# Lec12

- Integration of Rational Functions. §6.2.
  : Partial fractions.

$\int \frac{1}{x} dx = \ln|x| + C$

$\int \frac{1}{x+a} dx = \ln|x+a| + C$

**Ex** $\int \frac{x^2+1}{x^2+x} dx$

$= \int \frac{x^2+x+1-x}{x^2+x} dx$

$= \int \left[ 1 + \frac{1-x}{x^2+x} \right] dx$

Reduction of denominator.

$$\frac{P(x)}{Q(x)} = \text{polynomials} + \frac{P_1(x)}{Q(x)}$$

$P(x) = a_0 + a_1 x + \cdots + a_k x^k$ ← degree $k$

$Q(x) = b_0 + b_1 x + \cdots + b_\ell x^\ell$ ← degree $\ell$.

$\deg P_1(x) < \deg Q(x)$

To compute

$\int \frac{1-x}{x^2+x} dx$

$$\frac{1-x}{x^2+x} = \frac{1-x}{x(x+1)} = \frac{1}{x} + \frac{-2}{x+1}$$

$= \int \frac{1}{x} dx - 2\int \frac{1}{x+1} dx$

$= \ln|x| - 2\ln|x+1| + C.$

$\therefore \int \frac{x^2+1}{x^2+x} dx = \int 1\, dx + \int \frac{1-x}{x^2+x} dx$

$= x + \ln|x| - 2\ln|x+1| + C.$

Lesson: It will be good if

we can express $\frac{P_1(x)}{Q(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \cdots + \frac{A_\ell}{x-a_\ell}$.

"partial fractions"

since $\int \frac{1}{x-a_i} dx = \ln|x-a_i| + C.$

e.g. $\dfrac{x^2+1}{x(2x^2+1)^2} = \dfrac{2}{x} - \dfrac{4x}{2x^2+1} - \dfrac{3x}{(2x^2+1)^2}$.

<u>Fact</u> (Thm 1, §6.2) <u>Partial Fractions</u>

$P_1(x)$, $Q(x)$ polynomial $\deg P_1 < \deg Q$

and $Q(x)$

$$= (x-a_1)^{m_1}(x-a_2)^{m_2}\cdots(x-a_k)^{m_k}\left[(x+b_1)^2+c_1^2\right]^{n_1}\times\cdots\times\left[(x+b_l)^2+c_l^2\right]^{n_l}$$

e.g. $Q(x) = x^3(x-1)^2(x-2)(x^2+1)^3((x+2)^2+1)^4$

Then, we can express

$$\frac{P_1(x)}{Q(x)} = \frac{A_{11}}{(x-a_1)} + \frac{A_{12}}{(x-a_1)^2} + \cdots + \frac{A_{1m_1}}{(x-a_1)^{m_1}}$$

$$+ \cdots$$

$$+ \frac{B_{11}x + C_{11}}{(x+b_1)^2+c_1^2} + \cdots + \frac{B_{1n_1}x + C_{1n_1}}{\left[(x+b_1)^2+c_1^2\right]^{n_1}}$$

$$+ \cdots$$

With distinct zeros (distinct factors in the denominator).

e.g. $\dfrac{x^2+1}{(x-1)(x-2)(x-3)} = \dfrac{A}{x-1} + \dfrac{B}{x-2} + \dfrac{C}{x-3}$ from the previous fact

How do we find $A, B, C$ ?

Let $g(x) = \dfrac{x^2+1}{(x-1)(x-2)(x-3)}$

$$g(x)\cdot(x-1) = A + \underbrace{(x-1)\left[\frac{B}{x-2} + \frac{C}{x-3}\right]}_{\to 0 \text{ as } x \to 1}$$

$\therefore \lim_{x\to 1} g(x)(x-1) = A.$

$$\frac{1^2+1}{(1-2)(1-3)} = A. \qquad \therefore A = 1$$

Similarly, $\lim_{x\to 2} g(x)(x-2) = B$

$$= \frac{2^2+1}{(2-1)(2-3)} = B \qquad B = -5$$

$\lim_{x\to 3} g(x)(x-3) = C$

$$\frac{3^2+1}{(3-1)(3-2)} = C. \qquad C = 5.$$

$$\therefore \int \frac{x^2+1}{(x-1)(x-2)(x-3)}\,dx = \int\left[\frac{1}{(x-1)} + \frac{-5}{(x-2)} + \frac{5}{(x-3)}\right]dx$$

$$= \ln|x-1| - 5\ln|x-2| + 5\ln|x-3| + C.$$

With multiple zeros (repeated factors)

e.g.

$$g(x) = \frac{1}{(x-1)^2(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+1)}$$

$$C = \lim_{x \to -1} g(x)(x+1) = \frac{1}{(-1-1)^2} = \underline{\frac{1}{4}.}$$

$$g(x)(x-1)^2 = A(x-1) + B + \frac{C}{(x+1)}(x-1)^2$$

$$\therefore B = \lim_{x \to 1} g(x)(x-1)^2$$

$$= \frac{1}{1+1} = \underline{\frac{1}{2}.}$$

How to find A?

Note $\quad g(x)(x-1)^2 = A(x-1) + B + \frac{C}{(x+1)}(x-1)^2$

$$\frac{d}{dx}\left[g(x)(x-1)^2\right] = A + \underbrace{\left[\frac{C}{x+1}(x-1)^2\right]'}_{\to 0 \text{ as } x \to 1}$$

let $x \to 1$

$$\lim_{x \to 1}\left[g(x)(x-1)^2\right]' = A$$

since after derivative the factor $(x-1)$ still remains.

$$\lim_{x \to 1}\left(\frac{1}{x+1}\right)' = A$$

$$\lim_{x \to 1}\left(-\frac{1}{(x+1)^2}\right) = A \quad \therefore \underline{A = -\frac{1}{4}.}$$

TM