Lec 13

Midterm 1.
for material up to Fri. Jan 30.

- partial fractions : complete square case §6.2.
- Inverse substitutions §6.3.

With complete square factor (x^2+a^2).

e.g.

$$g(x) = \frac{1}{(x^2+1)(x-1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1}$$

$$C = \lim_{x\to 1} g(x)(x-1) = \frac{1}{1^2+1} = \frac{1}{2}$$

$A, B = ?$

$$g(x)(x^2+1) = Ax+B + \frac{C}{x-1}(x^2+1)$$

Can use complex numbers

note x^2+1 has zeros i & $-i$. $i = \sqrt{-1}$.

$i^2 = -1$.

$$\lim_{x\to i} g(x)(x^2+1) = \lim_{x\to i}\Big[Ax+B + \underbrace{\frac{C}{x-1}(x^2+1)}_{\to 0 \text{ as } x\to i}\Big]$$

$$= Ai+B$$

$$\therefore \lim_{x\to i} \frac{1}{x-1} = Ai+B$$

$$\frac{1}{i-1} = Ai+B$$

$$\frac{1}{i-1} = \frac{i+1}{(i-1)(i+1)} = \frac{i+1}{i^2-1} = \frac{i+1}{-1-1} = -\frac{1}{2}i - \frac{1}{2}$$

$\therefore A = -\frac{1}{2}, \quad B = -\frac{1}{2}.$

How to handle integrals $\int \frac{1}{x^2+a^2}\,dx$?

e.g. $\int \frac{1}{x^2+1}\,dx = \tan^{-1}x + C$ note We had $(\tan^{-1}x)' = \frac{1}{x^2+1}$

e.g. $\int \frac{1}{x^2+x+1}\,dx$

$= \int \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}\,dx \leftarrow x^2+x+1 = x^2+x+\frac{1}{4}-\frac{1}{4}+1$

$= (x+\frac{1}{2})^2+\frac{3}{4}$

completing the square.

$= \int \frac{4}{3} \frac{1}{\left[\sqrt{\frac{4}{3}}(x+\frac{1}{2})\right]^2+1}\,dx$

$\leftarrow$ Now $u = \sqrt{\frac{4}{3}}(x+\frac{1}{2})$

$du = \sqrt{\frac{4}{3}}\,dx$

$= \int \frac{4}{3} \frac{1}{u^2+1} \sqrt{\frac{3}{4}}\,du$

$= \sqrt{\frac{4}{3}} \cdot \tan^{-1} u + C$

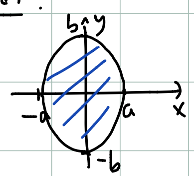
$= \sqrt{\frac{4}{3}} \tan^{-1}\left[\sqrt{\frac{4}{3}}(x+\frac{1}{2})\right] + C.$

. Inverse substitution § 6.3.

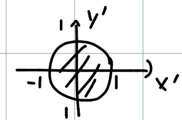
Trigonometric substitutions

Ex. Area of ellipse : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Method 1. "multivariable calculus".

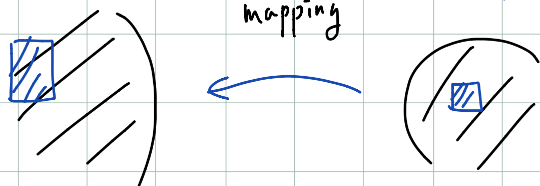


$x = a x'$
$y = b y'$
$\longleftarrow$



unit circle.

mapping

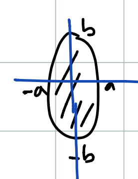


"You will learn more about this method in multi-variable calculus."

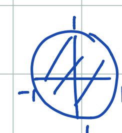
area change by the mapping

area(▨) = a·b area(▨)

So, the area of ellipse

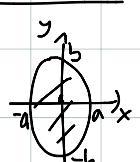


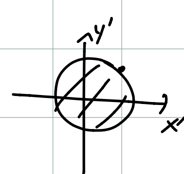
is a·b·area of



$\therefore$ The area $= \pi \cdot ab$. ▨

Method 2 IMPORTANT! Single variable calculus & substitution





$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \longleftrightarrow (x')^2 + (y')^2 = 1$

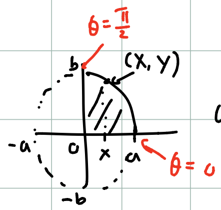
$x = a x'$
$y = b y'$

$x' = \cos\theta$, $y' = \sin\theta$

$$\boxed{\begin{aligned} x &= a\cos\theta \\ y &= b\sin\theta \end{aligned}}$$

area of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$:

$$\frac{y^2}{b^2} = 1-\frac{x^2}{a^2} \qquad y = \pm b\sqrt{1-\frac{x^2}{a^2}} . \qquad -a \le x \le a$$



$$\text{area} = 4\int_0^a y(x)\, dx$$

positive $y = b\sqrt{1-\frac{x^2}{a^2}}$

use $x = a\cos\theta$, $0 \le \theta \le \pi$

$dx = -a\sin\theta\, d\theta$

$$= 4\int_{x=0}^{x=a} b\sin\theta\cdot(-a\sin\theta)\,d\theta$$

$$= -4ab\int_{\theta=\frac{\pi}{2}}^{\theta=0} |\sin\theta|\sin\theta\, d\theta$$

$$= 4ab\int_0^{\frac{\pi}{2}} \sin^2\theta\, d\theta .$$

<u>note</u> $b\sqrt{1-\frac{x^2}{a^2}}$

$= b\sqrt{1-\cos^2\theta}$

$= b\sqrt{\sin^2\theta}$

$= b|\sin\theta|$

$= b\sin\theta$ for $\sin\theta \ge 0$.

with $0 \le \theta \le \pi$

To compute $\int_0^{\frac{\pi}{2}}\sin^2\theta\, d\theta$

① Use $\cos^2\theta - \sin^2\theta = \cos 2\theta$

double angle formula

$1-2\sin^2\theta = \cos 2\theta$ $\quad\therefore\quad \sin^2\theta = \frac{1-\cos 2\theta}{2}$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$$\therefore \int_0^{\frac{\pi}{2}}\sin^2\theta\, d\theta = \int_0^{\frac{\pi}{2}}\frac{1-\cos 2\theta}{2}\,d\theta = \int_0^{\frac{\pi}{2}}\frac12\, d\theta - \frac12\cancel{\int_0^{\frac{\pi}{2}}\cos 2\theta\, d\theta}$$

0 check!

$$= \underline{\frac{\pi}{4}}.$$

② $\int \sin^2\theta\, d\theta = \int \sin\theta\cdot\sin\theta\, d\theta = -\cos\theta\sin\theta - \int -\cos\theta\cdot\cos\theta$

Integration by parts

$$= -\cos\theta\sin\theta + \int\cos^2\theta\, d\theta$$

$$= -\cos\theta\sin\theta + \int(1-\sin^2\theta)\, d\theta$$

$$\therefore\ 2\int\sin^2\theta\, d\theta = -\cos\theta\sin\theta + \int d\theta$$

$$\therefore \int_0^{\frac{\pi}{2}}\sin^2\theta = \frac12\cancel{\Big[-\cos\theta\sin\theta\Big]_0^{\frac{\pi}{2}}} + \frac12\int_0^{\frac{\pi}{2}} d\theta = \underline{\frac{\pi}{4}}$$

0

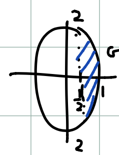
Conclusion : The area $= 4ab\cdot\frac{\pi}{4} = \underline{\pi ab}$.

The substitution $t = a\cos\theta$ (or $t = a\sin\theta$)

is suitable

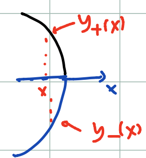
for the form $\sqrt{1-\frac{t^2}{a^2}}$

EX: (exercise)



← the area inside $x^2 + \frac{y^2}{4} = 1$ & $\frac{1}{2} \le x \le 1$.

$$\int_{\frac{1}{2}}^{1} (y_+(x) - y_-(x))\,dx = 2\int_{\frac{1}{2}}^{1} y_+(x)\,dx$$



$x^2 + \frac{y^2}{4} = 1$

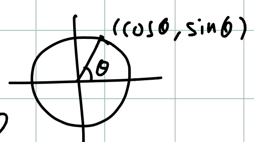
$y^2 = 4\sqrt{1-x^2}$

$y_+ = 2\sqrt{1-x^2}$

$$= 2\int_{\frac{1}{2}}^{1} 2\sqrt{1-x^2}\,dx$$

$$= 4\int_{\frac{1}{2}}^{1} \sqrt{1-x^2}\,dx \qquad x = \cos\theta,\ dx = -\sin\theta\,d\theta$$

$$= 4\int_{x=\frac{1}{2}}^{x=1} \sqrt{\sin^2\theta}\cdot(-\sin\theta)\,d\theta = -4\int_{x=\frac{1}{2}}^{x=1} |\sin\theta|\sin\theta\,d\theta$$

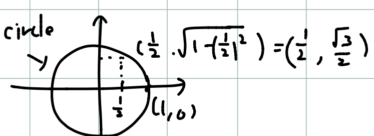


$$= -4\int_{\theta=\cos^{-1}(\frac{1}{2})}^{\cos^{-1}(1)} \sin^2\theta\,d\theta$$

← $\sin\theta \ge 0$ for $\frac{1}{2} \le x = \cos\theta \le 1$

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

unit circle

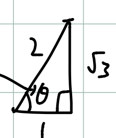


$$= -4\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_{\cos^{-1}(\frac{1}{2})}^{\cos^{-1}(1)}$$

$\sin 2\theta = 2\cos\theta\sin\theta$

$\theta = \cos^{-1}\frac{1}{2}$

$= \frac{\pi}{3}$

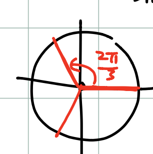


$$= -4\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_{\frac{\pi}{3}}^{0}$$

$$= -4\left[(0-0) - \left(\frac{\pi}{6} - \frac{\sin\frac{2\pi}{3}}{4}\right)\right]$$

$$= 4\left[\frac{\pi}{6} - \frac{1}{4}\cdot\frac{\sqrt{3}}{2}\right]$$

$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$

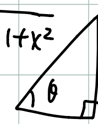


$$= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$

//

e.g. $\int \frac{1}{x^2+1}\,dx$.

$\sec\theta = \sqrt{1+x^2}$



$x = \tan\theta$

1

$\cos^2\theta + \sin^2\theta = 1$

(sol). $x = \tan\theta$ $\quad x^2+1 = \tan^2\theta + 1 = \sec^2\theta$ $\quad 1 + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$

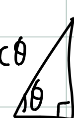
$dx = \sec^2\theta\, d\theta$

$$\int \frac{1}{x^2+1}\,dx = \int \frac{1}{\sec^2\theta}\cdot \sec^2\theta\, d\theta = \int d\theta = \theta + C = \tan^{-1}x + C$$

Ex. $\int \sqrt{1+x^2}\,dx$

(sol). $x = \tan\theta$ $\quad dx = \sec^2\theta\, d\theta$,

$\sec\theta$

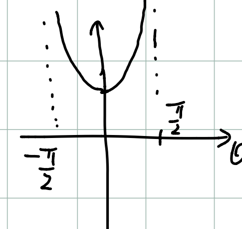


$x = \tan\theta$

1

$1+x^2 = \sec^2\theta$

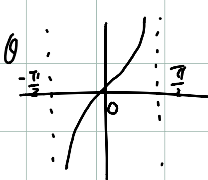
$\sec\theta$



$$\therefore \int \sqrt{1+x^2}\,dx = \int \sqrt{\sec^2\theta}\cdot \sec^2\theta\, d\theta$$

$$= \int |\sec\theta|\, \sec^2\theta\, d\theta$$

$\tan\theta$



$$= \int \sec^3\theta\, d\theta$$

$\sec\theta > 0$
for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

For $x = \tan\theta$,
can restrict $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
since then $\tan\theta$
can cover ALL real numbers.

. To compute $\int \sec^3 d\theta$ (consult WHW 3. Pro. 2.B. in class)

Can use integration by parts.

. Another method: substitution & partial fractions

$$\int \sec^3\theta\, d\theta = \int \frac{d\theta}{\cos^3\theta} = \int \frac{\cos\theta}{\cos^4\theta}\, d\theta = \int \frac{\cos\theta\, d\theta}{(1-\sin^2\theta)^2}$$

$$= \int \frac{du}{(1-u^2)^2} \quad \longleftarrow \text{ here } u = \sin\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$\therefore\ -1 < u < 1.$$

$$= \int \frac{du}{(1-u)^2(1+u)^2} = \int \left[\frac{A}{1-u} + \frac{B}{(1-u)^2} + \frac{C}{1+u} + \frac{D}{(1+u)^2} \right] du$$

Here,

$$B = \lim_{u\to 1} \frac{1}{(1-u)^2(1+u)^2}\cdot(1-u)^2 = \frac{1}{(1+u)^2}\Big|_{u=1} = \frac{1}{2^2} = \frac{1}{4}$$

$$-A = \lim_{u\to 1}\left[\frac{1}{(1-u)^2(1+u)^2}(1-u)^2\right]' = \left(\frac{1}{(1+u)^2}\right)'\Big|_{u=1} = -2\frac{1}{(1+u)^3}\Big|_{u=1}$$

$$= -\frac{1}{4}.$$

Can you see why?

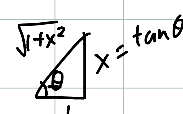
$$D = \frac{1}{(1-u)^2}\Big|_{u=-1} = \frac{1}{4}$$

$$C = \left(\frac{1}{(1-u)^2}\right)'\Big|_{u=-1} = 2\frac{1}{(1-u)^3}\Big|_{u=-1} = \frac{1}{4}$$

$$= \frac{1}{4}\ln|1-u| + \frac{1}{4}\cdot\left(\frac{1}{1-u}\right) + \frac{1}{4}\ln|1+u| + \frac{1}{4}\cdot\left(\frac{-1}{1+u}\right) + C$$

$$= \frac{1}{4}\ln|1-u^2| + \frac{1}{4}\frac{2u}{1-u^2} + C \qquad x = \tan\theta, \quad u = \sin\theta$$

$$= \frac{1}{4}\ln(\cos^2\theta) + \frac{1}{2}\cdot\frac{\sin\theta}{\cos^2\theta} + C$$



$$= \frac{1}{4}\ln(1+x^2) + \frac{1}{2}\frac{x}{(1+x^2)^{\frac{3}{2}}} + C.$$

$\boxed{//}$

Can try

- For $\sqrt{a^2+x^2}$, $\frac{1}{a^2+x^2}$ $\quad x = \tan\theta \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$