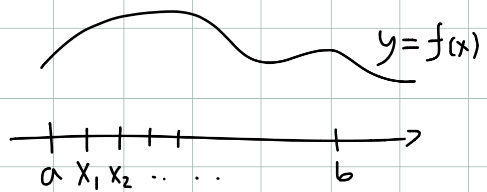
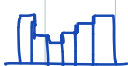
Lec 18 Error estimates §6.6.



$$\Delta x = h = \frac{b-a}{N}$$

Recall

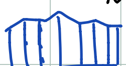
• Midpoint rule: $\int_a^b f(x)\,dx \approx M_N := \sum_{k=1}^{N} f\left(\frac{x_{k-1}+x_k}{2}\right)\cdot h$



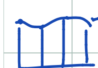
$$= h\cdot\left(f(m_1)+\cdots+f(m_N)\right)$$

• Trapezoid rule

$$T_N = h\cdot\left[\frac{f(x_0)}{2} + f(x_1)+\cdots+f(x_{N-1}) + \frac{f(x_N)}{2}\right]$$



• Simpson's rule


— parabolic pieces.

$$S_N = \frac{h}{3}\left[f(x_0)+f(x_N) + 4\sum_{\substack{0<k<N\\ \text{odd}}} f(x_k) + 2\sum_{\substack{0<k<N\\ \text{even}}} f(x_k)\right]$$

※ Notice that computing M_N, T_N, S_N requires computing the same number of $f(x_k)$'s.

$\therefore$ they require similar time.

Which one will be more effective than other?

How much precise the approximation?

• If we knew $f(x)$ is well-behaved (i.e. many (≥ 4) times differentiable)

then midpoint rule $\approx$ trapezoid rule $<$ Simpson's rule (significantly better)

for precision.

This is expectable since parabolic pieces (for Simpson's rule) approximates the graph

better than line segments.

error estimates

• Thm

• Mid point rule & trapezoid rule

Assume f is twice differentiable & f'' is continuous

& $|f''(x)| \leq K$ on $[a,b]$

Then, · $\left|\int_a^b f(x)dx - M_N\right| \leq K \cdot \dfrac{(b-a)^3}{24N^2} = K\dfrac{(b-a)}{24}h^2$

· $\left|\int_a^b f(x)dx - T_N\right| \leq K \cdot \dfrac{(b-a)^3}{12N^2} = K\dfrac{(b-a)}{12}h^2$

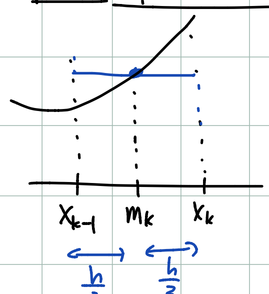
• Simpson's rule:

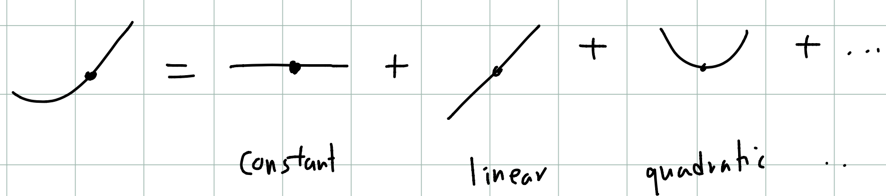
If $f^{(4)}$ is continuous & $|f^{(4)}(x)| \leq K$ on $[a,b]$,

then $\left|\int_a^b f(x)dx - S_N\right| \leq K\dfrac{(b-a)}{180}h^4 = K\dfrac{(b-a)^5}{180N^4}$

the error goes to zero much faster as $h \to 0$ than Mid point or trapezoid rule.

Proof of Mid point rule error estimate.





For $|x - m_k| \leq \frac{h}{2}$ & since f'' is continuous, $f(x) = \underbrace{f(m_k)+ \quad f'(m_k)(x-m_k)}_{\text{Correspond to mid point rule.}} + E_k(x)$

where

$$|E_k(x)| \leq \frac{1}{2}\max_{[x_{k-1},x_k]}|f''|\,|x-m_k|^2$$

Your exercise to show this.

$$\therefore \int_{x_{k-1}}^{x_k} f(x)\,dx = f(m_k)\,h + 0 + \int_{x_{k-1}}^{x_k} E_k(x)\,dx$$

$$\therefore \left| \int_{x_{k-1}}^{x_k} f(x)\,dx - f(m_k)\cdot h \right| = \left| \int_{x_{k-1}}^{x_k} E_k(x) \right| dx$$

$$\le \int_{x_{k-1}}^{x_k} \frac{1}{2} \max_{[x_{k-1},x_k]} |f''| \, (x-m_k)^2\,dx$$

$$= \frac{1}{2} \underbrace{\max_{[x_{k-1},x_k]} |f''|}_{\text{assumption.}} \cdot \left[\frac{(x-m_k)^3}{3} \right]_{x_{k-1}}^{x_k}$$

$$\le \frac{1}{2} \cdot K \cdot \frac{2\cdot \left(\frac{h}{2}\right)^3}{3}$$

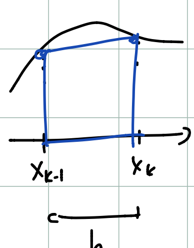
$$= K \cdot \frac{h^3}{24}.$$

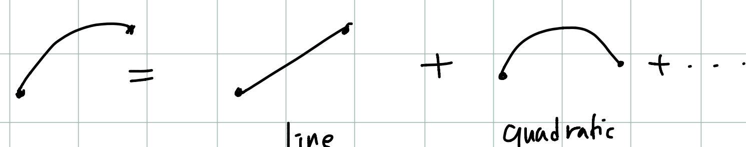
$$\therefore \left| \int_a^b f(x)\,dx - \underbrace{\sum_{k=1}^{N} f(m_k)\cdot h}_{M_N} \right| \le \sum_{k=1}^{N} K \cdot \frac{h^3}{24} \quad \leftarrow \quad h = \frac{b-a}{N}$$

$$= K \cdot \frac{h^3}{24} \cdot N$$

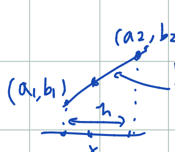
$$= K \cdot \frac{(b-a)}{24} \cdot h^2. \quad \blacksquare$$

Proof of trapezoid rule error estimate.





$\downarrow$



$b_1 + \frac{b_2 - b_1}{h}\cdot(x - a_1)$: $f(x_{k-1}) + \frac{f(x_k) - f(x_{k-1})}{h}(x - x_{k-1})$

$$f(x) = \overbrace{f(x_{k-1}) + \frac{f(x_k) - f(x_{k-1})}{h}(x - x_{k-1})}^{\text{correspond to trapezoid rule}} + \tilde{E}_k(x)$$

similar to HW 4, problem 4*. **Exercise:** $\left|\tilde{E}_2(x)\right| \le \frac{1}{2} \max_{[x_{k-1},x_k]} |f''(x)| \, \left|(x - x_{k-1})(x - x_k)\right|$ for $x_{k-1} \le x \le x_k$

$$\int_{x_{k-1}}^{x_k} f(x)\,dx = f(x_{k-1}) \cdot h + \frac{f(x_k) - f(x_{k-1})}{h} \cdot \frac{\overbrace{(x_k - x_{k-1})^2}^{h^2}}{2} + \underbrace{\int_{x_{k-1}}^{x_k} \tilde{E}_k(x)\,dx}$$

$$= f(x_{k-1})h + \frac{f(x_k) - f(x_{k-1})}{2h} + \int_{x_{k-1}}^{x_k} \tilde{E}_k(x)\,dx$$

$$\int_{x_{k-1}}^{x_k} \tilde{E}_k(x)\,dx \le \frac{1}{2} \max_{[x_{k-1},x_k]} |f''| \int_0^h x(x-h)\,dx$$

$$\le \frac{1}{2} K \left[\frac{x^3}{3} - h\frac{x^2}{2}\right]_0^h$$

$$= \frac{1}{2} K \cdot \left(\frac{h^3}{3} - \frac{h^3}{2}\right) = \frac{1}{2} K \cdot \frac{h^3}{6} = \frac{1}{12} K \cdot h^3.$$

$$\left|\int_a^b f(x)\,dx - T_N\right| \le \sum_{k=1}^{N} \frac{1}{12} K h^3 = \frac{1}{12} K \cdot h^3 \cdot \overbrace{\frac{(b-a)}{h}} \leftarrow N = \frac{b-a}{h}.$$

$$= \frac{1}{12} K (b-a) h^2.$$

$\blacksquare$