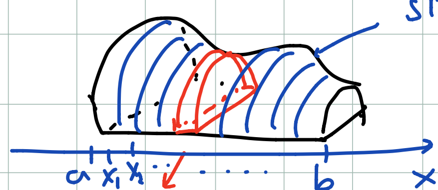
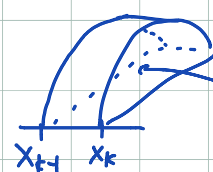
Lec19. Volumes §7.1, 7.2.

Volumes by slicing.

slice it into N pieces





A(x)
Area of cross section at x.

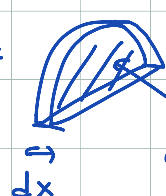
slice  with volume $\Delta V_k \approx A(x_k^*)\Delta x_k$, $x_{k-1} \leq x_k^* \leq x_k$

Total volume $V = \sum_{k=1}^{N} \Delta V_k$

$$\approx \sum_{i=1}^{N} A(x_k^*)\Delta x_k$$ ← This is a Riemann sum.

Slice it into "infinitely" many infinitely small pieces $\Delta V_i \to 0$.

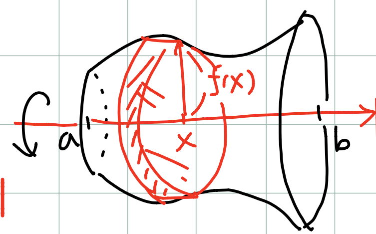
$$V = \int_{x=a}^{x=b} dV = \int_{x=a}^{x=b} A(x)\,dx.$$

dV := volume of 

Area A(x) of the cross section

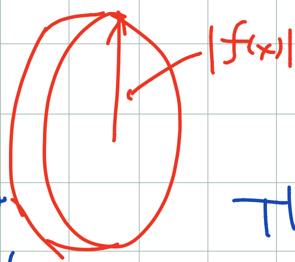
Volumes of solids of revolution

thin slice



$a \le x \le b$.

let volume ΔV →



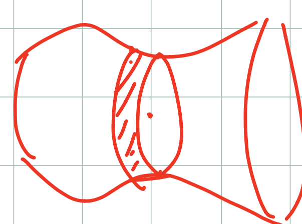
This is approximately a thin cylinder

Δx

This slice has volume, say, ΔV, approximately, $\pi [f(x)]^2 \Delta x$

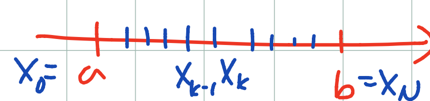
i.e. $\Delta V \approx \pi [f(x)]^2 \Delta x$

The total volume is the sum of the volumes of the thin slices



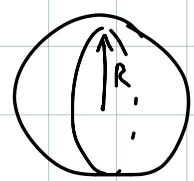
$$\text{Total volume} = \sum_{k=1}^{N} \Delta V$$

$$\approx \sum_{k=1}^{N} \pi [f(x_k)]^2 \Delta x.$$

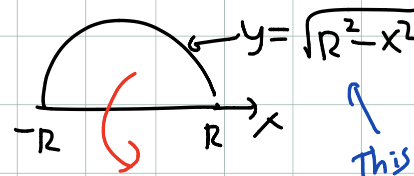


Taking $N \to \infty$ (more & more slices),
thinner & thinner

$\therefore$ $$\text{Total volume} = \int_a^b \pi [f(x)]^2 \, dx.$$

e.g. 

Volume of the sphere of radius R



This formula is from
$x^2+y^2=R^2$
$\Rightarrow y^2=R^2-x^2$.

$$V=\int_{-R}^{R}\pi\left(\sqrt{R^2-x^2}\right)^2dx$$

$$=\int_{-R}^{R}\pi\left(R^2-x^2\right)dx$$

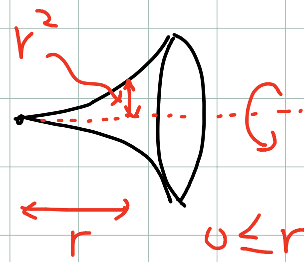
$$=\pi\int_{-R}^{R}\underbrace{(R^2-x^2)}_{}dx \quad \leftarrow \text{even function.}$$

$$=\pi\cdot 2\int_{0}^{R}(R^2-x^2)\,dx$$

$$=2\pi\left[R^2x-\frac{x^3}{3}\right]_0^R=2\pi\left(R^3-\frac{R^3}{3}\right)$$

$$=\frac{4\pi}{3}R^3 \quad \blacksquare$$

Ex (mass)



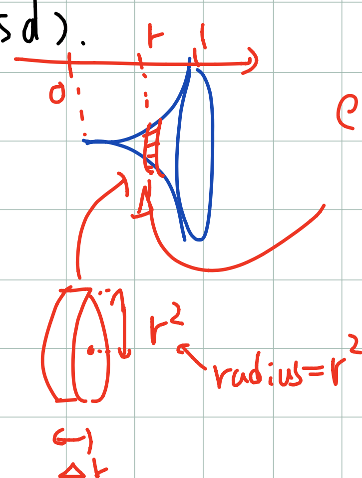
Suppose mass density per volume

$$\rho(r) = r$$

$0 \le r \le 1$.

Compute the total mass.

<sol>.



each thin slice has mass Δm

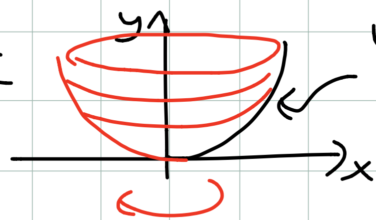
$$\approx \begin{pmatrix}\text{density}\\ \text{per volume}\end{pmatrix} \times \begin{pmatrix}\text{volume}\\ \text{of the thin slice}\end{pmatrix}$$

$$\Delta m \approx \rho(r) \cdot \pi \left[r^2\right]^2 \Delta r$$

i.e. $\Delta m \approx r \cdot \pi (r^2)^2 \Delta r = \pi r^5 \Delta r$

$\therefore$ Total mass $\left(= \lim_{N\to\infty} \sum_{k=1}^{N} \Delta m = \lim_{N\to\infty} \sum_{k=1}^{N} \pi (r_k)^5 \Delta r \right)$

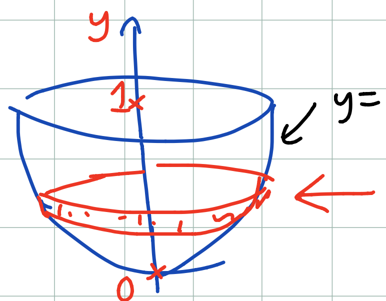
$$= \int_0^1 \pi r^5 \, dr = \pi \left[\frac{r^6}{6}\right]_0^1 = \underline{\frac{\pi}{6}}. \quad \boxed{//}$$

Ex  $y = x^2$ $\quad 0 \le y \le 1$

the solid of revolution about the y-axis.

Compute the volume.

<sol).



$y = x^2$

view x as a function of y

$x = \sqrt{y}$ (because $y = x^2$)

$\updownarrow \Delta y$

has volume

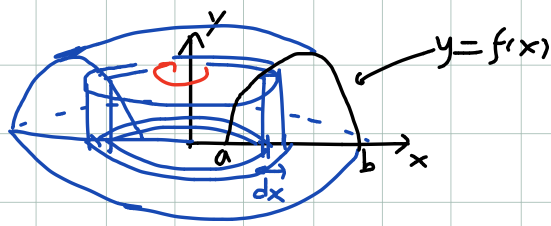
$\approx \pi x^2 \Delta y = \pi (\sqrt{y})^2 \Delta y$

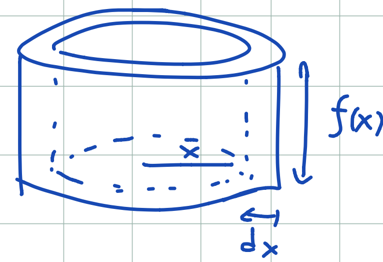
$0 \le y \le 1$.

$$\text{volume} = \int_0^1 \pi (\sqrt{y})^2 \, dy = \int_0^1 \pi y \, dy$$

$$= \pi \left[\frac{y^2}{2} \right]_0^1 = \frac{\pi}{2}$$

cylindrical shells

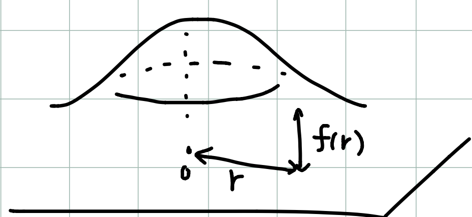




$$dV = 2\pi x \cdot dx \cdot f(x)$$
$$= 2\pi x f(x)\,dx$$

$$V = \int_a^b 2\pi x f(x)\,dx.$$

Ex. 2-dim'l Gaussian distribution.

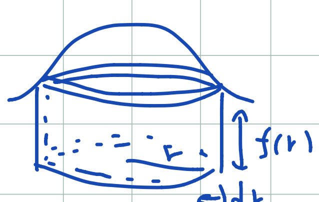


r = distance from 0.

$f(r) = e^{-r^2}$ height.

<sol>. ※ In the following, the improper integrals involve e^{-r^2} times some factor like r, r^2 etc, so they (e.g. $\int_0^\infty e^{-r^2}dr$, $\int_0^\infty re^{-r^2}dr$, $\int_{-\infty}^\infty e^{-x^2}dx$, etc) converge.

Method ①. Cylindrical shells $dV = 2\pi r\,dr\, f(r)$



$$\therefore\ V = \int_0^\infty 2\pi r f(r)\,dr \qquad f(r) = e^{-r^2}$$

$$= \lim_{R\to\infty} \left[-\pi e^{-r^2}\right]_0^R$$

$$= \lim_{R\to\infty} \left[-\pi e^{-R^2} + \pi\right] = \pi.$$ ▨

Method ② horizontal slicing

$$\therefore \quad dV = \pi r^2 \cdot df = \pi r^2 \cdot \frac{df}{dr} \cdot dr$$

$$\therefore V = \int_0^{\infty} \pi r^2 \cdot f'(r)\, dr \qquad f(r) = e^{-r^2}$$

$$= \int_0^{\infty} \pi \cdot r^2 (-2r) \cdot e^{-r^2} dr \qquad f'(r) = -2r e^{-r^2}$$

$$= -\pi \int_0^{\infty} u \cdot e^{-u} du \qquad u = r^2, \quad du = 2r\,dr$$

Integration by parts

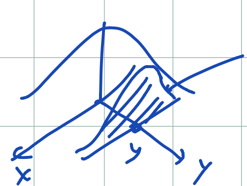
$$= -\pi \left[-u e^{-u} \right]_0^{\infty} + \pi \int_0^{\infty} e^{-u} du$$

$\leftarrow$ you can make this rigorous by taking $\lim\limits_{R \to \infty} \int_0^R \ldots$

$$= \pi(0+0) + \pi[0+1]$$

$$= \pi.$$

Method ③ veritical slicing.



$e^{-x^2+y^2}$

$$V = \int_{y=-\infty}^{y=\infty} dV(y) = \int_{y=-\infty}^{y=\infty} \underbrace{\int_{x=-\infty}^{x=\infty} e^{-(x^2+y^2)} dx\, dy}_{dV(y)}$$

$$= \int_{y=0}^{y=\infty} e^{-y^2} \left[\underbrace{\int_{x=-\infty}^{x=\infty} e^{-x^2} dx}_{} \right] dy$$

Since it is a constant.

$$= \underbrace{\int_{x=-\infty}^{x=\infty} e^{-x^2} dx}_{} \int_{y=-\infty}^{y=\infty} e^{-y^2} dy$$

This improper integral converges but e^{-x^2} is something impossible to explicitly integrate.

$$= \left[\int_{-\infty}^{\infty} e^{-x^2} dx \right]^2$$

From ① or ②, we know, $V = \pi$, thus as a consequence,

$$\underline{\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}}.$$

$\blacksquare$