Lec35 part 1

Sequences

TODAY: Iterated maps.

discrete time dynamical systems

Iterated maps.

- cobwebbing.
- Stability of fixed points
- periodic cycles:
  - e.g. logistic maps.

§. Iterated maps.

Some sequences are given by a special type of recursive relation in the form

$$x_{n+1} = g(x_n), \quad g(t) \text{ a given function.}$$

- **Iterated maps** appear:

$$x_{n+1} = g(x_n)$$

$$\Rightarrow \quad x_0, \; x_1 = g(x_0), \; x_2 = g(x_1) = g(g(x_0))$$

$$x_3 = g(x_2) = g(g(x_1)) = g(g(g(x_0)))$$

$$\vdots$$

$$x_n = \underbrace{g(g(g \cdots g(x_0)) \cdots))}_{\text{iterated } n\text{-times}} = g^{[n]}(x_0)$$

notation for n-th iteration

e.g. $x_{n+1} = \frac{1}{2} x_n \Rightarrow x_n = \left(\frac{1}{2}\right)^n x_0$

- **A geometric way to understand iterated maps:**

Use the graph of $y = g(x)$ & $y = x$ to keep track of the sequence

n-th iteration of g to v

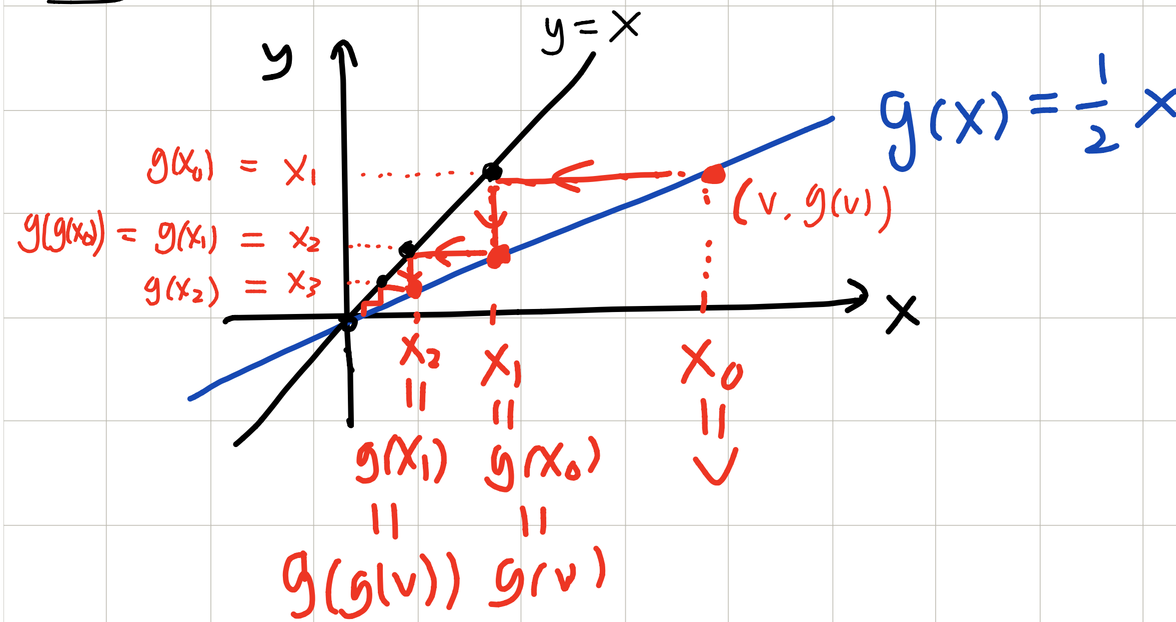
$$x_0 = v, \; x_1 = g(v), \; x_2 = g(g(v)), \cdots, \; x_n = g^{[n]}(v), \cdots$$

(equivalently, $x_0 = v$

$x_{n+1} = g(x_n)$, for all $n \geq 1$.)

「 We draw some "Cobwebs" 」

e.g. $X_n = \left(\frac{1}{2}\right)^n v \iff X_0 = v$ & $X_{n+1} = g(X_n)$, $g(x) = \frac{1}{2}x$.

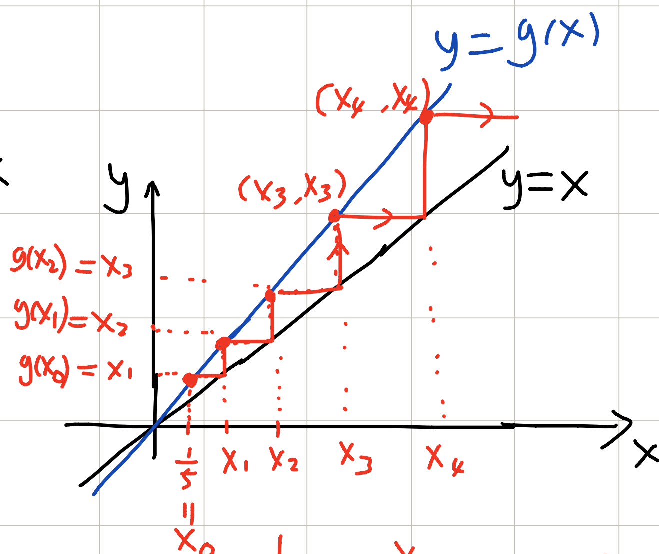


Here, we see that $\lim_{n\to\infty} X_n = 0$

e.g. $g(x) = \frac{6}{5}x$

$X_0 = \frac{1}{5}$

$X_{n+1} = g(X_n)$



We see that $\lim_{n\to\infty} X_n = \infty$.

Cobwebbing

(1) start with $(v, 0)$. $x_0 = v$

(2) move vertically to $(v, g(v))$

on the graph of $y = g(x)$.

(3) move horizontally to $(g(v), g(v))$

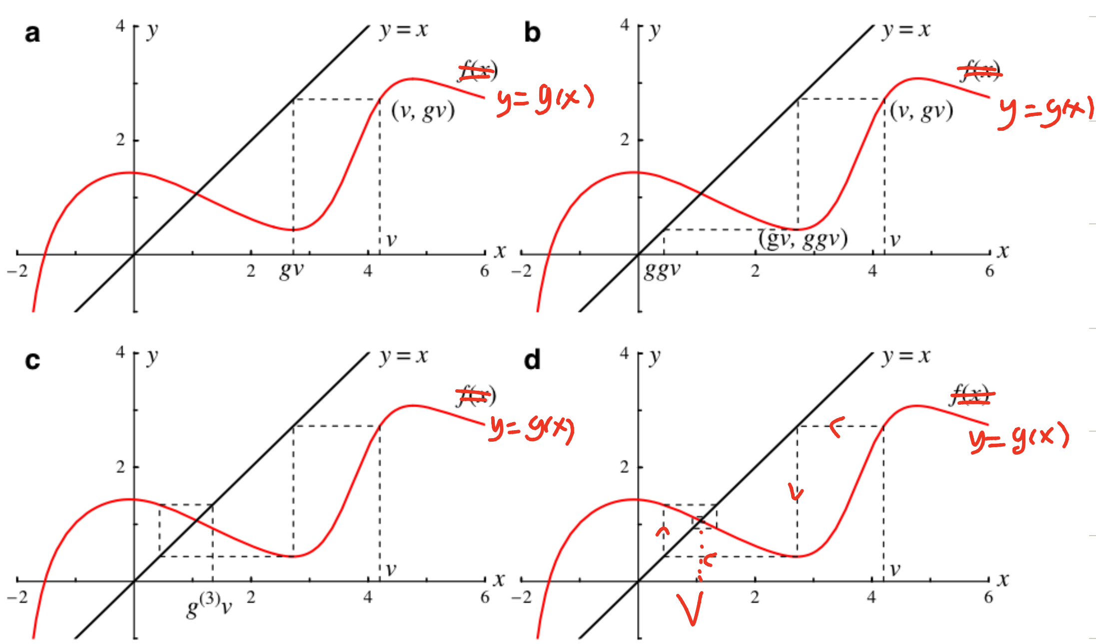
on the line $y = x$

(4) move vertically to $(g(v), g(g(v)))$

on the graph $y = g(x)$.

(5) repeat (3) & (4).

※ Use ARROW to show direction



- <u>limits of sequences given by iterated maps</u>

Here, the limit

$$\lim_{n\to\infty} g^{[n]}(v) = \lim_{n\to\infty} \underbrace{g(g(g\cdots(g(v))\cdots)))}_{n\text{-iteration}}$$

is a point, $\overline{V}$, where the graph $y = g(x)$ intersects with the line $y = x$.

In other words,

$$\boxed{\text{If } \lim_{n\to\infty} g^{[n]}(v) = \overline{V} \text{ then } g(\overline{V}) = \overline{V}.}$$

* **Fixed point of g**

= a point $\overline{V}$ with $g(\overline{V}) = \overline{V}$

= a point where $y = g(x)$ & $y = x$ intersect.

- $\overline{V} = \lim_{n\to\infty} g^{[n]}(x_0) \implies g(\overline{V}) = \overline{V}$

i.e. $\overline{V}$ is a fixed point under the map g.

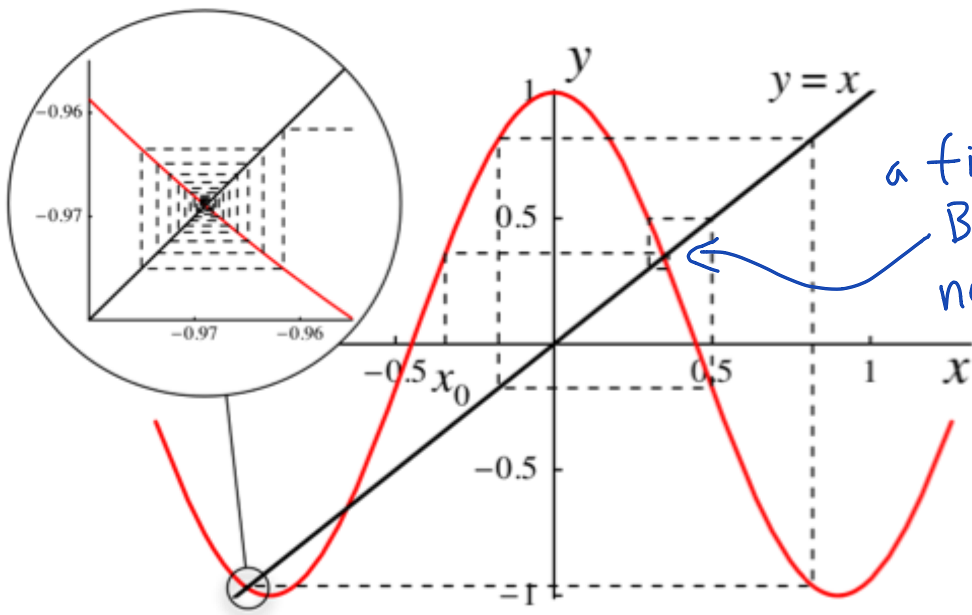
BUT,

- $g(\bar{v}) = \bar{v} \;\cancel{\Rightarrow}\; \bar{v} = \lim_{n\to\infty} g^{[n]}(x_0)$ for some x_0.

WRONG

※ a fixed point may <u>NOT</u> be a limit of a sequence given by the iterated maps

e.g.



a fixed pt BUT, not a limit.

Figure 10.16. *After some detours, the cobweb converges to the leftmost point of 'on $\tilde{z}$ between y = $\cos(\frac{7}{2}x)$ and the line $y = x$ for the initial value $x_0 = -0.343$.*

a fixed point & the limit of a sequence

When draw cobwebs,

it is IMPORTANT to identify

the fixed points, i.e.

where $y = g(x)$ & $y = x$

intersect,

because they are candidates for limits

※ Fixed points are important (for finding limits)

for iterated maps,

like critical points are important for

finding max/min.

Fixed points are not necessarily limits,

like critical points are not necessarily max/min.

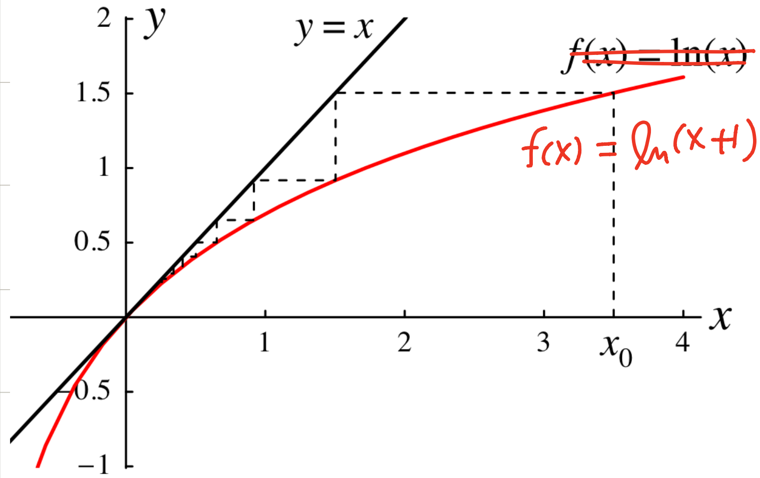
- Cobwebs may be used to find limits.

e.g.

$x_0 = 3.5$

$x_{n+1} = f(x_n)$

$f(x) = \ln(x+1)$.



$\lim_{n\to\infty} x_n = 0$.

Application

Find limits of sequences given by some recursion relations of type $x_{n+1} = f(x_n)$.

Ex. $a_{n+1} = \frac{1}{3} a_n^2 + \frac{2}{3}$

Find the $\lim_{n\to\infty} a_n$

(a) If $a_0 = 1.5$

(b) If $a_0 = 3$

<sol>. • We view this as iterated maps

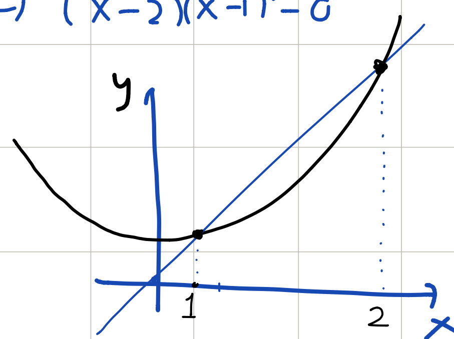
$a_{n+1} = f(a_n)$, $f(x) = \frac{1}{3}x^2 + \frac{2}{3}$.

• To find fixed points

$\boxed{\text{Solve } x = f(x)}$, i.e. $x = \frac{1}{3}x^2 + \frac{2}{3}$

$3x = x^2 + 2 \iff x^2 - 3x + 2 = 0 \iff (x-2)(x-1) = 0$

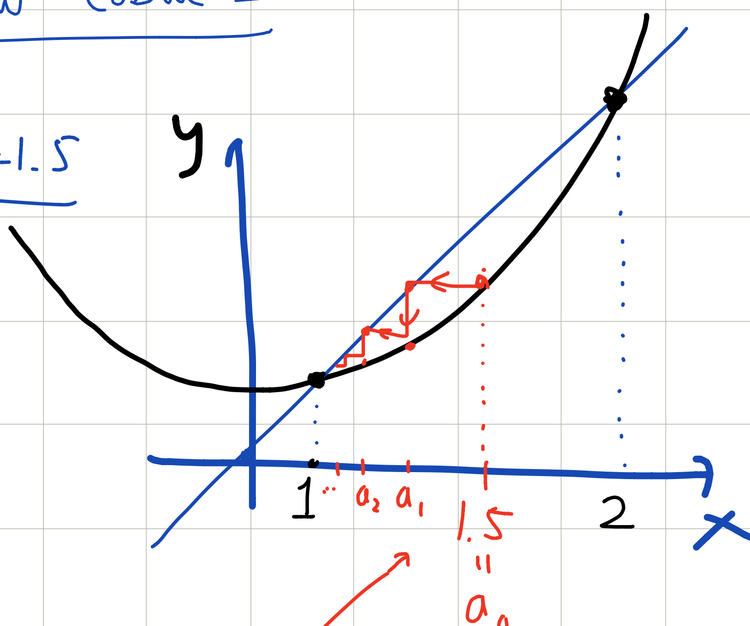
So, fixed points $x = 1, 2$.



• Draw cobwebs

(a)

If $a_0 = 1.5$

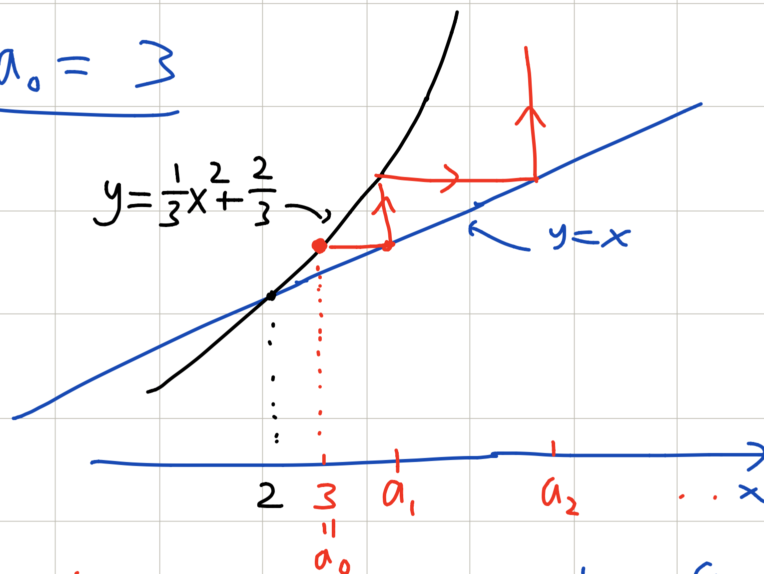


If $a_0 = 1.5$, a_k will converge to 1.

$$\lim_{k \to \infty} a_k = 1.$$

▨ (a)

(b) If $\underline{a_0 = 3}$



If $a_0=3$, then $a_k \longrightarrow +\infty$. $\underline{\lim_{k\to\infty} a_k = \infty}$ ▨ (b)

<u>WARNING</u> Finding limits using cobwebs

works well

ONLY for the case

- drawing the graph of $y=g(x)$
- find the fixed points

are <u>NOT</u> too complicated/difficult.

Now, we find a condition when a fixed point can be a limit.

§. Stability of a fixed point.

DEF

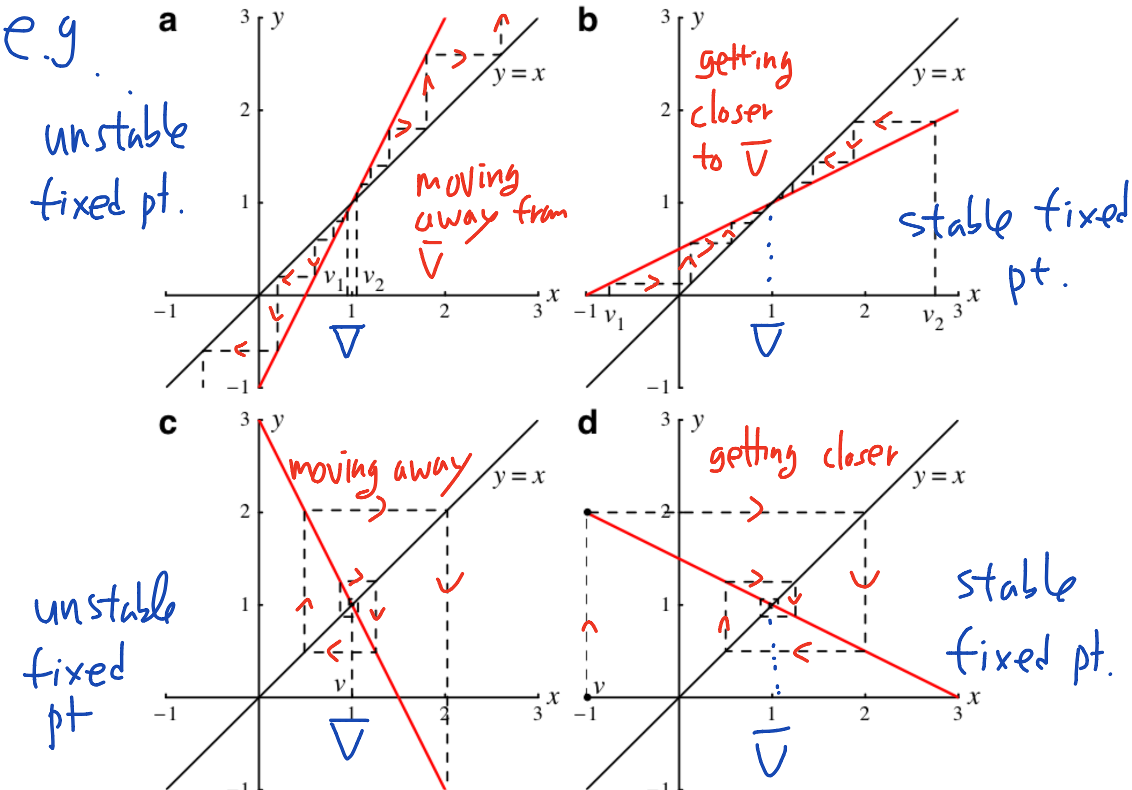
A fixed point $\bar{v}$ for $y = g(x)$ is called **stable** if once $x_k = g^{[k]}(v)$ is closed enough to $\bar{v}$ for some k, then $\lim_{n\to\infty} g^{[k]}(v) = \bar{v}$.

Otherwise, called **unstable**.

※ ONLY stable points can be the limit of an iterated map sequence.

unless the sequence is constant after certain n, (i.e. $a_n = a_{n+1} = \cdots = \cdots \ \forall n \geq N$)

e.g.



※ From these examples, we see the stability is related to the angle of intersection (between $y=g(x)$ & $y=x$)

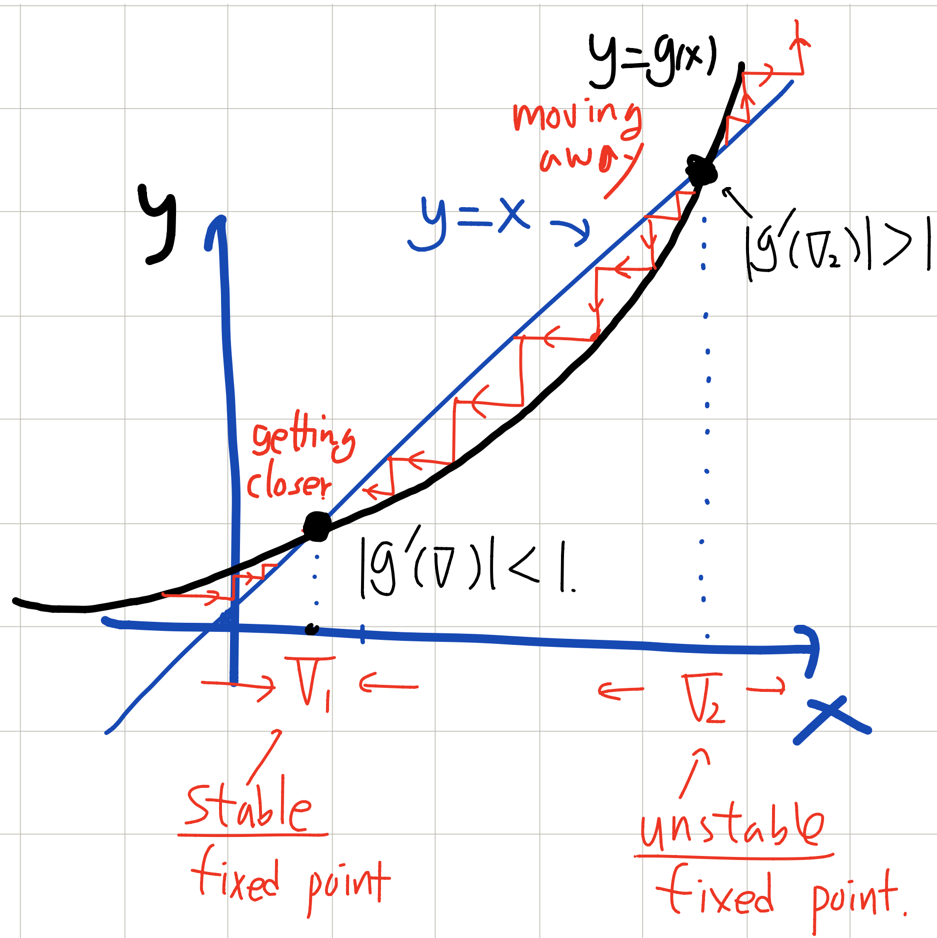
- A test to check stability.

$\bar{v}$ a fixed point, $g(\bar{v})=\bar{v}$

strict inequalities

- stable if $-1 < g'(\bar{v}) < 1$.
- unstable if $g'(\bar{v}) > 1$ or $g'(\bar{v}) < -1$.

※ A point of this test : No Need to draw graphs to check stability. Compute the derivative.



- Near a fixed pt $\bar{v}$, $g(\bar{v}) = \bar{v}$.

for x close to $\bar{v}$

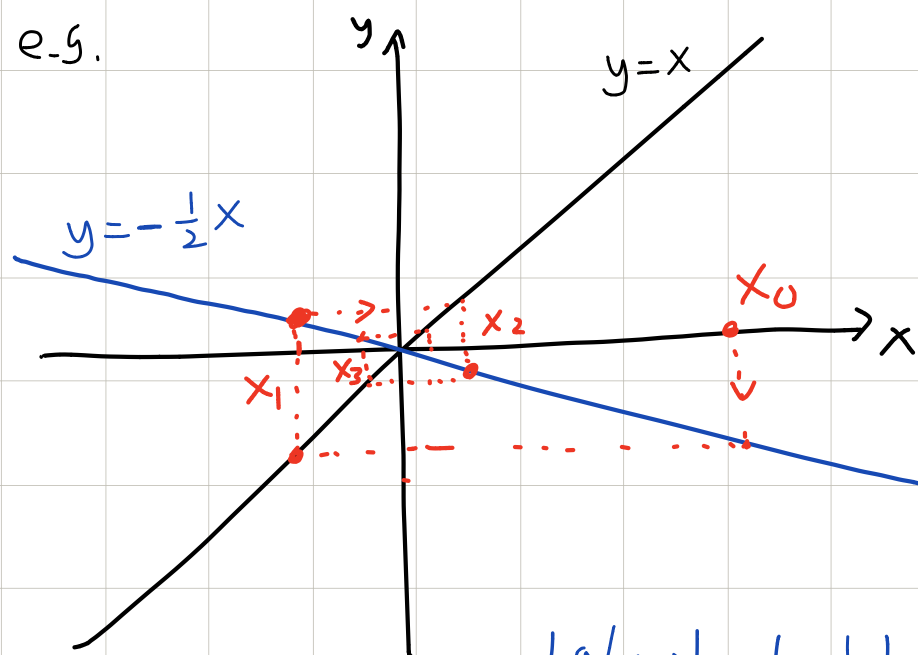
$$\frac{|g(x) - \bar{v}|}{|x-\bar{v}|} = \frac{|g(x) - g(\bar{v})|}{|x-\bar{v}|} \approx |g'(\bar{v})|$$

for $|x-\bar{v}|$ small

Therefore, (for x close to $\bar{v}$)

- $|g'(\bar{v})|<1 \Longrightarrow |g(x)-\bar{v}| < |x-\bar{v}|$ — get closer to $\bar{v}$ $\therefore$ stable

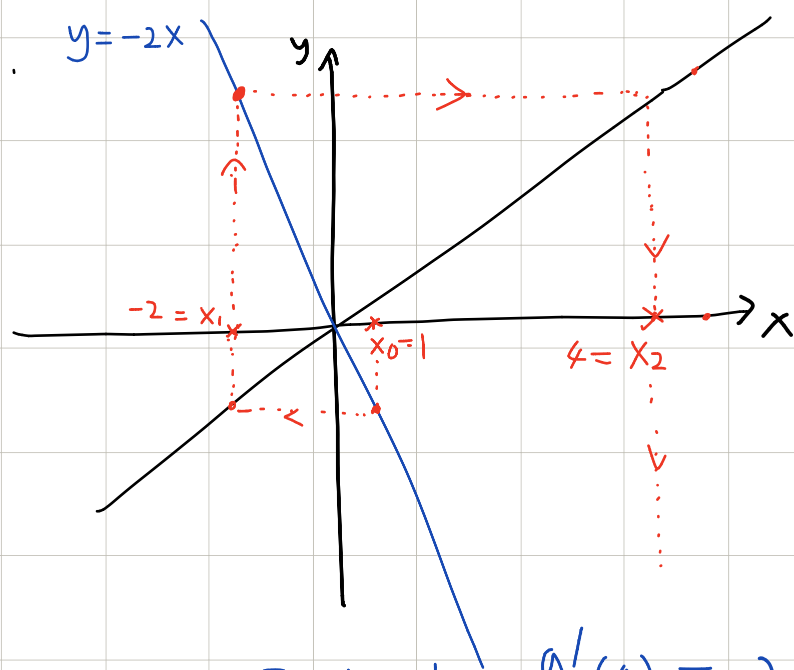
- $|g'(\bar{v})|>1 \Longrightarrow |g(x)-\bar{v}| > |x-\bar{v}|$ — move away from $\bar{v}$. $\therefore$ unstable e.g.



Fixed point $\overline{V} = 0$. $|g'(0)| = |-\frac{1}{2}| = \frac{1}{2} < 1$

$\therefore$ stable fixed pt.

e.g.



Fixed point $\overline{V} = 0$. $g'(0) = -2 < -1$.

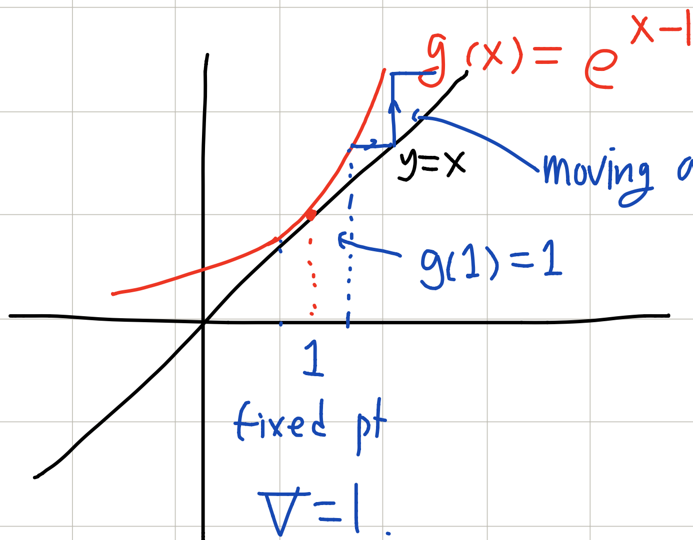
$\therefore$ unstable fixed pt.

BE CAREFUL

The above test does **NOT** work

if $g'(\nabla) = 1$ or -1

e.g.



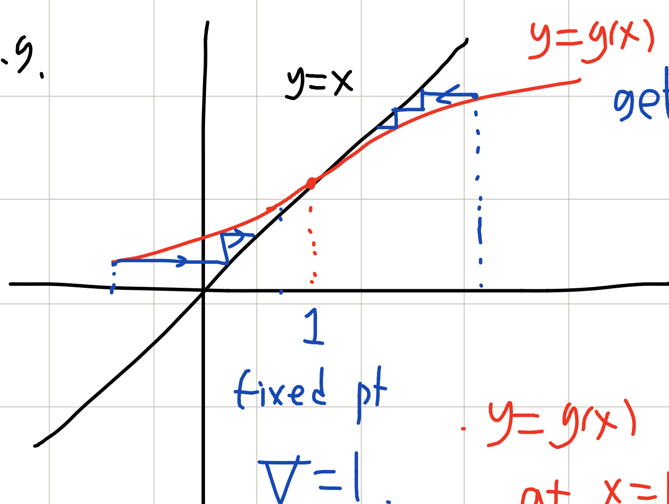
moving away,

$\therefore$ UNSTABLE

$g'(x) = e^{x-1}$

$g'(1) = 1$.

e.g.



getting closer.

$\therefore$ STABLE

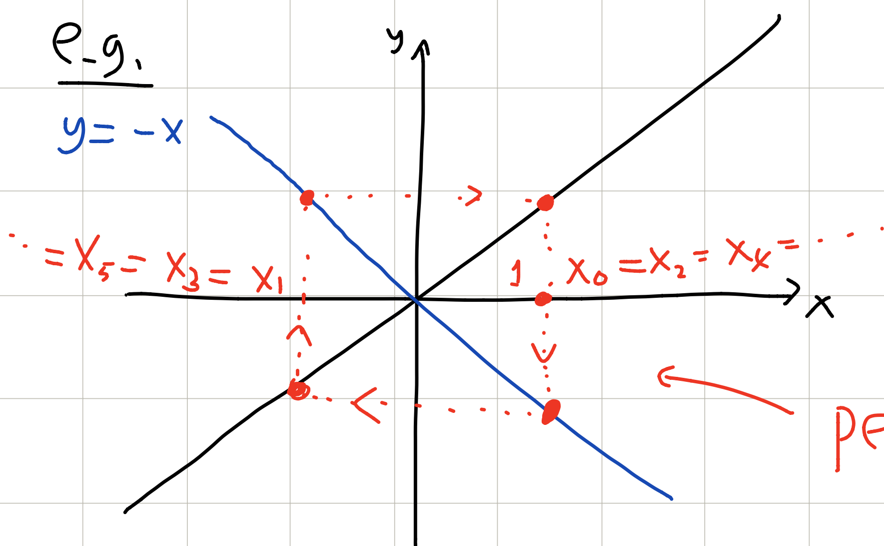
. $y=g(x)$ is tangent to $y=x$ at $x=1$

So, $g'(1) = 1$

An interesting case

unstable, but has periodic orbits nearby.

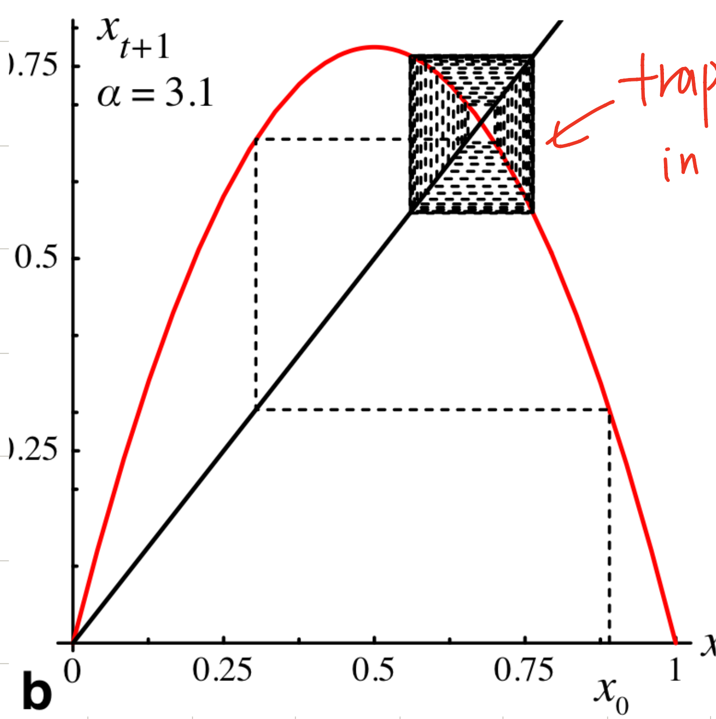
e.g.



periodic orbit.

In this case, the sequence is
$(1, -1, 1, -1, 1, -1, \ldots)$

e.g. unstable, but still traps the sequence nearby.



traps the sequence in this box.