Lec 10.

- Substitution §5.6.
- Areas between two curves §5.7.
- Integration by parts §6.1

Substitution

$$u = g(x), \quad du = g'(x)\,dx$$

$$\int_{g(a)}^{g(b)} f(u)\,du = \int_a^b f(g(x))\,g'(x)\,dx$$

for Rieman integrable f & g'.

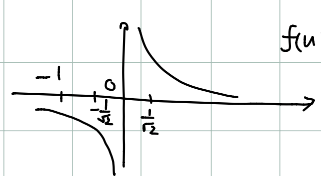
Ex $\displaystyle\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$ $\qquad \cos x = u$, $\; du = -\sin x\,dx$

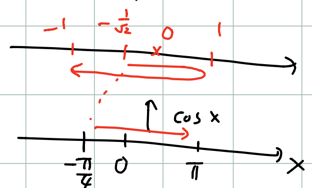
$$= \int \frac{-du}{u} = -\ln|u| + C = -\ln|\cos x| + C$$

Be CAREFUL when you compute definite integrals (using substitution)

- $\displaystyle\int_{-\frac{\pi}{4}}^{\pi} \tan x\,dx \;\cancel{=}\; \Big[-\ln|\cos x|\Big]_{-\frac{\pi}{4}}^{\pi} = \cancel{-\ln|\cos\pi|}^{\,0} + \ln\left|\cos\left(\tfrac{\pi}{4}\right)\right| = \ln\frac{1}{\sqrt{2}}$

wrong.



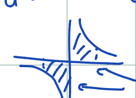


range of $u = \cos x$ for $-\frac{\pi}{4} \le x \le \pi$,

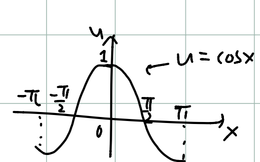
is $-1 \le u \le 1$.

$\frac{1}{u}$ is not integrable on $[-1, 1]$.

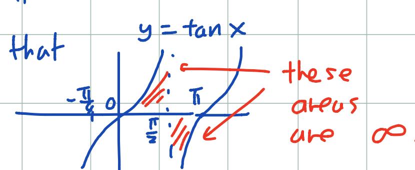
In fact $\displaystyle\int_{-1}^{0} \frac{1}{u}\,du = -\infty$, $\displaystyle\int_0^1 \frac{1}{u}\,du = +\infty$



both infinite area.



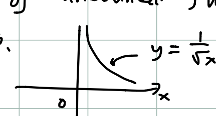
Or note that



Rmk In §6.5 "Improper integrals"

we will learn more about integrals of unbounded functions

e.g.



Substitution technique requires practice! Do MANY exercises.

Ex $\int \frac{1}{\cos x}\,dx$ ← Anti derivative of $\frac{1}{\cos x}$

$$= \int \frac{\cos x}{\cos x \cos x}\,dx$$

$$= \int \frac{\cos x}{\cos^2 x}\,dx$$

← $\cos^2 x = 1 - \sin^2 x$

$$= \int \frac{\cos x}{1-\sin^2 x}\,dx$$

$u = \sin x$ $\quad du = \cos x\,dx$

$$= \int \frac{du}{1-u^2}$$

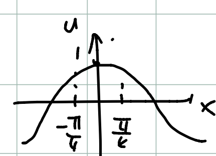
$$= \int \frac{1}{2}\left(\frac{1}{1-u} + \frac{1}{1+u}\right)du$$

$$= \frac{1}{2}\left[\int \frac{1}{1-u}\,du + \int \frac{1}{1+u}\,du\right]$$

- $\left(\ln|1+u|\right)' = \frac{1}{1+u}$
- $\left(\ln|1-u|\right)' = -\frac{1}{1-u}$.

$$= \frac{1}{2}\left(-\ln|1-u| + \ln|1+u|\right) + C$$

$$= \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right| + C. = \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + C.$$



$u = \cos x$ $\quad -\frac{\pi}{4} \le x \le \frac{\pi}{4}$

e.g. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{\cos x}\,dx$

$\frac{1}{u}$ is continuous

on $\cos(\frac{\pi}{4}) \le u \le 1$.

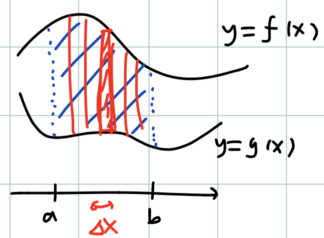
$$= \frac{1}{2}\ln\frac{1+\sin x}{1-\sin x}\Bigg]_{x=-\frac{\pi}{4}}^{x=\frac{\pi}{4}}$$

← can apply substitution

$$= \frac{1}{2}\left(\ln\frac{1+\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}} - \ln\frac{1-\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}}\right) = \ln\frac{1+\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}} = \underline{\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)} \quad \blacksquare$$

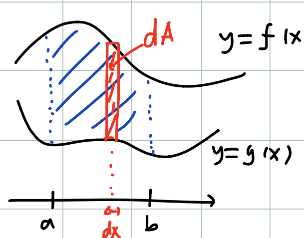
§ 5.7 Area between two curves.

Area:



$$\text{Area} \approx \sum_{i=1}^{N} \Delta A_i$$

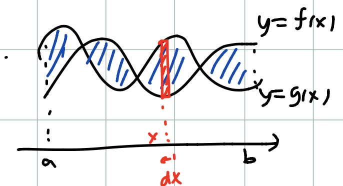
$$\approx \sum_{i=1}^{N} |f(x_i) - g(x_i)|\, \Delta x_i$$



⊂ "infinitesimal width".

"area element" ← not a quantity but an abstract notion.

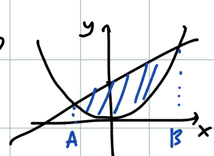
$dA = |f(x) - g(x)|\, dx$



$$\boxed{\text{Area} = \int_{x=a}^{x=b} dA = \int_a^b |f(x) - g(x)|\, dx.}$$

EX Area of the region bounded by $y = x^2$, $y = x + 2$

<sol>



$x^2 = x + 2 \implies x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

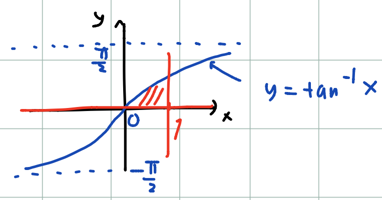
$x = -1$, $x = 2$
$\uparrow$ A $\quad \uparrow$ B

$$\therefore \text{area} = \int_{-1}^{2} (\underbrace{x+2 - x^2}_{>0 \text{ on } [-1,2]})\, dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-1}^{2} = 2 + 4 - \frac{8}{3} - \frac{1}{2} + 2 - \frac{1}{3}$$

$$= 4 + \frac{1}{2}.$$

<u>Ex</u>. Area of the region bounded by $y = \tan^{-1} x$, $x = 1$, $y = 0$.

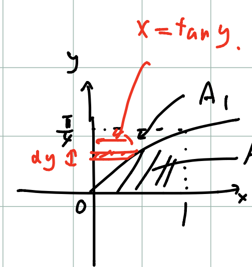
<sol>



$$\int_0^1 \tan^{-1} x \, dx = ?$$

An alternative method.

different point of view :



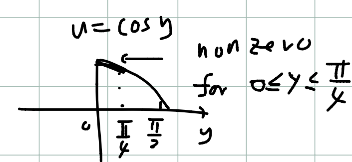
// Sometimes, it is useful to view the curve as the graph of the function $x = g(y)$!

$y = \tan^{-1} x$
$x = \tan y$.

$$A = \frac{\pi}{4} - A_1$$

$$= \frac{\pi}{4} - \int_0^{\frac{\pi}{4}} \tan y \; dy$$

$$= \frac{\pi}{4} - \int_0^{\frac{\pi}{4}} \frac{\sin y}{\cos y} dy$$



$$= \frac{\pi}{4} - \int_{y=0}^{y=\frac{\pi}{4}} \frac{-du}{u}$$

$u = \cos y$
$du = -\sin y \, dy$

$$= \frac{\pi}{4} + \int_{y=0}^{y=\frac{\pi}{4}} \frac{du}{u}$$

<u>Note</u> $\frac{1}{u}$ is continuous for $u = \cos y$ while $0 \le y \le \frac{\pi}{4}$

$$= \frac{\pi}{4} + \Big[\ln|u| \Big]_{y=0}^{y=\frac{\pi}{4}}$$

$$= \frac{\pi}{4} + \Big[\ln|\cos y| \Big]_{y=0}^{y=\frac{\pi}{4}}$$

$$= \frac{\pi}{4} + \underbrace{\ln \cos\frac{\pi}{4}}_{<0} - \cancel{\ln \cos 0}^{\,0} = \frac{\pi}{4} + \ln\left(\cos\frac{\pi}{4}\right) = \frac{\pi}{4} + \ln\frac{1}{\sqrt{2}}$$

$\cos 0 = 1$

$$= \underline{\frac{\pi}{4} - \frac{1}{2}\ln 2} \quad \blacksquare$$