Lec 27 §7.9. Differential Equations.

Differential Equation (DE):

$$F(x, y(x), y'(x), y''(x), \ldots) = g(x).$$

Variable x.

Unknown : $y(x)$.

We consider DE's of the form $F(x, y(x), y'(x)) = g(x)$

"1st order DE".

Ex. Solve $\underbrace{\frac{dy}{dx}} = \ln x \cdot e^{y}$

ratio between increment in y & increment in x

<sol>.

"Separation of variable"

$e^{-y} dy = \ln x \, dx.$

balance between increment in y & increment in x.

antiderivative in y

antiderivative in x.

$$\Leftrightarrow \quad \int e^{-y} dy = \int \ln x \, dx$$

$$e^{-y}\frac{dy}{dx} = \ln x$$

$$\int e^{-y}\frac{dy}{dx}\,dx = \int \ln x \, dx$$

substitution $dy = \frac{dy}{dx}dx$.

$$\int e^{-y} dy = \int \ln x \, dx$$

$$\Leftrightarrow \quad -e^{-y} = x\ln x - x + C$$

undetermined constant.

$$\Leftrightarrow \quad e^{-y} = -x\ln x + x + C$$

$$\Leftrightarrow \quad -y = \ln(-x\ln x + x + C)$$

$$y = -\ln(-x\ln x + x + C) \quad \square$$

Separable equation: $\frac{dy}{dx} = f(y)\,g(x)$

$$\Rightarrow \quad \frac{dy}{f(y)} = g(x)\,dx$$

$$\Rightarrow \int \frac{dy}{f(y)} = \int g(x)\,dx$$

From this can relation between x & y WITHOUT $\frac{dy}{dx}$.

There are only a few types of DE's that we know how to solve explicitly & separable eqns are one of them.

EX Initial Value problem.

Solve
$$\begin{cases} \frac{dy}{dx} = e^{-y}\ln x \\ y(1) = 0. \leftarrow \text{initial condition (IC)} \end{cases}$$

<sol>. We have from previous calculation, the general solution is

$$y = -\ln(-x\ln x + x + C).$$

To satisfy IC

$$0 = y(1) = -\ln(-1\cdot \cancel{\ln 1}^{\,0} + 1 + C)$$

$$1 + C = 1 \quad \therefore \quad \underline{C = 0}$$

$\therefore$ Sol. $y = -\ln(-x\ln x + x)$. $\boxed{/\!/}$

EX (Logistic eqn) $\begin{cases} \frac{dy}{dx} = y - y^2. \\ y(0) = 2 \end{cases}$ from population models.

Solve the initial value prob. (IVP)

$$\frac{dy}{dx} = k\, y\left(1 - \frac{y}{L}\right)$$

fertility (k) capacity (L)

<sol> $\frac{dy}{dx} = y - y^2$

$$\frac{dy}{y-y^2} = dx \qquad \int \frac{dy}{y-y^2} = \int dx$$

LHS: $$\int \frac{dy}{y-y^2} = \int \frac{dy}{y(1-y)} = \int \left[\frac{1}{1-y} + \frac{1}{y}\right] dy$$

$$= -\ln|1-y| + \ln|y| + C_1$$

RHS: $\int dx = x + C_2$

$$\therefore \quad -\ln|1-y| + \ln|y| = x + C \quad \leftarrow \quad C = C_2 - C_1$$

an arbitrary const.

$$\ln\left|\frac{y}{1-y}\right| = x + C$$

$$\therefore \quad \frac{y}{1-y} = ke^x \quad \leftarrow \quad k = \pm e^C$$

an arbitrary constant.

So, $y = ke^x - ke^x y$

$$(1 + ke^x)y = ke^x$$

So, $$y = \frac{ke^x}{1+ke^x}$$ k: arbitrary constant.

$y(0) = 2 \Rightarrow 2 = \frac{k}{1+k}$, $k = -2$. $\therefore y = \frac{-2e^x}{1-2e^x}$

Ex. Solve $\begin{cases} \dfrac{dy}{dx} = y - y^2 \\ y(0) = 1 \end{cases}$

<sol>. $\dfrac{dy}{y-y^2} = dx$ is <u>not</u> valid at $x=0$.
since $y(0)=1$.

But in this case,

$y(x) \equiv 1$ is a solution

since $\dfrac{dy}{dx} = 0$. , $y - y^2 = 1 - 1^2 = 0$, & $y(0)=1$.

Such a solution $\dfrac{dy}{dx} \equiv 0$ is called a "steady state".

In $\dfrac{dy}{dx} = \underbrace{f(y)}\, g(x)$

For a zero y_0 of $f(y) = 0$

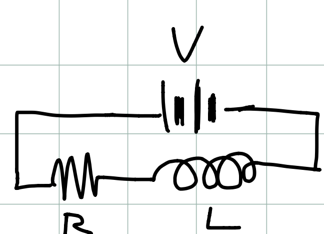
makes steady state $y \equiv y_0$.

Another type that we can solve explicitly.

: First order <u>linear</u> equations

$$\frac{dy}{dx} + p(x)\, y = q(x)$$

Ex A electric circuit.



t : time

$I = I(t)$: current.

$V = V(t)$: voltage

$R = R(t)$: resistance

$L = L(t)$: inductance.

$$\frac{dI}{dt} + \frac{R}{L} I = \frac{V}{L}.$$

Case $q(x) \equiv 0$. (homogeneous eqn.)

$\frac{dy}{dx} + p(x)\, y = 0$ is a separable eqn. $\frac{dy}{dx} = -p(x)\, y$.

$$\int \frac{dy}{y} = \int -p(x)\,dx$$

$$\ln|y| = -\int p(x)\,dx + C$$

$$y = K e^{-\int p(x)dx}$$

K = undetermined constant.

or $e^{\int p(x)dx} y = K$.

· General case $q(x) \not\equiv 0$. (Inhomogeneous eqn.)

Method 1 (Integrating factor) Let $\mu(x) = \int p(x)dx$

$e^{\mu(x)}$ to y :

multiply to DE :

$$\left(e^{\mu(x)} y\right)' = e^{\mu(x)} y' + p(x) e^{\mu(x)} y = q(x) e^{\int p(x)dx}$$

$$\therefore\ e^{\mu(x)} y = \int \left[q(x) e^{\mu(x)} \right] dx$$

$$\therefore\ y = e^{-\mu(x)} \int q(x) e^{\mu(x)} dx.$$

Method 2 (Variation of parameter)

Try to use the solution to $y' + py = 0$.

Try $y = \underbrace{k(x)}_{\text{a function to be determined}} \underbrace{e^{-\mu(x)}}_{\text{solution to } y'+py=0}$ $\qquad \mu(x) = \int p(x)\,dx$

"By putting $k(x)$ (an unknown function) as an additional factor, we can generate a nontrivial class of functions for possible solutions to $y' + py = q$, which are close to the solution to $y' + py = 0$."

Plug-in $y = k(x)e^{-\mu(x)}$ in the DE. (we will get eqn for $k(x)$).

$$(ke^{-\mu})' + p \cdot ke^{-\mu} = q$$

$$k'e^{-\mu} + \cancel{k \cdot (-p) e^{-\mu}} + \cancel{p \cdot k e^{-\mu}} = q$$

$$k' = e^{\mu} \cdot q \qquad \therefore \; k(x) = \int e^{\mu(x)} q(x)\,dx.$$

$$\therefore \; y(x) = e^{-\mu(x)} \int e^{\mu(x)} q(x)\,dx.$$

e.g. $\frac{dy}{dx} + \frac{2y}{x} = \frac{1}{x^2}$

<sd>. $p(x) = \frac{2}{x}$. $\mu(x) = \int p(x)\,dx = 2\ln|x| + C$

can choose $C=0$.

Method 1 $(e^{\mu(x)} y)'$ $\therefore e^{\mu(x)} = e^{2\ln|x|} = x^2$

$= (x^2 \cdot y)' = x^2 y' + 2x y$

$= x^2 \left(y' + \frac{2}{x} y \right)$

$= x^2 \frac{1}{x^2} = 1$

$x^2 y = \int dx = x + C.$

$\therefore \underline{y = \frac{x+C}{x^2}}.$

Method 2 $e^{-\mu(x)} = \frac{1}{e^{\mu(x)}} = \frac{1}{x^2}.$

Try $y = k(x) e^{-\mu(x)} = k(x) \frac{1}{x^2}$ to solve

$\left(k(x) \frac{1}{x^2} \right)' + \frac{2}{x} \cdot k(x) \frac{1}{x^2} = \frac{1}{x^2}$

$k'(x) \frac{1}{x^2} + \cancel{k(x) \cdot \frac{-2}{x^3}} + \cancel{\frac{2}{x^3} k(x)} = \frac{1}{x^2}$

$\therefore k'(x) = 1.$ $\therefore k(x) = x + C.$

$\therefore \underline{y = \frac{x+C}{x^2}}.$ ☑