Lec 2

- More on Σ. §5.1
- Areas §5.2

- $\sum_{j=1}^{n} 1 = \underbrace{1+1+\cdots+1}_{n \text{ terms}} = n.$

- $\sum_{j=1}^{n} j = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$

pf
$$\begin{aligned} & 1+2+3+\cdots+n-2+n-1+n \\ + & \; n+n-1+n-2+\cdots+3+2+1 \end{aligned} = 2\sum_{j=1}^{n} j \qquad \therefore \; 2\sum_{j=1}^{n} j = n\cdot(n+1) \quad \boxed{/\!/}$$

$$= \underbrace{(n+1)+(n+1)+\cdots+(n+1)+(n+1)+(n+1)}_{n\text{-terms}}$$

- $\sum_{j=0}^{n-1} r^j = 1+r+r^2+\cdots+r^{n-1} = \frac{1-r^n}{1-r}$ for $r \neq 1$.

pf Let $S = \sum_{j=0}^{n-1} r^j$.

$$\begin{aligned} S &= 1+r+r^2+\cdots+r^{n-1} \\ -\; rS &= r+r^2+\cdots+r^n \end{aligned} \quad \swarrow \text{Cancellations!}$$

$$S - rS = 1 - r^n$$

$\therefore$ For $r \neq 1$, $(1-r)S = 1-r^n \Rightarrow S = \frac{1-r^n}{1-r}.$ $\boxed{/\!/}$

- $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$

pf $(k+1)^3 - k^3 = 3k^2 + 3k + 1$

Thus. $\sum_{k=1}^{n}\left[(k+1)^3 - k^3\right] = 3\underbrace{\sum_{k=1}^{n} k^2}_{\text{What we want to compute.}} + 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$

$$\begin{aligned} \text{L.H.S} = & \; \cancel{2^3} - 1^3 \\ & + \cancel{3^3} - \cancel{2^3} \\ & + \cancel{4^3} - \cancel{3^3} \\ & \vdots \\ & + (n+1)^3 - \cancel{n^3} \end{aligned}$$

"telescoping sum".

$\therefore$ L.H.S $= (n+1)^3 - 1$

But R.H.S $= 3\sum_{k=1}^{n} k^2 + 3\cdot\frac{n(n+1)}{2} + n$

$$\therefore \sum_{k=1}^{n} k^2 = \frac{1}{3}\left[(n+1)^3 - 1 - 3\frac{n(n+1)}{2} - n\right]$$

$$= \frac{1}{3}\left[(n+1)^3 - (n+1) - 3\frac{n(n+1)}{2}\right]$$

$$= \frac{(n+1)}{3}\left[(n+1)^2 - 1^2 - \frac{3n}{2}\right]$$

$$= \frac{n+1}{3}\left[(n+1+1)(n+\cancel{1}-\cancel{1}) - \frac{3n}{2}\right] = \frac{n+1}{3}\cdot n\cdot\left[n+2-\frac{3}{2}\right] = \frac{(n+1)n\cdot(2n+1)}{6}. \quad \boxed{/\!/}$$

Ex $\displaystyle\sum_{k=1}^{100} \frac{1}{k(k+1)} = \sum_{k=1}^{100}\left[\frac{1}{k} - \frac{1}{k+1}\right] = 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \frac{1}{4} + \cdots + \underset{k=99}{\cancel{\frac{1}{99}} - \cancel{\frac{1}{100}}} + \underset{k=100}{\cancel{\frac{1}{100}} - \frac{1}{101}}$

$$= 1 - \frac{1}{101} = \frac{100}{101}. \quad \blacksquare$$

- $\displaystyle\sum_{k=1}^{N} k^3 = \left[\frac{N(N+1)}{2}\right]^2$: can be shown using $(k+1)^4 - k^4$.

: Exercise.

Definite integral as area
& Area as limits of Sums.

Read §5.2

Definite integral as area.

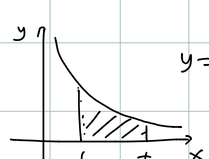
Let f be continuous & >0 on $[a,b]$

the area $= \displaystyle\int_a^b f(x)\,dx$

For the moment this is just the notation of the area.

Area as limits of sums

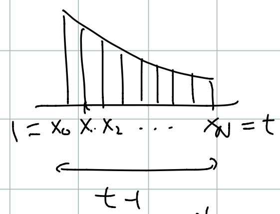
e.g.

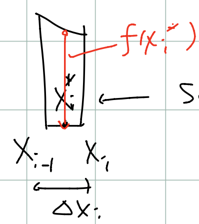


$y = f(x) = \frac{1}{x}$.

using the integral notation

the area $= \displaystyle\int_1^t \frac{1}{x}\,dx$.





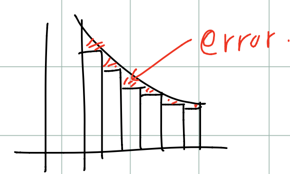
sample point

$x_i^* \in [x_{i-1}, x_i]$

$$\text{area} \underset{\text{approx}}{\approx} \sum_{i=1}^{N} f(x_i^*)\Delta x_i$$

Suppose. $\Delta x_i = \dfrac{t-1}{N}$.

$x_i^* = 1 + i\,\dfrac{t-1}{N}$.



Then, $\displaystyle \text{area} \approx \sum_{i=1}^{N} \left(\frac{1}{1 + i\frac{(t-1)}{N}} \right) \frac{t-1}{N}$

The error of approx. $\longrightarrow 0$ as $N \to \infty$. (Why? We will give rigorous explanation later.)

$$\therefore \text{area} = \lim_{N\to\infty} \sum_{i=1}^{N} \frac{1}{1 + i\frac{(t-1)}{N}} \frac{t-1}{N}$$

So, $\displaystyle \int_1^t \frac{1}{x}\,dx = \lim_{N\to\infty} \sum_{i=1}^{N} \frac{1}{1 + i\left(\frac{t-1}{N}\right)} \frac{t-1}{N}.$

Here, observe the correspondence of the notation

$$\int_1^t \longleftrightarrow \lim_{N\to\infty} \sum_{i=1}^{N}$$

$$x \longleftrightarrow 1 + i\left(\frac{t-1}{N}\right)$$

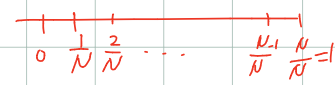
$$dx \longleftrightarrow \frac{t-1}{N}.$$

e.g. $\displaystyle \lim_{N\to\infty} \sum_{k=1}^{N} \frac{1}{N^2}\sqrt{N^2 - k^2} = ?$

(sol) $\displaystyle = \lim_{N\to\infty} \sum_{k=1}^{N} \sqrt{\left(1 - \left(\frac{k}{N}\right)^2\right) N^2}\ \frac{1}{N^2}$

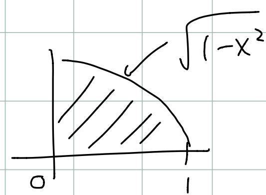
$$\frac{N}{N^2}$$

$$= \lim_{N\to\infty} \sum_{k=1}^{N} \underbrace{\sqrt{1-\left(\frac{k}{N}\right)^2}}_{f\left(\frac{k}{N}\right)} \cdot \underbrace{\frac{1}{N}}_{\Delta x_k}$$



$$f(x) = \sqrt{1-x^2}$$

= area of

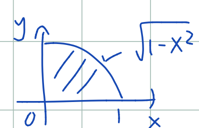


$$= \int_0^1 \sqrt{1-x^2}\,dx.$$

$$= \frac{\pi}{4}$$

Here, we had.

$$\lim_{N\to\infty} \sum_{k=1}^{N} \sqrt{1-\left(\frac{k}{N}\right)^2}\,\frac{1}{N} = \int_0^1 \sqrt{1-x^2}\,dx$$



Observe the correspondence of the notation

$$\lim_{N\to\infty} \sum_{k=1}^{N} \longrightarrow \int_0^1$$

$$\frac{k}{N} \longrightarrow x$$

$$\frac{1}{N} \longrightarrow dx$$

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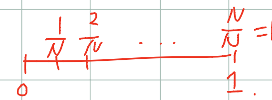
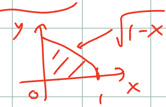
e.g.

$$\lim_{N\to\infty} \sum_{k=1}^{N} \frac{1}{N}\sqrt{N-k} = \lim_{N\to\infty} \sum_{k=1}^{N} \sqrt{1-\left(\frac{k}{N}\right)}\;\frac{\sqrt{N}}{N}$$

$$= \lim_{N\to\infty} \sum_{k=1}^{N} \sqrt{1-\left(\frac{k}{N}\right)} \cdot \frac{1}{N} \cdot \sqrt{N}$$

$$= \lim_{N\to\infty} \underbrace{\sqrt{N}}_{\downarrow \text{ as } N\to\infty,\ \infty} \cdot \underbrace{\sum_{k=1}^{N} \underbrace{\sqrt{1-\frac{k}{N}}}_{f(x)=\sqrt{1-x}} \cdot \underbrace{\frac{1}{N}}_{\Delta x}}_{\to \text{ area}}$$





$$= +\infty .$$