Lec 37 Series.

Wed:
- Series
- Convergence / divergence of series.
  - Basic examples - divergence test

Today: §9.3.
- Methods to check convergence/divergence
  - comparision test

Next Mon: (§9.3~9.4)
  - Comparison to integrals (Integral test)
  - ratio test.
  - more complicated examples.

Today: Methods for checking Convergence/divergence of series $\sum_{n=0}^{\infty} a_n$

- Comparison test: try to compare with a related but simpler series
- Divergence test.: look at what happens to a_n as $n \to \infty$.
- Integral test: for some type of series can compare with integrals.

After knowing some basic examples one can consider convergence/divergence of more complicated series by comparing those to the known & simpler series.

- "Comparison test"

Thm (Comparison test) Given $\sum_{n=0}^{\infty} a_n$, $\sum_{n=0}^{\infty} b_n$,

(divergent case)

- $\sum_{n=0}^{\infty} a_n = \infty$ if $\begin{cases} \cdot\ a_n \geq b_n \text{ for all } n \geq M \text{ for some } M. \\ \sum_{n=0}^{\infty} b_n = \infty \end{cases}$

- $\sum_{n=0}^{\infty} a_n = -\infty$ if $\begin{cases} a_n \leq b_n \text{ for all } n \geq M \text{ for some } M \\ \sum_{n=0}^{\infty} b_n = -\infty. \end{cases}$

Thm (Comparison test) Given $\sum_{n=0}^{\infty} a_n$, $\sum_{n=0}^{\infty} b_n$,

(convergent case)

- $\sum_{n=0}^{\infty} a_n$ converges & $\sum_{n=0}^{\infty} a_n \leq \sum_{n=0}^{\infty} b_n$ if $\begin{cases} \cdot\ |a_n| \leq b_n \text{ for all } n \geq M \text{ for some } M. \\ \cdot\ \sum_{n=0}^{\infty} b_n \text{ converges} \end{cases}$

- How to use comparison test.

Idea: for comparison

- Identify the dominant terms (most rapidly increasing)
- Try to find a simpler series with the same dominant terms, where the convergence / divergence is easier.
- Compare the original series with the simpler series.

e.g. Does $\sum_{n=1}^{\infty} \frac{1}{n+e^{-n}}$ converge? Did in Lec 36.

<sol>.

Note, in $\frac{1}{n+e^{-n}}$ $e^{-n} \to 0$ as $n \to \infty$.

In particular, for $n \geq 1$

$e^{-n} \leq 1$. (since $e > 1$)

* The dominant term in the denominator is n.

- For $n \geq 1$, $n+e^{-n} \leq n+1$

$$\frac{1}{n+e^{-n}} \geq \frac{1}{n+1}$$

simpler but, related.

- Can compare it with the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n+1} = \sum_{k=2}^{\infty} \frac{1}{k} = \infty.$$

• So, $\frac{1}{n+e^{-n}} \geq \frac{1}{n+1}$ for $n \geq 1$

& $\sum_{n=1}^{\infty} \frac{1}{n+1} = \infty$.

By "comparison test",

$$\sum_{n=1}^{\infty} \frac{1}{n+e^{-n}} = \infty.$$ It diverges (to ∞). $\boxed{//}$

• EX Does $\sum_{n=1}^{\infty} \frac{n}{n^2+\sin(n^2)}$ converge?

<sol>.

• Note $-1 \leq \sin(n^2) \leq 1$.

$\therefore$ $n^2 + \sin(n^2) \leq n^2+1$.

dominating term.

$\therefore$ $\frac{n}{n^2+\sin(n^2)} \geq \frac{n}{n^2+1}$ ← simpler. but, related.

• $\sum_{n=1}^{\infty} \frac{n}{n^2+1} = ?$

• Note $\frac{n}{n^2+1} = \frac{n}{n^2(1+\frac{1}{n^2})}$ ← factor out dominating terms

$$= \frac{1}{n(1+\frac{1}{n^2})}$$

this behaves like $\frac{1}{n}$.

(since $1+\frac{1}{n^2} \to 1$ as $n\to\infty$)

At this moment

can guess $\sum_{n=0}^{\infty} \frac{n}{n^2+1}$ is similar to $\sum_{n=0}^{\infty} \frac{1}{n} = \infty$.

More rigorously,

$$\begin{cases} \dfrac{n}{n^2+1} = \dfrac{1}{n(1+\frac{1}{n^2})} \geq \dfrac{1}{n(1+1)} = \dfrac{1}{2n} \\ \displaystyle\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} = \infty \end{cases}$$

by comparison test. $\sum_{n=1}^{\infty} \frac{n}{n^2+1} = \infty$.

- So, $\begin{cases} \dfrac{n}{n^2+\sin(n^2)} \geq \dfrac{n}{n^2+1} \quad \text{for } n \geq 1 \\ \displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1} = \infty. \end{cases}$

By comparison test,

$$\sum_{n=1}^{\infty} \frac{n}{n^2+\sin(n^2)} = \infty$$

It diverges (to ∞) $\blacksquare$

For $\sum_{n=1}^{\infty} \frac{1}{n^p}$, comparing with integrals works well.

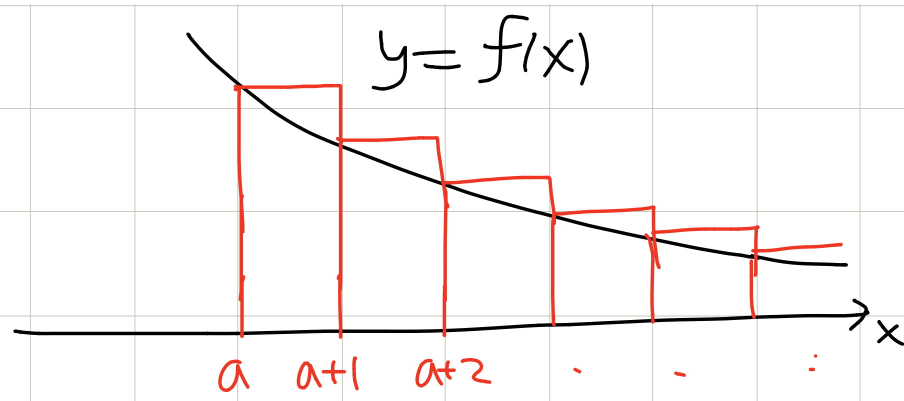
Integral test For some type of a_n

- Compare the series $\sum_{n=0}^{\infty} a_n$

  with an improper integral

  $$\int_a^{\infty} f(x)\,dx$$

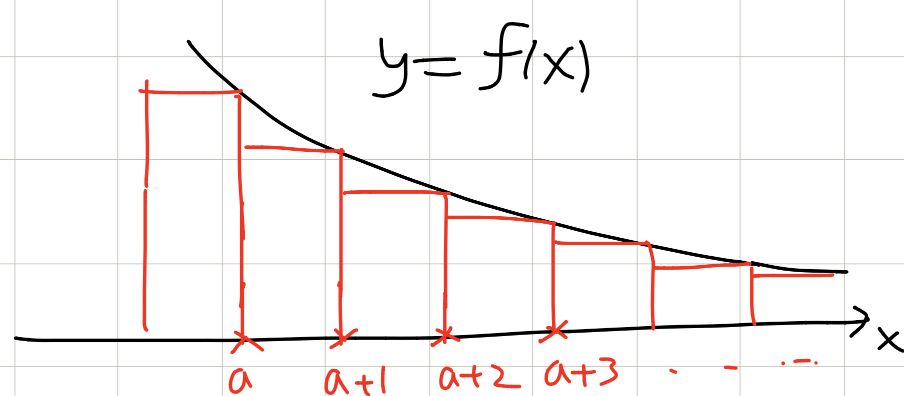
  to check convergence/divergence of $\sum_{n=0}^{\infty} a_n$.



If $f(x) \geq 0$ & $f(x)$ is monotonically decreasing for all $x \geq a$,

then $\sum_{n=a}^{N} f(n) \geq \int_a^N f(x)\,dx$ for all $N \geq a$

so $\sum_{n=a}^{\infty} f(n) \geq \int_a^{\infty} f(x)\,dx$



If $f(x) \geq 0$ & $f(x)$ is monotonically decreasing for all $x \geq a$,

then $\sum_{n=a}^{N} f(n) \leq f(a) + \int_a^N f(x)\,dx$ for all $N \geq a$.

so $\sum_{n=a}^{\infty} f(n) \leq f(a) + \int_a^{\infty} f(x)\,dx$

Integral test

If $f(x) \geq 0$ & $f(x)$ monotonically decreasing for all $x \geq a$,

then $\sum_{n=a}^{\infty} f(n)$ converges

$\iff \int_a^{\infty} f(x)\,dx$ converges

Moreover, if converges,

The difference is at most $f(a)$

$$\int_a^{\infty} f(x)\,dx \leq \sum_{n=a}^{\infty} f(n) \leq f(a) + \int_a^{\infty} f(x)\,dx$$

e.g. IMPORTANT.

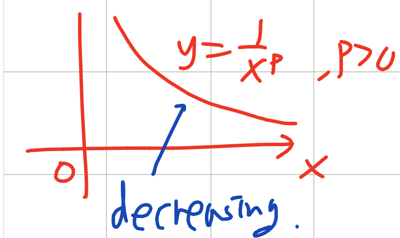
Let $p > 0$.

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ "p-series" (e.g. $p=1$ $\sum_{n=1}^{\infty} \frac{1}{n}$)

- converges if $p > 1$
- diverges if $p \leq 1$.

<u>Reason</u> $f(x)=\frac{1}{x^p}$, $x>0$

For $p>0$, if $0<x_1<x_2$



then $x_1^p < x_2^p$

since $x_1^p, x_2^p > 0$.

so, $\frac{1}{x_1^p} > \frac{1}{x_2^p}$.

inequality reversed!

• Therefore $f(x)$ is monotonically decreasing for $x>0$.

(Other way to see this:

$$f'(x) = (x^{-p})' = -p\,x^{p-1} < 0 \text{ for } x>0$$

so, decreasing on $x>0$)

• For $x>0$,

· $f(x)=\frac{1}{x^p}>0$

· $f(x)$ decreasing

$$\left.\begin{array}{l} \cdot\ f(x)=\frac{1}{x^p}>0 \\ \cdot\ f(x) \text{ decreasing} \end{array}\right\} \xRightarrow{\boxed{\text{Integral test}}} \int_1^\infty \frac{1}{x^p}\,dx \le \boxed{\sum_{n=1}^\infty \frac{1}{n^p}}$$

$$\boxed{\sum_{n=1}^\infty \frac{1}{n^p}} \le 1 + \int_1^\infty \frac{1}{x^p}\,dx$$

$f(1)$ for $f(x)=\frac{1}{x^p}$

<u>Case</u> $p=1$: $\int_1^\infty \frac{1}{x}\,dx = \lim_{\alpha\to\infty} [\ln x]_1^\alpha$

$$= \lim_{\alpha\to\infty} (\ln\alpha - 1) = \infty. \quad \text{So, } \sum_{n=1}^\infty \frac{1}{n} = \infty$$

Case $p>0$ & $p \neq 1$

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{\alpha\to\infty}\left[\frac{1}{1-p}x^{1-p}\right]_1^\alpha$$

$$= \lim_{\alpha\to\infty}\left(\frac{\alpha^{1-p}}{1-p} - \frac{1}{1-p}\right)$$

$$= \begin{cases} \infty & \text{if } 1-p>0 \\ \dfrac{1}{p-1}. & \text{if } 1-p<0 \end{cases}$$

So

$$\boxed{\sum_{n=1}^\infty \frac{1}{n^p} \quad \begin{array}{l} \cdot \text{ converges for } 1-p<0, \text{ i.e. } p>1 \\ \cdot \text{ diverges to } \infty, \text{ for } p\leq 1 \end{array}}$$

For $p>1$,

$$\boxed{\frac{1}{p-1} < \sum_{n=1}^\infty \frac{1}{n^p} < 1+\frac{1}{p-1}}$$

this gives an estimate of the value of $\sum_{n=1}^\infty \frac{1}{n^p}$

EX Check convergence of $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)}$

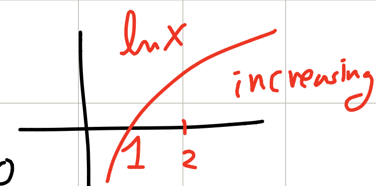
<sol>

- Observe for $x \geq 2$, $\ln x > 0$.

- For $x \geq 2$, x is increasing.

  $\ln x$ is increasing & >0.

  So, $x \ln x$ is increasing & >0.



- So, $f(x) = \frac{1}{x \ln x}$ is decreasing & >0

- $\int_2^{\infty} \frac{1}{x \ln x} dx = \lim_{\alpha \to \infty} \int_2^{\alpha} \frac{1}{x \ln x} dx$

$= \lim_{\alpha \to \infty} \Big[\ln(\ln x) \Big]_2^{\alpha}$

$= \lim_{\alpha \to \infty} \big[\ln(\ln \alpha) - \ln(\ln 2) \big]$

$= \infty$.

- By Integral test,

$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges to ∞. $\square$

For $\int \frac{1}{x \ln x} dx$

use $u = \ln x$

$du = \frac{1}{x} dx$

So $\int \frac{1}{x \ln x} dx$

$= \int \frac{1}{u} du$

$= \ln u + C$

$= \ln [\ln x] + C$.

Summary.

By comparing to integral $\int_1^\infty \frac{1}{x^p}\,dx$,

We have

- $\sum_{n=1}^\infty \frac{1}{n} = \infty$
- $\sum_{n=1}^\infty \frac{1}{n^2}$ converges
- $\sum_{n=1}^\infty \frac{1}{n^p}$ • converges for $p > 1$
  • diverges for $p \le 1$
- $\sum_{n=1}^\infty \frac{1}{n \ln n}$ d.

We also have

- $\sum_{n=0}^\infty r^n$ • converges for $|r| < 1$
  • diverges for $|r| \ge 1$.

• Also for some series (e.g. $\sum_{n=1}^\infty \frac{1}{n \ln n}$)
We may apply integral test.

Now,

• How to handle more complicated examples of series?

For more complicated looking series one may try

- Find a simpler & related series & try Comparison test.

EX. $\sum_{n=1}^{\infty} \frac{n+e^{-n}}{1+n^3}$. Check convergence.

<sol>.

- rough estimate & get an idea.

Considering dominating terms

$$\frac{n+e^{-n}}{1+n^3} \underset{\text{behaves like}}{\sim} \frac{n}{n^3} = \frac{1}{n^2}.$$

& $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, from integral test.

- So, can guess that the series $\sum_{n=1}^{\infty} \frac{n+e^{-n}}{1+n^3}$ also converges.
- also, we see that it is natural to TRY Comparison test (comparing with something similar to $\frac{1}{n^2}$)

- Find simpler & related series for comparison

$$0 \leq \frac{n+e^{-n}}{1+n^3} \leq \frac{n+1}{1+n^3} \qquad (e^{-n} \leq 1)$$

$$\leq \frac{2n}{1+n^3} \qquad (n+1 \leq 2n \text{ for } n \geq 1)$$

$$\leq \frac{2n}{n^3} \qquad \left(\frac{1}{1+n^3} \leq \frac{1}{n^3} \text{ for } n \geq 1\right)$$

$$= \frac{2}{n^2}.$$

- $\left\{ \begin{array}{l} \cdot\ 0 \leq \dfrac{n+e^{-n}}{1+n^3} \leq \dfrac{2}{n^2} \\ \cdot\ \displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2} \text{ converges.} \end{array} \right. \xRightarrow[\text{COMPARISON TEST}]{} \displaystyle\sum_{n=1}^{\infty} \frac{n+e^{-n}}{1+n^3}$ converges. ∎

EX Check convergence : $\displaystyle\sum_{n=2}^{\infty} \frac{1+\sin(e^n)}{n(\ln n)^2}$.

<sol>

- Observe $-1 \leq \sin(e^n) \leq 1$.

So $\left| \dfrac{1+\sin(e^n)}{n(\ln n)^2} \right| \leq \dfrac{2}{n(\ln n)^2}$ for $n \geq 2$.

- Try comparison test.

- Need to know convergence of $\sum_{n=2}^{\infty} \frac{2}{n(\ln n)^2}$.

  - Try integral test.

These are the two conditions for the integral test

$f(x) = \frac{2}{x(\ln x)^2}$, $x \geq 2$: $\begin{cases} \cdot\ f(x) > 0 \text{ since } \ln x > 0 \text{ for } x \geq 2. \\ \cdot\ f(x) \text{ decreasing} \end{cases}$

$$\int_2^\infty f(x)\,dx = 2\int_2^\infty \frac{1}{x(\ln x)^2}\,dx$$ (since $x(\ln x)^2$ is increasing).

$$= 2\int_{\ln 2}^\infty \frac{1}{u^2}\,du \qquad \begin{cases} u = \ln x & x = 2 \Rightarrow u = \ln 2 \\ du = \frac{1}{x}\,dx & x \to \infty \Rightarrow u \to \infty \end{cases}$$

$< \infty$. converges!

So, by integral test, $\sum_{n=2}^{\infty} \frac{2}{n(\ln n)^2}$ converges.

- Finally, $\begin{cases} \cdot\ \left| \frac{1+\sin(e^n)}{n(\ln n)^2} \right| \leq \frac{2}{n(\ln n)^2} \text{ for } n \geq 2 \\ \cdot\ \sum_{n=2}^{\infty} \frac{2}{n(\ln n)^2} \text{ converges} \end{cases}$

Two conditions for "convergence" holds in the comparison test

$$\Longrightarrow \quad \sum_{n=2}^{\infty} \frac{1+\sin(e^n)}{n(\ln n)^2} \text{ converges.}$$

COMPARISON TEST

$\blacksquare$