Lec 28. Parametric Curves.

Today: - examples §8.2
- slopes, tangents, normals §8.3

Wed: - arc lengths, areas §8.4

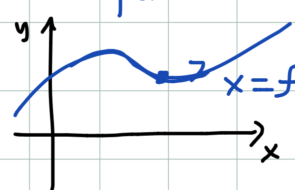
Mar 6, Fri. - Polar curves : § 8.5 ~ 8.6.
slopes / arc-le lengths / areas.

Mon: - § 8.6.

Tue: possible review / Q&A.

March 11, Wed: Midterm 2 (material up to & including March 6.)

a parametric curve



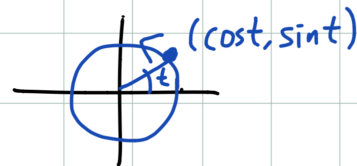
$x = f(t)$, $y = g(t)$ t : parameter.

Basic questions

- Given • a geometric curve, • a motion of a particle.
or a curve given as a solution to $F(x,y) = 0$, etc

- find a corresponding parametric expression. "parametrization"
(parametric expression can be useful to study the curves.)
e.g. computing arc-length, etc.

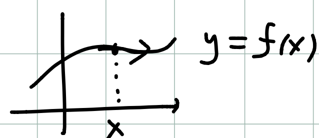
e.g. $x^2+y^2=1$:



$x = \cos t$, $y = \sin t$

$0 \le t \le 2\pi$

e.g.



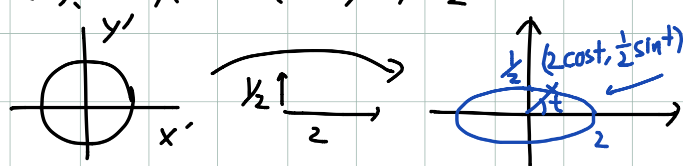
$x = x$, $y = f(x)$, x parameter.

- Given a parametric curve, find the corresponding geometric curve.

e.g. can appear as a solution to DE.
e.g. motion of a particle

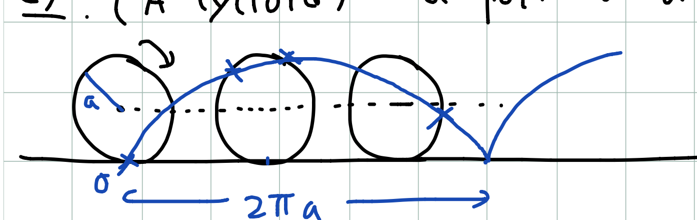
e.g. to study the geometry of the trajectory of a particle.

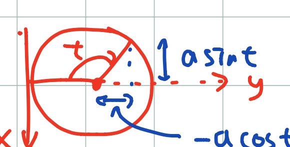
e.g. $x = 2\cos t$, $y = \frac{1}{2}\sin t$



ellipse $\frac{x^2}{2^2} + \frac{y^2}{(\frac{1}{2})^2} = 1$

EX. (A cycloid) a point on a rolling circle, without slipping.

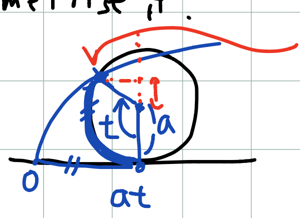




Parametrize it.

<Sol>.

$t=0$
$x=0$
$y=0$.

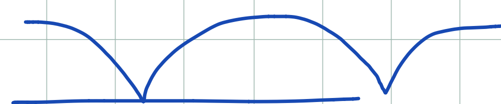


$x = at - a\sin t = a(t - \sin t)$

$y = a - a\cos t = a(1 - \cos t)$.

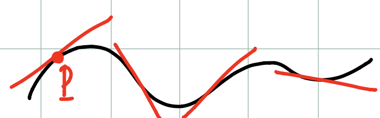
$x = a(t - \sin t)$
$y = a(1 - \cos t)$

← it is interesting to see that this parametrization with smooth functions yields a curve having non smooth points.

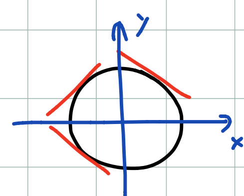


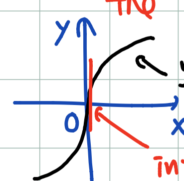
the curve is not smooth here!

- smooth (plane) curve. = A curve whose tangent lines change continuously along the curve.



e.g. smooth curve

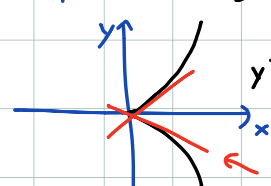




$y = \sqrt[3]{x}$

infinite slope of the tangent line, at $x=0$, but the tangent lines changes continuously.

e.g. nonsmooth curve.



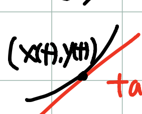
$y^2 = x^3$

the tangent lines have "sudden" change

Slope of a parametric curve.

$x = f(t), \quad y = g(t)$

$$\frac{dy}{dx} \overset{\text{chain rule}}{=} \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{g'(t)}{f'(t)}. \qquad \text{if } f'(t) \neq 0.$$



tangent line with slope $\frac{dy}{dx}$

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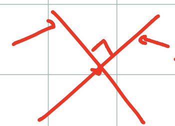
normal line with slope $-\frac{dx}{dy}$

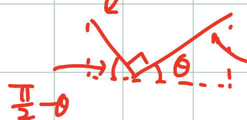
slope $-\frac{1}{p}$.

$= -\frac{dx}{dy}$.



slope $p = \frac{dy}{dx}$

slope $= -\tan(\frac{\pi}{2}-\theta) = -\frac{\sin(\frac{\pi}{2}-\theta)}{\cos(\frac{\pi}{2}-\theta)} = \frac{-\cos\theta}{\sin\theta} = -\frac{1}{\tan\theta}$.



slope $= \tan\theta$

$$-\frac{dx}{dy} = -\frac{dx}{dt}\Big/\frac{dy}{dt} = -f'(t)/g'(t), \quad \text{if } g'(t) \neq 0.$$

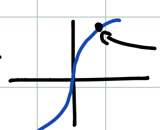
chain rule

※ The plane curve described by $x = f(t)$, $y = g(t)$,

is smooth on those $(x(t), y(t))$

where $f'(t)$, $g'(t)$ are continuous, and not both zero.

(A plane curve is smooth if either the slope of normal line or the slope of the tangent line changes continuously.)

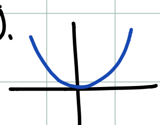
e.g. 

$x(t) = t^3$ $\quad x'(t) = 3t^2$

$y(t) = t$ $\quad y'(t) = 1$

※ At t where $f'(t) = 0 = g'(t)$,

the curve may or may not be smooth.

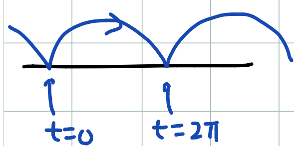
e.g. 

$x = t^3$ $\quad x'(0) = 0$

$y = t^6$ $\quad y'(0) = 0$

e.g. $x(t) = t - \sin t$ $\quad x'(t) = 1 - \cos t$ $\quad x'(0) = 0$

$y(t) = 1 - \cos t$ $\quad y'(t) = \sin t$ $\quad y'(0) = 0$



not smooth at $t = 0, \pm 2\pi, \pm 4\pi, \cdots$,

- Shape of a parametric curve : Concavity / convexity

For $x = f(t)$, $y = g(t)$. $\quad \dfrac{dy}{dx} = \dfrac{g'(t)}{f'(t)}$ for $f'(t) \neq 0$.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dt}\left[\frac{dy}{dx}\right] \Big/ \frac{dx}{dt}$$

$$= \left[\frac{g'(t)}{f'(t)}\right]' \Big/ f'(t) = \frac{g''(t) f'(t) - g'(t) f''(t)}{[f'(t)]^2} \Big/ f'(t)$$

$$= \frac{1}{[f'(t)]^3}\left[g''(t) f'(t) - g'(t) f''(t) \right] \quad \text{for} \quad f'(t) \neq 0.$$

- sketching a parametric curve using second derivatives.

Ex. $x = t^2 - t$ $\quad y = t^3$. $\quad -1 \le t \le 1$.

$f(t) = t^2 - t$ $\quad f'(t) = 2t - 1$.

$g(t) = t^3$ $\quad g'(t) = 3t^2$.

t		0		$\frac{1}{2}$	
$f'(t)$		$-$		0	$+$
$g'(t)$	$+$	0		$+$	
x		$\leftarrow$		$\cdot$	$\rightarrow$
y	$\uparrow$	$\cdot$		$\uparrow$	
curve	$\nwarrow$	$\leftarrow$	$\nwarrow$	$\uparrow$	$\nearrow$

For concavity

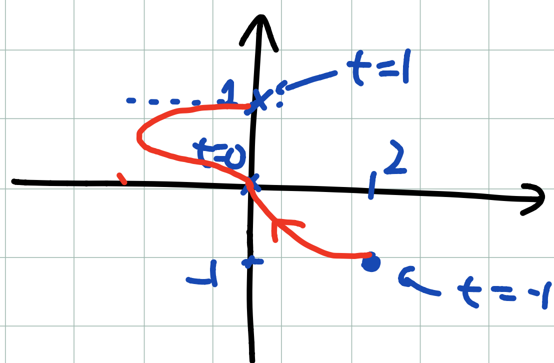
$f''(t) = 2$

$g''(t) = 6t$

$$\frac{d^2y}{dx^2} = \frac{1}{(f')^3}\left[g''f' - g'f''\right] = \frac{1}{(2t-1)^3}\left[6t\cdot 2 - 3t^2\cdot 2\right]$$

$$= \frac{6t(2-t)}{(2t-1)^3}$$

t		0		$\frac{1}{2}$		2	
$\frac{d^2y}{dx^2}$	$+$	0	$-$	$\pm\infty$	$-$	0	$+$
curve	concave up		concave down		concave down		concave up



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