Lec 11.
- integration by parts §6.1
- Integrals of rational functions §6.2
  partial fractions.

Integration by Parts §6.1

Practice!

Note $\int \frac{dU}{dx} dx = U + C$

using infinitesimal notation : $\int dU = U + C$.

$$\boxed{dU = \frac{dU}{dx} dx.}$$

$$\frac{d}{dx}\left(U(x)V(x)\right) = U(x)\frac{dV}{dx} + V(x)\frac{dU}{dx}$$

using "infinitesimal" notation

$$d(UV) = UdV + VdU$$

$$\therefore \int d(UV) = \int U\,dV + \int V\,dU$$

$$\boxed{UV = \int U\,dV + \int V\,dU}$$

additive constant C is absorbed in these indefinite integrals (which are determined up to additive constants).

$$\therefore \boxed{\int U\,dV = UV - \int V\,dU.}$$

$$\boxed{\int U(x)\,V'(x)\,dx = U(x)V(x) - \int V(x)\,U'(x)\,dx}$$

$$\boxed{\int_a^b U(x)\,V'(x)\,dx = \Big[U(x)\,V(x)\Big]_a^b - \int_a^b V(x)\,U'(x)\,dx.}$$

When computing $\int f(x)dx$, if difficult, we can try to represent $\int f(x)dx$ as $\int U(x)\,V'(x)\,dx$ by finding U & V, in such a way that $\int V(x)\,U'(x)\,dx$ is easier to compute.

Try to find · U with simpler U'
· V' with easy V.

Ex. $\int x e^x \, dx$

<sol>. $U = x$, $V' = e^x$

$U' = 1$ $\quad V = e^x$

$$\int x e^x dx = \int U V' = UV - \int U'V$$

$$= x e^x - \int 1 \cdot e^x \, dx$$

$$= x e^x - e^x + C. \ \boxed{/}$$

Ex. $\int \ln x \, dx$.

<sol>. $U = \ln x$, $V' = 1$

$U' = \frac{1}{x}$ $\quad V = x$

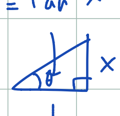
$$\int \underset{1 = V' \ \ U}{\ln x \, dx} = \underset{V \ \ U}{x \ln x} - \int \underset{U' \ , \ V}{\frac{1}{x} \cdot x} \, dx = \underline{x \ln x - x + C}.$$

Ex a. $\int \tan^{-1} x \, dx$

b. $\int_1^{\sqrt{3}} \tan^{-1} x \, dx$

<sol>. Notice $(\tan^{-1} x)' = \frac{1}{1+x^2}$

$\theta = \tan^{-1} x$



$\tan \theta = x$

differentiate in x $\quad \frac{d \tan \theta}{d\theta} \cdot \frac{d\theta}{dx} = 1.$

$$\tan' \theta = \left(\frac{\sin \theta}{\cos \theta} \right)'$$

$$= \frac{\cos^2 \theta + \sin^2 \theta}{\cos^2 \theta}$$

$$= \frac{1}{\cos^2 \theta}$$

$$\therefore \frac{d\theta}{dx} = \frac{1}{\tan'\theta} = \cos^2\theta.$$

$$\therefore \frac{d\tan^{-1}x}{dx} = \frac{1}{1+x^2}$$

a.)

Now, $\int \tan^{-1}x \, dx = x\tan^{-1}x - \int x \frac{1}{1+x^2}\,dx$

$V' = 1 \quad U$

$V = x \quad U' = \frac{1}{1+x^2}$

$$= x\tan^{-1}x - \int \frac{1}{u}\frac{du}{2}$$

$u = 1+x^2$

$du = 2x\,dx$

$$= x\tan^{-1}x - \frac{1}{2}\ln|u| + C$$

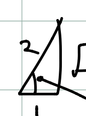
$$\underline{= x\tan^{-1}x - \frac{1}{2}\ln|1+x^2| + C}$$

▨

b). $\int_1^{\sqrt{3}} \tan^{-1}x \, dx = \left[x\tan^{-1}x - \frac{1}{2}\ln|1+x^2|\right]_1^{\sqrt{3}}$

$$= \sqrt{3}\cdot\frac{\pi}{3} - \frac{1}{2}\ln(1+\sqrt{3}^2)$$

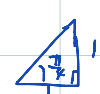
$$- 1\cdot\frac{\pi}{4} + \frac{1}{2}\ln(1+1^2)$$



$\theta = \frac{\pi}{3}$

$\tan^{-1}\sqrt{3} = \frac{\pi}{3}.$

$$= \left(\frac{\sqrt{3}}{3} - \frac{1}{4}\right)\pi - \frac{1}{2}\ln 4 + \frac{1}{2}\ln 2$$



$\tan^{-1}1 = \frac{\pi}{4}$

$$\underline{= \left(\frac{\sqrt{3}}{3} - \frac{1}{4}\right)\pi - \frac{1}{2}\ln 2.}$$

▨

Ex $\int_0^{\pi} e^x \cos x \, dx$

<Sol>. Note $e^x \xrightarrow{\frac{d}{dx}} e^x \xrightarrow{\frac{d}{dx}} e^x$

$\cos x \xleftarrow{\int dx} -\sin x \xleftarrow{\int dx} -\cos x$

The functions do not get simpler, but cyclical!

Let $U = \cos x$, $V' = e^x$, $V = e^x$.

$$\int_0^{\pi} e^x \cos x \, dx = \int_0^{\pi} V' \cdot U \, dx$$

$$= V \cdot U \Big]_0^{\pi} - \int_0^{\pi} V U' dx$$

$$= e^x \cos x \Big]_0^{\pi} - \int_0^{\pi} e^x \cdot (-\sin x) \, dx$$

$$= e^x \cos x \Big]_0^{\pi} + \int_0^{\pi} e^x \sin x \, dx$$

Integration by parts $F' = e^x$, $G = \sin x$

$$= \left[e^x \cos x \right]_0^{\pi} + \left[e^x \sin x \right]_0^{\pi} - \int_0^{\pi} e^x \cos x \, dx$$

$\therefore$ $$2 \int_0^{\pi} e^x \cos x \, dx = \left[e^x \cos x \right]_0^{\pi} + \left[e^x \sin x \right]_0^{\pi}$$

$$= \left[e^x (\cos x + \sin x) \right]_0^{\pi}$$

$$= e^{\pi} (\cos \pi + \cancel{\sin \pi}^{0}) - e^0 (\cos 0 + \cancel{\sin 0}^{0})$$

$$= -e^{\pi} - e^0 = -e^{\pi} - 1.$$

$\therefore$ $$\int_0^{\pi} e^x \cos x \, dx = \underline{\frac{1}{2} (-e^{\pi} - 1)}.$$ ▨

Ex

- $\int x e^x \, dx = x e^x - \int e^x \, dx = x e^x - e^x + C$
- $\int x^2 e^x \, dx = x^2 e^x - \int 2x \, e^x \, dx$
- $\int x^3 e^x \, dx = x^3 e^x - \int 3x^2 e^x \, dx$

So, to compute

$$I_n = \int x^n e^x \, dx$$

can use $\quad I_n = x^n e^x - n \int x^{n-1} e^x \, dx$

$$= x^n e^x - n I_{n-1}$$

Reduction formula: $\boxed{\text{“} I_n = \text{something with } I_{n-1} \text{”}}$

To compute I_n, compute I_{n-1} ← simpler

To compute I_{n-1}, compute I_{n-2} ← simpler

$\quad\vdots \qquad \vdots$

$$\begin{aligned}
\int x^4 e^x \, dx = I_4 &= x^4 e^x - 4 I_3 \\
&= x^4 e^x - 4(x^3 e^x - 3 I_2) \\
&= x^4 e^x - 4x^3 e^x + 12(x^2 e^x - 2 I_1) \\
&= \underline{x^4 e^x - 4x^3 e^x + 12 x^2 e^x + 12 x e^x - 12 e^x + C}
\end{aligned}$$

Ex $\int \frac{dx}{(x^2+1)^3} = ?$

<sol>. Notice $\frac{1}{(x^2+1)^3} = \frac{1+x^2-x^2}{(x^2+1)^3} = \frac{1}{(x^2+1)^2} - \frac{x^2}{(x^2+1)^3}$

lower degree

And,

$$\int \frac{x^2}{(x^2+1)^3}dx = \int x \cdot \boxed{\frac{x}{(x^2+1)^3}} dx$$

can apply substitution

$u = x^2+1$

$$\int \frac{x}{(x^2+1)^3}dx = \int \frac{\frac{1}{2}du}{u^3}$$

$$= \frac{1}{-4}u^{-2} + C.$$

$$= -\frac{1}{4}\frac{1}{(x^2+1)^2} + C$$

by parts

$$= x \cdot \left(-\frac{1}{4}\frac{1}{(x^2+1)^2}\right)$$

$$- \int 1 \cdot \left[-\frac{1}{4}\frac{1}{(x^2+1)^2}\right] dx.$$

lower degree

So we see

if we let $I_n = \int \frac{dx}{(x^2+1)^n}$.

then we can get an equation.

A reduction formula. " I_n = something with I_{n-1} "

To compute I_3, can compute I_2 ← simpler

To compute I_2, can compute I_1 ← simpler

$$I_1 = \int \frac{1}{1+x^2}dx$$

$$= \tan^{-1}x + C.$$

See $\begin{cases} \text{page } 336-337 \\ \text{exercise } 31 \sim 35, \ \S 6.1. \end{cases}$

e.g. Note x^2+1 cannot be factored out in real numbers.

($x^2+1=0$ does <u>not</u> have real solutions)

<u>Note</u> but it cannot be factored out in complex numbers.