \leq \boxed{\frac{n^2}{n!} = b_n}$$

for $n \geq 2$

($\sin(n) \leq 1$, $e^{-n} > 0$) (b_n ← name)

At this moment,

we want to use comparison test.

- For convergence of $\sum_{n=1}^{\infty} \frac{n^2}{n!} = \sum_{n=1}^{\infty} b_n$ ← name.

apply ratio test. (natural because we see $n!$).

$$\frac{b_{n+1}}{b_n} = \frac{(n+1)^2}{(n+1)!} \Big/ \frac{n^2}{n!}$$

$$= \frac{(n+1)^2}{n^2} \Big/ \frac{(n+1)!}{n!}$$

$$= \frac{(n+1)^2}{n^2} \Big/ n+1$$

$$= \frac{n+1}{n^2} \longrightarrow 0 < 1 \quad \text{as } n \to \infty.$$

So, by ratio test,

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{n^2}{n!} \quad \text{converges.}$$

- Now for the original series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n^2 + \sin(n)}{n! + e^{-n}}$

$\begin{cases} \bullet\ |a_n| \leq b_n \quad n \geq 2 \\ \bullet\ \sum_{n=1}^{\infty} b_n \text{ converges} \end{cases}$ $\Longrightarrow$ (Comparison test) $\sum_{n=1}^{\infty} a_n$ converges. ∎

Absolute / Conditional Convergence. §9.4.

- Alternating series test.

Recall from Lec 38.

Thm Suppose $\sum_{n=1}^{\infty} |a_n|$ converges.
Then $\sum_{n=1}^{\infty} a_n$ converges.

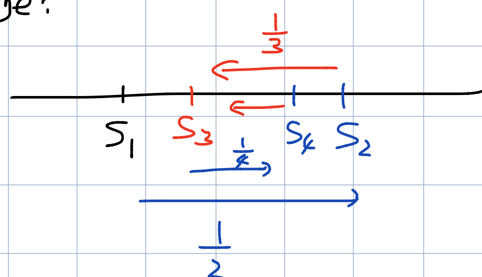
<u>Absolute Convergence</u>: $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, if $\sum_{n=1}^{\infty} |a_n|$ converges.

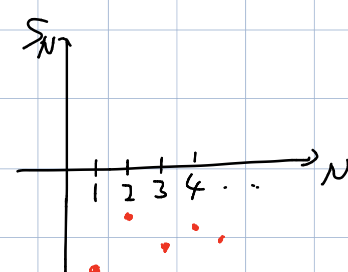
<u>conditional convergence</u>: $\sum_{n=1}^{\infty} a_n$ is conditionally convergent if $\sum_{n=1}^{\infty} a_n$ converges, but not $\sum_{n=1}^{\infty} |a_n|$.

e.g. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$ converges absolutely.

<u>EX</u>. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ converges conditionally. ($\sum_{n=1}^{\infty} \frac{1}{n} = \infty$, so, <u>not</u> absolutely convergent.)

Why does it converge? Let $a_n = (-1)^n \frac{1}{n}$. $S_N = \sum_{n=1}^{N} a_n$.





↑ the oscillation becomes smaller & smaller.

Thm (Alternating series test).

Assume for some $N>0$:
- $a_n a_{n+1} < 0$ for $n \geq N$
- $|a_n| \geq |a_{n+1}|$ decreasing
- $\lim_{n\to\infty} |a_n| = 0$.

Then $\sum_{n=1}^{\infty} a_n$ converges.

(proof).

Can assume $N=1$ & $a_1>0$.

Let $S_n = \sum_{k=1}^{n} a_k$.

Then, $S_2 < S_1$

$$S_2 \leq S_3 = \underbrace{a_3 + a_2}_{\leq 0} + a_1 \leq S_1$$

$$S_2 \leq S_4 = \underbrace{\overset{\leq 0}{a_4} + a_3}_{\geq 0} + S_2 \leq S_3$$

$\vdots$

$n=$odd : $$S_{n+1} \leq S_{n+2} = \underbrace{\underset{>0}{a_{n+2}} + \underset{<0}{a_{n+1}}}_{\leq 0} + S_n \leq S_n$$ (with $a_{n+1} + S_n = S_{n+1}$)

$n=$even : $$S_n \leq S_{n+2} = \underbrace{\underset{<0}{a_{n+2}} + \underset{>0}{a_{n+1}}}_{\geq 0} + S_n \leq S_{n+1}$$ (with $a_{n+1} + S_n = S_{n+1}$)

Thus

$$S_2 \leq S_4 \leq \cdots \leq S_{2n-2} \leq S_{2n} \leq S_{2n-1} \leq S_{2n-2} \leq S_{2n-3} \leq \cdots \leq S_3 \leq S_1$$

$\{S_{2k}\}$ monotonically increasing & bounded
$\{S_{2k-1}\}$ // decreasing & // .

Limits exist:

Let $\lim_{k\to\infty} S_{2k} = S_{even}$, $\lim_{k\to\infty} S_{2k-1} = S_{odd}$

Now, want to show $S_{even} = S_{odd}$

Easy b/c $S_{2k} - S_{2k-1} = a_{2k} \to 0$ as $k\to\infty$.

Thus $S_{even} - S_{odd} = 0$.

$\therefore \lim_{n\to\infty} S_n$ exists.

& moreover, the above shows

$$S_{2k} \leq \lim_{n\to\infty} S_n \leq S_{2k-1} \text{ for any } k\geq 1$$

in particular,

$$\left|\lim_{n\to\infty} S_n - S_k\right| \leq |S_{k+1} - S_k| = |a_k|.$$

□