HOLOMORPHIC EXTENSIONS OF LAPLACIANS AND THEIR DETERMINANTS

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Abstract. The Teichmüller space $Teich(S)$ of a surface S in genus $g > 1$ is a totally real submanifold of the quasifuchsian space $QF(S)$. We show that the determinant of the Laplacian $\det'(\Delta)$ on $Teich(S)$ has a unique holomorphic extension to $QF(S)$. To realize this holomorphic extension as the determinant of differential operators on S, we introduce a holomorphic family $\{\Delta_{\mu,\nu}\}$ of elliptic second order differential operators on S whose parameter space is the space of pairs of Beltrami differentials on S and which naturally extends the Laplace operators of hyperbolic metrics on S. We study the determinant of this family $\{\Delta_{\mu,\nu}\}$ and show how this family realizes the holomorphic extension of $\det'(\Delta)$ as its determinant.

1. Introduction

In this paper, we discuss determinants of Laplacians of Riemann surfaces and their holomorphic extensions.

Given a closed Riemannian manifold X with metric m, its corresponding Laplacian Δ is a self-adjoint positive definite elliptic second order differential operator on functions on X, which has discrete spectrum

$$\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \to \infty.$$

The determinant of the operator Δ may be defined formally as the product of the nonzero eigenvalues of Δ. A regularization $\det'(\Delta)$ of this product was defined by Ray and Singer [RS1] [RS2], using the zeta function of Δ.

This determinant $\det'(\Delta)$ has appeared to be very important in mathematics. For example, in [OPS1], (see also [Sa2]), Osgood, Phillips and Sarnak studied $-\log\det'(\Delta)$ as a "height" function on the space of metrics on a compact orientable smooth surface S of genus g. For $g > 1$, they showed that when restricted to a given conformal class of metrics on S, it attains its minimum at the unique hyperbolic metric in this conformal class, and has no other critical points. Thus, to find Riemannian metrics on S which are extremal, in the sense that they minimize $-\log\det'(\Delta)$, it suffices to consider its restriction to the moduli space $\mathcal{M}_g$ of hyperbolic metrics on a Riemann surface S of genus g. It was shown by Wolpert that this restriction is a proper function (see [W4]), which was used also by Osgood, Phillips and Sarnak to show that the isospectral sets (with respect to the Laplacian) of isometry classes of metrics on S are all compact in the C^∞ topology (see [OPS2]).

Date: February 27, 2006.

2000 *Mathematics Subject Classification.* 32G15; 58J52.

This research was partially supported by the NSF under grant DMS-0072508.

The universal cover of the orbifold $\mathcal{M}_g$, with covering group the mapping class group Γ_g, is the Teichmüller space *Teich*(S). The function $-\log\det'(\Delta)$ lifts to a function on the Teichmüller space *Teich*(S) invariant under Γ_g. In the first part of this paper, we are interested in the function theoretic properties of $\log\det'(\Delta)$ on *Teich*(S).

1.1. **Holomorphic extensions of determinants of Laplacians.** Before stating our first main theorem, consider the special case of genus 1.

Example ([RS2] or [Sa1], p. 33, (A.1.7)). For $z \in \mathbb{H}$, let T_z be the flat torus obtained by the lattice of $\mathbb{C}$ generated by 1 and z. Then the determinant of Laplacian of this flat torus is

$$\log\det'(\Delta)(z) = \log(2\pi(\mathrm{Im}\ z)^{1/2}|\eta(z)|^2)$$

where $\eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ for $q = e^{2\pi iz}$ is the Dedekind eta function; this is a modular form of weight 1/2.

The manifold $\mathbb{H}$ has a complexification $\mathbb{H}\times\overline{\mathbb{H}}$, and the function $\log\det'(\Delta)(z)$ on the diagonal $\{w = \overline{z}\}$ has a unique holomorphic extension to $\mathbb{H}\times\overline{\mathbb{H}}$, namely,

$$\log\bigl(2\pi(\frac{z-w}{2i})^{1/2}\,\eta(z)\,\overline{\eta(\overline{w})}\bigr).$$

We show that even in higher genus $g > 1$, the function $\log\det'(\Delta)$ has a unique holomorphic extension. In higher genus, the objects corresponding to $\mathbb{H}$ and $\mathbb{H}\times\overline{\mathbb{H}}$ are the Teichmüller space *Teich*(S) and the quasifuchsian space

$$QF(S) = Teich(S) \times Teich(\overline{S}) \cong Teich(S) \times \overline{Teich(S)},$$

respectively where the real analytic manifold *Teich*(S) imbeds as the diagonal in $QF(S)$. Bers's "simultaneous uniformization theorem" [Be] identifies the quasifuchsian space $QF(S)$ with the space of hyperbolic metrics modulo isotopies on the 3-manifold $S\times\mathbb{R}$, whose ideal boundary at infinity is conformally isomorphic to a pair of Riemann surfaces. McMullen recently used the quasifuchsian space to study the geometry of the Teichmüller space via the above complexification [Mc].

Now let us state our first main result.

Theorem 1.1. *The function* $\log\det'(\Delta)$ *on Teich*(S) *has a unique holomorphic extension to the quasifuchsian space* $QF(S)$.

Remark. Historically, the first result in the spirit of Theorem 1.1 is due to Fay [F] who obtained a holomorphic extension of the analytic torsion from the Picard variety of a compact Riemann surface to the space of $\mathbb{C}^*$-representations of its fundamental group.

Remark. We note that the holomorphic extension of $\log\det'(\Delta_n)$ of the Laplacian acting on the $(n,0)$-forms for $n \geq 2$ is given by McIntyre and Teo [TM] using the holomorphic extension of Selberg's zeta function. Their method does not work in our case of $\log\det'(\Delta)$ $=\log\det'(\Delta_0)= \log\det'(\Delta_1)$.

In the proof of Theorem 1.1, we use the Belavin-Knizhnik formula (see Theorem 2.6, also see [W3] and [ZT]) and the holomorphic extension of the Weil-Petersson form constructed by Platis [Pl] (see Theorem 2.3).

We remark that the asymptotic behavior of $\log\det'(\Delta)$ near the boundary of Teichmüller space is important in both geometry and physics and was studied in [W4] and [BB]. It would be interesting to understand the asymptotic behavior of the holomorphic extension of $\log\det'(\Delta)$ near the boundary of the quasifuchsian space.

In view of Theorem 1.1, it is natural to ask whether there is an actual family of elliptic differential operators on S whose determinant realizes the holomorphic extension of $\det'(\Delta)$. To address this question we introduce a family $\{\Delta_{\mu,\nu}\}$ of elliptic second order differential operators on S which is holomorphic with respect to its parameter (μ,ν), the pair of Beltrami differentials and which uniquely extends the Laplacians of hyperbolic metrics. Because of holomorphy of this family, the differential operators $\Delta_{\mu,\nu}$ cannot be self-adjoint off the diagonal $\{\mu=\nu\}$. These operators $\Delta_{\mu,\nu}$ are new examples of non-self-adjoint elliptic second order differential operators with a natural geometric origin!

1.2. **Holomorphic extensions of Laplacians and their determinants.** To state our theorem on the holomorphic extension $\Delta_{\mu,\nu}$ of Laplacians we need a few terminologies. Recall that a marking on S is a Riemann surface X_0 together with an oriented diffeomorphism between X_0 and S. A Beltrami differential μ on X_0 is a complex $(-1,1)$-form which in one (and hence all) local representations

$$\mu = \mu(z)\frac{d\overline{z}}{dz}$$

satisfies $\|\mu\|_\infty < 1$. The space $M(X_0)$ of smooth Beltrami differentials on X_0 is a contractible complex analytic manifold modeled on a Fréchet space. Denote by $M(S)$ the space of smooth complex structures on S, which is equivalent by the uniformization theorem to the space of hyperbolic metric on S. Then $M(X_0)$ gives a complex coordinate chart on $M(S)$, in which the origin $0 \in M(X_0)$ corresponds to $X_0 \in M(S)$ (see [EE]). Denote the complex conjugate of $M(X_0)$ by $\overline{M(X_0)}$. The diagonal

$$\{(\mu,\mu) \mid \mu \in M(X_0)\} \subset M(X_0) \times \overline{M(X_0)}$$

is a totally real submanifold. Given $0 < k < 1$ and $E > 0$, we introduce the space of Beltrami differentials

$$M_{k,E}(X_0) = \{\mu \in M(X_0) \mid \|\mu\|_\infty < k \text{ and } \|\mu\|_{C^2(X_0)} < E\}.$$

where the C^2-norm $\|\cdot\|_{C^2(X_0)}$ is defined by the hyperbolic metric on X_0.

The upper-half plane $\mathbb{H}$ with its standard hyperbolic metric $y^{-2}(dx^2+dy^2)$ is the Riemannian universal cover of X_0; the covering transformation group G is called the Fuchsian group of X_0. The Laplacian of $\mathbb{H}$ is given by the formula

$$\Delta_{\mathbb{H}} = (z-\overline{z})^2\frac{\partial^2}{\partial z \partial \overline{z}},$$

where z is the standard coordinate of $\mathbb{H}$, and it induces the Laplacian Δ of the hyperbolic surface $X_0 = \mathbb{H}/G$.

Denote by M^G the set of Beltrami differentials on $\mathbb{H}$ which transform as

$$\mu(z) = \mu(g(z))\frac{\overline{\partial g}}{\partial g}$$

for all $g \in G$. Then $M(X_0)$ is identified with M^G. It is well known that for each Beltrami differential μ on $\mathbb{H}$ there exists unique quasiconformal homeomorphism $f^\mu : \mathbb{H} \to \mathbb{H}$ satisfying the Beltrami differential equation

$$\overline{\partial} f = \mu\, \partial f$$

whose continuous extension to the real axis fixes $0, 1, \infty$.

We are now ready to state our second main theorem.

Theorem 1.2. *There exists unique family of elliptic second order differential operators $\Delta_{\mu,\nu}$ on S parametrized by $(\mu, \nu) \in M(X_0) \times \overline{M(X_0)}$, with the following properties:*

1. *$\Delta_{\mu,\nu}$ depends holomorphically on (μ, ν);*
2. *the lift of $\Delta_{\mu,\mu}$ to $\mathbb{H}$ is the pull-back of the Laplacian $\Delta_{\mathbb{H}}$ by the quasiconformal mapping $f^\mu : \mathbb{H} \to \mathbb{H}$, i.e., $\Delta_{\mu,\mu}$ is the Laplacian of the hyperbolic metric on S induced by the pullback hyperbolic metric on $\mathbb{H}$ by the map f^μ;*
3. *given $0 < k < 1$ and $E > 0$, there exists a constant $\epsilon > 0$ such that if $\mu,\, \nu \in M_{k,E}(X_0)$ and*

$$\|\mu - \nu\|_{C^2(X_0)} < \epsilon,$$

the determinant $\det'(\Delta_{\mu,\nu})$ is defined, and depends holomorphically on (μ, ν).

The operator $\Delta_{\mu,\nu}$ is constructed by modifying the explicit expression for $(f^\mu)^*\Delta_{\mathbb{H}}$, incorporating the quasifuchsian parameter (μ, ν) and corresponding quasiconformal mapping $f_{\mu,\nu}$. We use a result of Ahlfors and Bers [AB], that the unique normalized solution of Beltrami differential equation depends analytically on the Beltrami differential.

To establish property (3), we apply the definition of determinant using complex powers of elliptic operators due to Seeley ([Se1], [Se2], [Sh], and [KV]). The restriction $\|\mu-\nu\|_{C^2(X_0)} < \epsilon$ is introduced to satisfy the conditions for the construction of complex power.

Denote by $\widetilde{\det'(\Delta)}$ the holomorphic extension of $\det'(\Delta)$ to $QF(S)$ obtained in Theorem 1.1. We have the principal fiber bundle

$$\begin{array}{ccc} \mathit{Diff}_0(S) & \longrightarrow & M(X_0) \\ & & \big\downarrow \pi \\ & & \mathit{Teich}(S), \end{array} \tag{1.1}$$

where the projection π is known to be holomorphic (see [EE]). This gives rise to the principal fiber bundle

$$\begin{array}{ccc} \mathit{Diff}_0(S) \times \mathit{Diff}_0(S) & \longrightarrow & M(X_0) \times \overline{M(X_0)} \\ & & \big\downarrow \pi\times\bar{\pi} \\ & & QF(S). \end{array} \tag{1.2}$$

The lift $(\pi \times \bar{\pi})^*\widetilde{\det'(\Delta)}$ is holomorphic on $M(X_0) \times \overline{M(X_0)}$. We know by Theorem 1.2 (2) that

$$\det'(\Delta_{\mu,\mu}) = (\pi \times \bar{\pi})^*\widetilde{\det'(\Delta)}(\mu, \mu),$$

and by Theorem 1.2 (3) that the determinant $\det'(\Delta_{\mu,\nu})$ is defined and holomorphic on some open neighborhood N of the diagonal in $M(X_0) \times \overline{M(X_0)}$. Therefore, by analytic continuation, we have the equality

$$\det'(\Delta_{\mu,\nu}) = (\pi \times \bar{\pi})^* \widetilde{\det'(\Delta)}(\mu, \nu) \quad \text{for } (\mu, \nu) \in N,$$

and we may regard the holomorphic function $(\pi \times \bar{\pi})^* \widetilde{\det'(\Delta)}$ as the determinant of $\Delta_{\mu,\nu}$ even for those (μ, ν) to which Theorem 1.2 (3) does not apply. That is, on all of $M(X_0) \times \overline{M(X_0)}$, we may define

$$\det'(\Delta_{\mu,\nu}) = (\pi \times \bar{\pi})^* \widetilde{\det'(\Delta)}(\mu, \nu). \tag{1.3}$$

Remark. From the family $\{\Delta_{\mu,\nu}\}$, we may construct holomorphic families of elliptic operators in a neighborhood of each Teichmüller point $([X_0], [X_0])$ in $QF(S)$, using the Ahlfors-Weill section s of the fibre bundle (1.1) (see [AW] or [IT] pp. 153-157). This induces a holomorphic section $s \times \bar{s}$ of fibration $\pi \times \bar{\pi}$ of (1.2), defined in a neighborhood U of the point $([X_0], [X_0])$ in $QF(S)$. Clearly, by (1.3),

$$\det'(\Delta_{(s[X], \bar{s}[Y])}) = \widetilde{\det'(\Delta)}([X], [Y]) \quad \text{on } U.$$

However, this method does not give rise to a family of operators over all of $QF(S)$, since by Earle [Ea], there is no global holomorphic cross-section for the fibre bundle $\pi : M(X_0) \to Teich(S)$ of (1.1).

Plan of the paper. In Section 2, we prove Theorem 1.1 and in Section 3, we prove Theorem 1.2. In subsequent sections, we provide proof of the results used in Section 3.

Acknowledgment. This paper is author's 2005 thesis for Northwestern University. He deeply thanks his thesis advisor Ezra Getzler for guidance and support. He is also grateful to Curtis McMullen, Peter Sarnak, András Vasy, and Jared Wunsch for helpful discussions or comments. He thanks anonymous referees for valuable remarks.

2. Holomorphic Extensions of Determinants of Laplacians

In this section, we use several fundamental facts of Teichmüller spaces and the determinant of Laplacians to prove Theorem 1.1.

2.1. **Preliminaries.** In this subsection, we review the facts that we need on Teichmüller spaces and quasifuchsian spaces, including the Belavin-Knizhnik formula and Platis's theorem. In the next subsection, we prove Theorem 1.1.

Determinants of Laplacians. Let Δ be the Laplace-Beltrami operator on functions on a compact Riemannian manifold M. Let

$$\zeta_\Delta(s) = \sum_{\lambda \in \mathrm{Spec}(\Delta) \setminus \{0\}} \lambda^{-s} \tag{2.1}$$

be the zeta-function of Δ. The determinant $\det'(\Delta)$ is defined (see [RS1]) as

$$-\log \det'(\Delta) = \frac{d\zeta_\Delta(0)}{ds}. \tag{2.2}$$

The sum in (2.1) is absolutely convergent for $\operatorname{Re} s > \frac{\dim M}{2}$ sufficiently large, and has a meromorphic extension to the whole complex plane. This meromorphic extension is regular at $s = 0$, and so there is no difficulty in taking the derivative at $s = 0$ in (2.2).

Teichmüller spaces. A general reference for this section is [IT].

Let S be an oriented closed surface with genus $g > 1$. The Teichmüller space $\mathit{Teich}(S)$ of S is the space of isotopy classes of hyperbolic Riemannian metrics on S, that is, metrics with Gaussian curvature -1. By uniformization theorem, $\mathit{Teich}(S)$ is also the space of isotopy classes of complex structures on S.

The set of equivalence classes of hyperbolic metrics (or equivalently complex structures) under orientation preserving diffeomorphisms on S forms the moduli space $\mathcal{M}_g$ of compact Riemann surfaces of genus g.

Denote the group of orientation preserving diffeomorphisms on S by $\mathit{Diff}^+(S)$, and the group of isotopies by $\mathit{Diff}_0(S)$. The mapping class group

$$\Gamma_g = \mathit{Diff}^+(S)/\,\mathit{Diff}_0(S)$$

is a discrete group which acts properly discontinuously on $\mathit{Teich}(S)$. Thus $\mathit{Teich}(S)$ is almost a covering space of $\mathcal{M}_g$, with covering transformation group Γ_g:

$$\begin{array}{ccc} \Gamma_g & \longrightarrow & \mathit{Teich}(S) \\ & & \big\downarrow \\ & & \mathcal{M}_g = \ \Gamma_g \backslash \mathit{Teich}(S) \end{array}$$

The only caveat is that the action of Γ_g is not free, i.e. there are points in $\mathit{Teich}(S)$ which are fixed under some finite subgroups of Γ_g. These points descend to $\mathcal{M}_g$ as orbifold singularities.

Fixing a hyperbolic metric on S, we may decompose S into $2g-2$ pairs of pants, separated by closed geodesics $\gamma_1, \ldots, \gamma_{3g-3}$. A hyperbolic pair of pants is determined up to isometry by the lengths of its boundary geodesics. Given the combinatorial pants decomposition of S, we get a hyperbolic metric by specifying the lengths l_i ($l_i > 0$) of the geodesics γ_i and the angle θ_i by which they are twisted along γ_i before gluing. Let $\tau_i = l_i\theta_i/2\pi$, $i = 1, \ldots, 3g-3$. Then the system of variables

$$(l_1, \ldots, l_{3g-3}, \tau_1, \ldots, \tau_{3g-3})$$

is a real analytic coordinate system on $\mathit{Teich}(S)$, called the *Fenchel-Nielsen coordinates* of $\mathit{Teich}(S)$. This coordinate system gives a diffeomorphism

$$\mathit{Teich}(S) \approx \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}\,.$$

There is a a natural symplectic form ω_{WP} on $\mathit{Teich}(S)$, called the Weil-Petersson form. By a theorem of Wolpert ([W1], [W2]; see also [IT]), this form is given in Fenchel-Nielsen coordinates by the formula

$$\omega_{WP} = \sum_{i=1}^{3g-3} dl_i \wedge d\tau_i. \tag{2.3}$$

The Teichmüller space $Teich(S)$ has a natural complex structure, for which ω_{WP} is a Kähler form. The following theorem is well known. (See, for example, [Ah].)

Theorem 2.1. *For a closed surface S with genus $g > 1$, $Teich(S)$ is biholomorphic to a bounded open contractible domain in $\mathbb{C}^{3g-3}$.*

Corollary 2.2. *There are global holomorphic coordinates $z = (z^1, \ldots, z^{3g-3})$ on $Teich(S)$.*

Quasifuchsian spaces. While Teichmüller space is a space of Riemann surfaces, the quasifuchsian space defined by Lipman Bers (See [Be]) is a space of pairs of Riemann surfaces. The quasifuchsian space $QF(S)$ of the surface S may simply be defined as

$$QF(S) = Teich(S) \times Teich(\overline{S}).$$

Here, $\overline{S}$ denotes the surface S with the opposite orientation.

The complex conjugate $\overline{X}$ of a Riemann surface X is defined by the following diagram:

$$\begin{array}{ccc} \mathbb{H} & \longrightarrow & \overline{\mathbb{H}} \\ \downarrow & & \downarrow \\ X & \dashrightarrow & \overline{X} \end{array} \tag{2.4}$$

The upper arrow is complex conjugation, and the vertical arrows are the universal coverings given by the uniformization theorem for Riemann surfaces. There is a canonical map from $Teich(S)$ to $Teich(\overline{S})$ defined by sending a Riemann surface $X \in Teich(S)$ to its complex conjugate $\overline{X} \in Teich(\overline{S})$. As complex manifolds, $Teich(\overline{S}) \cong \overline{Teich(S)}$, where $\overline{Teich(S)}$ is the complex conjugate of $Teich(S)$, i.e. the holomorphic structure of $\overline{Teich(S)}$ is the anti-holomorphic structure of $Teich(S)$.

The diagonal map

$$Teich(S) \hookrightarrow Teich(S) \times \overline{Teich(S)}$$

sending $X \in Teich(S)$ to (X, X) embeds $Teich(S)$ as a totally real submanifold into $QF(S)$. The action of Γ_g on $Teich(S)$ extends to $QF(S)$ by the diagonal action: for $\rho \in \Gamma_g$ and $(X, Y) \in QF(S) = Teich(S) \times \overline{Teich(S)}$,

$$\rho \cdot (X, Y) = (\rho \cdot X, \rho \cdot Y).$$

By Corollary 2.2, $QF(S) = Teich(S) \times \overline{Teich(S)}$ has global holomorphic coordinates

$$(z^1, \ldots, z^{3g-3}, w^1, \ldots, w^{3g-3}).$$

We abbreviate this coordinate system to (z, w). Then $Teich(S) = \{w = \overline{z}\} \subset QF(S)$.

Holomorphic extension of Weil-Petersson form. The following result is due to Platis ([Pl], Theorems 6 and 8).

Theorem 2.3. *The differential form $i\omega_{WP}$ on the Teichmüller space $Teich(S)$ has an extension Ω to the quasifuchsian space $QF(S)$ which is a holomorphic non-degenerate closed $(2, 0)$-form whose restriction to the diagonal $Teich(S) \subset QF(S) \cong Teich(S) \times \overline{Teich(S)}$ is $i\omega_{WP}$.*

The following lemma is elementary.

Lemma 2.4. *Let $U \subset \mathbb{C}^n$ be a connected complex domain, and let ϕ be a holomorphic function on $U \times \overline{U}$ whose restriction to the diagonal $U \subset U \times \overline{U}$ vanishes. Then ϕ vanishes on all of $U \times \overline{U}$.*

We can now prove the following result.

Proposition 2.5. *In terms of the holomorphic coordinate system*

$$(z, w) = (z^1, \ldots, z^{3g-3}, w^1, \ldots, w^{3g-3})$$

on Teich$(S) \times \overline{\mathit{Teich}(S)}$, the 2-form Ω of Theorem 2.3 may be written locally as

$$\Omega = \sum_{i,j} \Omega_{ij}\, dz^i \wedge dw^j.$$

Proof. Since Ω is $(2,0)$ form, we may write

$$\Omega = \sum_{i,j} \big(A_{ij}\, dz^i \wedge dz^j + B_{ij}\, dz^i \wedge dw^j + C_{ij}\, dw^i \wedge dw^j \big).$$

Because the restriction $i\omega_{WP}$ of Ω to the diagonal $\{w = \overline{z}\}$ is $(1,1)$-form, we see that A_{ij} and C_{ij} vanish on the diagonal. Since Ω is holomorphic, Lemma 2.4 shows that A_{ij} and C_{ij} vanish. □

The Laplacian on hyperbolic surfaces and the Belavin-Knizhnik formula. Let X be a compact hyperbolic surface of genus $g > 1$, and let Δ be the Laplacian on scalar functions on X. On the universal cover $\mathbb{H}$ of X, the pull-back of Δ by the covering map may be written as

$$\Delta = (z - \overline{z})^2 \frac{\partial^2}{\partial z \partial \overline{z}}.$$

The *Siegel upper half space* P_g is the space of complex symmetric matrices in $\mathbb{C}^{g\times g}$ with positive definite imaginary part. The period matrix τ is a holomorphic map from *Teich*(S) to P_g.

We will use the Belavin-Knizhnik formula. (See the article by Wolpert [W3] and the one by Zograf and Takhtajan [ZT].) We only need the following special case of this theorem ([ZT], Theorem 2).

Theorem 2.6. *In Teich(S),*

$$\partial\overline{\partial} \log \left(\frac{\det'(\Delta)}{\det(\operatorname{Im}\tau)} \right) = -\frac{i}{6\pi}\, \omega_{WP},$$

where $\operatorname{Im}\tau$ is the imaginary part of the period matrix τ. The differential operator $\partial\overline{\partial}$ comes from the complex structure on Teich(S).

This formula and the result of the next section together with the theorem of Platis are the key ingredients in the construction of the holomorphic extension of $\log\det'(\Delta)$.

2.2. **Holomorphic extension of** $\log\det'(\Delta)$**.** The following is a key step in the proof of Theorem 1.1.

Proposition 2.7. *Let V and W be domains in the complex space $\mathbb{C}^n$ diffeomorphic to the open unit ball. Consider $V \times W \subset \mathbb{C}^n \times \mathbb{C}^n$, with holomorphic coordinates (z, w), and let $\partial_z = dz^i\,\partial_{z_i}$ and $\partial_w = dw^j\,\partial_{w_j}$. Suppose Ω is a holomorphic closed 2-form on $V \times W$ which is locally written as*

$$\Omega = \sum_{i,j} \Omega_{ij} dz^i \wedge dw^j.$$

Then there is a holomorphic function q on $V \times W$ such that $\partial_z \partial_w q = \Omega$.

Proof. Choose smooth polar coordinates on V and W, and denote the centers of these coordinate systems by z_0 and w_0 respectively. Denote the radial line in polar coordinates from z_0 to the point $z \in V$ by $\mathsf{v}(z)$; similarly, denote the radial line in polar coordinates from w_0 to the point $w \in W$ by $\mathsf{w}(w)$. More generally, if c is a smooth chain in V, let $\mathsf{v}(c)$ denote the cone on c with vertex z_0, and similarly if c is a smooth chain in W, let $\mathsf{w}(c)$ denote the cone on c with vertex w_0.

Define $q(z, w)$ by the formula

$$q(z, w) = \int_{\mathsf{v}(z)\times\mathsf{w}(w)} \Omega.$$

Since the chain $\mathsf{v}(z) \times \mathsf{w}(w)$ varies smoothly as (z, w) varies, the function $q(z, w)$ is smooth. Observe that q is unchanged by isotopies of the coordinate systems on V and W which fix the centers z_0 and w_0, and that q vanishes on $V \times \{w_0\}$ and on $\{z_0\} \times W$.

If c is a differentiable curve in W parametrized by the interval $[0, t]$, we have by Stokes's theorem

$$q(z, c(t)) - q(z, c(0)) = \int_{\mathsf{v}(z)\times c} \Omega + \int_{\{z\}\times\mathsf{w}(c)} \Omega - \int_{\{z_0\}\times\mathsf{w}(c)} \Omega - \int_{\mathsf{v}(z)\times\mathsf{w}(c)} d\Omega.$$

The second and third terms on the right-hand side vanish, since Ω vanishes when restricted to the 2-simplex $\{z\} \times \mathsf{w}(c)$, and the last term vanishes since $d\Omega = 0$. Taking the limit $t \to 0$, we see that

$$\iota(0, c'(0)) dq(z, c(0)) = -\int_{\mathsf{v}(z)\times c(0)} \iota(0, c'(0))\Omega. \tag{2.5}$$

Since Ω is holomorphic along $\{z\} \times W$, it follows that q is holomorphic along $\{z\} \times W$ as well. A similar argument shows that q is holomorphic along $V \times \{w\}$; combining these two calculations, we see that q is holomorphic on $V \times W$.

We now calculate $\partial_w \partial_z q$. By (2.5),

$$\partial_w q(z, w) = -\sum_{i=1}^n dw^i \int_{\mathsf{v}(z)\times\{w\}} \iota(\partial_{w^i})\Omega.$$

If c is a differentiable curve in V, parametrized by the interval $[0,t]$, we have by Stokes's theorem

$$(\partial_w q)(c(t),w) - (\partial_w q)(c(0),w) = \sum_{i=1}^{n} dw^i \left(-\int_{c\times\{w\}} \iota(\partial_{w^i})\Omega + \int_{\mathsf{v}(c)\times\{w\}} d\iota(\partial_{w^i})\Omega \right). \tag{2.6}$$

The second term on the right-hand side vanishes. Indeed,

$$\begin{aligned} d\iota(\partial_{w^i})\Omega &= -\partial_{z^k}\Omega_{ji} dz^k \wedge dz^j - \partial_{w^k}\Omega_{ji} dw^k \wedge dz^j \\ &= -\sum_{j<k} \big(\partial_{z^k}\Omega_{ji} - \partial_{z^j}\Omega_{ki}\big)\, dz^k \wedge dz^j - \partial_{w^k}\Omega_{ji} dw^k \wedge dz^j \\ &= -\partial_{w^k}\Omega_{ji} dw^k \wedge dz^j. \end{aligned}$$

Restricting to $\mathsf{v}(c)\times\{w\}$, this differential form vanishes.

Taking $t \to 0$ in (2.6), we see that

$$\iota(c'(0),0) d(\partial_w q)(c(0),w) = -\sum_{i=1}^{n} dw^i\, \iota(c'(0),0)\iota(\partial_{w^i})\Omega(c(0),w),$$

or in other words, $\partial_z\partial_w q = \Omega$. □

From Proposition 2.5, we know that the holomorphic 2-form Ω of Theorem 2.3 satisfies the hypotheses of Theorem 2.7. Restricted to the diagonal $Teich(S) = \{w = \overline{z}\} \subset QF(S)$, the differential equation in Theorem 2.7 for the holomorphic function q on $QF(S)$ becomes

$$\partial\overline{\partial} q = i\omega_{WP}.$$

Thus, the proof of Theorem 2.7 gives a method of constructing a Kähler potential for the Kähler form $i\omega_{WP}$ on the Teichmüller space, using the extended form Ω to quasifuchsian space.

Example. (See p. 214 in [IT]) When S has genus 1, the Teichmüller space $Teich(S)$ may be identified with the upper half plane $\mathbb{H}$, and

$$\omega_{WP} = -i(z-\overline{z})^{-2}\, dz \wedge d\overline{z}.$$

One easily finds the Kähler potential $q(z) = \log(z-\overline{z})$. The method used in the proof of Theorem 2.7, applied to the 2-form $\Omega = (z-w)^{-2}\, dz \wedge dw$, yields the holomorphic function

$$q(z,w) = \log(z-w) - \log(z_0-w) - \log(z-w_0) + \log(z_0-w_0)$$

on the quasifuchsian space $\mathbb{H} \times \overline{\mathbb{H}}$.

Using the holomorphic function q on $QF(S)$, we now construct the holomorphic extension of $\log\det'(\Delta)$. The holomorphic function

$$\tilde{q}(z,w) = \frac{1}{2}\big(q(z,w) + \overline{q}(\overline{w},\overline{z})\big)$$

on $QF(S)$ restricts to a real function $\tilde{q}$ on the diagonal such that

$$\partial\overline{\partial}\tilde{q} = i\omega_{WP}.$$

Theorem 2.8. *There exists a unique holomorphic extension of* $\log \det'(\Delta)$ *to the quasifuchsian space* $QF(S)$. *In coordinates* (z, w) *on* $QF(S) \cong Teich(S) \times \overline{Teich(S)}$, *this extension has the form*

$$\log \det'(\Delta)(z,w) = -\frac{1}{6\pi}\tilde{q}(z,w) + \log\det((\tau(z) - \overline{\tau}(\overline{w}))/2i) + f(z) + \overline{f}(\overline{w}).$$

Proof. By Theorem 2.6, the one-form

$$\alpha = \partial\big(\log\det'(\Delta) + \frac{1}{6\pi}\tilde{q} - \log\det(\operatorname{Im}\tau)\big)$$

is holomorphic. Since $Teich(S)$ is simply connected, it follows that there is a differentiable function f such that

$$df = \alpha.$$

Since $\overline{\partial} f = \alpha^{0,1} = 0$, f is seen to be holomorphic. The theorem is now proved by analytically extending each of the functions $\det(\operatorname{Im}\tau)$, $\tilde{q}$, f and $\overline{f}$ in the holomorphic factorization

$$\log\det'(\Delta) = \log\det(\operatorname{Im}\tau) + C_g\tilde{q} + f + \overline{f}$$

on $Teich(S)$ to $QF(S)$. The holomorphic extension of $\tilde{q}$ is evident, since it is by construction the restriction of the holomorphic function $\tilde{q}$ on $QF(S)$. The function f is extended to $f(z)$, the function $\overline{f}$ to $\overline{f}(\overline{w})$, and the function $\det(\operatorname{Im}\tau)$ to

$$\log\det((\tau(z) - \overline{\tau}(\overline{w}))/2i).$$

(Note that the matrix $\tau(z) - \overline{\tau}(\overline{w})$ is everywhere invertible on $QF(S)$.) The uniqueness of the holomorphic extension of $\log\det'(\Delta)$ follows from Lemma 2.4. $\square$

It would not be hard, using this theorem, to give an explicit lower bound for the radius of convergence of the real analytic function $\log\det'(\Delta)$ on $Teich(S)$.

3. Holomorphic Extensions of Laplacians and Their Determinants

In this section, we prove Theorem 1.2. In Section 3.1, we construct the family $\{\Delta_{\mu,\nu}\}$, and show that it satisfies properties (1) and (2) in Theorem 1.2. In Section 3.2, we show the property (3) of Theorem 1.2. In Section 4, we provides several necessary estimates on quasiconformal mappings. Using the results of Section 4, we prove in Sections 5 and 6 the results which are used in Section 3.2. From now on, we denote by ∂ and $\overline{\partial}$ the Cauchy-Riemann operators $\frac{1}{2}(\partial_x - i\partial_y)$ and $\frac{1}{2}(\partial_x + i\partial_y)$, respectively.

3.1. **The holomorphic extension $\Delta_{\mu,\nu}$ of the Laplacian.** In this subsection, we construct the family $\{\Delta_{\mu,\nu}\}$ of elliptic second order differential operators of Theorem 1.2, and demonstrate properties (1) and (2).

Unless otherwisely stated, we restrict our domain to $\mathbb{H}$, and denote by μ and ν smooth Beltrami differentials on $\mathbb{H}$ (that is, smooth complex valued functions on $\mathbb{H}$ satisfying $\|\mu\|_\infty$, $\|\nu\|_\infty < 1$). By $\hat{\mu}$ we denote a Beltrami differential on the lower half plane $\overline{\mathbb{H}}$ defined by $\hat{\mu}(z) = \overline{\mu}(\overline{z})$. Denote by $\overline{\partial}_\mu$ the operator $\overline{\partial} - \mu\partial$, and by $\partial_{\overline{\mu}}$ the operator $\partial - \overline{\mu}\overline{\partial}$.

The following definition is due to Ahlfors and Bers.

Definition 3.1. Given a pair (μ,ν) of Beltrami differentials on $\mathbb{H}$, denote by $f_{\mu,\nu}:\mathbb{C}\to\mathbb{C}$ the unique continuous normalized solution (i.e. fixing 0, 1 and ∞) of the Beltrami equation on $\mathbb{C}$,

$$\begin{cases}\overline{\partial}_\mu f_{\mu,\nu} &= 0, \qquad \operatorname{Im} z>0,\\ \overline{\partial}_{\hat\nu} f_{\mu,\nu} &= 0, \qquad \operatorname{Im} z<0.\end{cases}$$

Let $f^\mu = f_{\mu,\mu}$.

We have the following result of Ahlfors and Bers [AB].

Lemma 3.1. *$f_{\mu,\nu}$ is a homeomorphism of the Riemann sphere $\widehat{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$. In particular, it is an open embedding of $\mathbb{H}$ into $\mathbb{C}$, and $\partial f_{\mu,\nu}$ is nowhere zero on $\mathbb{H}$.*

By complex conjugation of the Beltrami differential equation in Definition 3.1, we see that

$$\overline{f_{\nu,\mu}(z)} = f_{\mu,\nu}(\overline{z}). \tag{3.1}$$

In particular, $\overline{f^\mu}(z) = f^\mu(\overline{z})$ and thus f^μ maps $\mathbb{H}$ onto $\mathbb{H}$. In fact, $f_{\mu,\nu}$ maps $\mathbb{H}$ onto $\mathbb{H}$ if and only if $f^\mu|_{\mathbb{R}}\equiv f^\nu|_{\mathbb{R}}$.

In our construction of $\Delta_{\mu,\nu}$, we use the result of Ahlfors and Bers that the normalized solutions of the Beltrami equations depend analytically on the Beltrami differentials. The following lemma summarizes what we need (see [AB]).

Lemma 3.2. *For each $z\in\mathbb{H}$, $f_{\mu,\nu}(z)$, $\overline{f_{\nu,\mu}}(z)$, $\partial f_{\mu,\nu}(z)$ and $\partial\overline{f_{\nu,\mu}}(z)$, depend holomorphically on μ and anti-holomorphically on ν.*

Now, we start with the following key calculation in our construction of $\Delta_{\mu,\nu}$. By Lemma 3.1 and the inequality $|\mu|<1$, the function

$$\alpha = \frac{1}{(1-|\mu|^2)\partial f^\mu} \tag{3.2}$$

is bounded on $\mathbb{H}$.

Proposition 3.3.

$$(f^\mu)^*\partial\overline{\partial} = |\alpha|^2\left(-\mu\,\partial^2 + (1+|\mu|^2)\,\overline{\partial}\partial - \overline{\mu}\,\overline{\partial}^2 + (\overline{\partial}_\mu\log\alpha)\,\partial + (\partial_{\overline{\mu}}\log\overline{\alpha})\,\overline{\partial}\right).$$

One easily sees that when $\mu=0$, the above formula for $(f^\mu)^*\partial\overline{\partial}$ reduces to $\partial\overline{\partial}$.

In the proof of Proposition 3.3, we denote f^μ by f, and $(f^\mu)^{-1}$ by h. By the chain rule applied to the equations $h\circ f = z$ and $\overline{h}\circ f=\overline{z}$, and the Beltrami equation, we see that

$$\begin{pmatrix}\partial h\circ f & \partial\overline{h}\circ f\\ \overline{\partial}h\circ f & \overline{\partial}\overline{h}\circ f\end{pmatrix} = \begin{pmatrix}\partial f & \partial\overline{f}\\ \overline{\partial}f & \overline{\partial f}\end{pmatrix}^{-1} = \begin{pmatrix}\alpha & -\overline{\mu}\,\alpha\\ -\mu\,\overline{\alpha} & \overline{\alpha}\end{pmatrix}. \tag{3.3}$$

By the chain rule applied to the equation $\partial h\circ f=\alpha$, we see that

$$\begin{pmatrix}\partial f & \partial\overline{f}\\ \overline{\partial}f & \overline{\partial f}\end{pmatrix}\begin{pmatrix}\partial^2 h\circ f\\ \overline{\partial}\partial h\circ f\end{pmatrix} = \begin{pmatrix}\partial\alpha\\ \overline{\partial}\alpha\end{pmatrix}.$$

Applying (3.3), we see that

$$\overline{\partial}\partial h\circ f = |\alpha|^2\,\overline{\partial}_\mu\log\alpha \tag{3.4}$$

Proof of Proposition 3.3. If u is a C^∞ function on $\mathbb{H}$, then

$$(f^\mu)^*\partial\overline{\partial}u = (\partial\overline{\partial}(u\circ h))\circ f.$$

We have

$$\begin{aligned}\partial\overline{\partial}(u\circ h) &= \partial\left((\partial u\circ h)\overline{\partial}h + (\overline{\partial}u\circ h)\overline{\partial h}\right)\\ &= (\partial^2 u\circ h)\partial h\overline{\partial}h + (\overline{\partial}\partial u\circ h)\partial\overline{h}\overline{\partial}h\\ &+ (\partial\overline{\partial}u\circ h)\partial h\overline{\partial h} + (\overline{\partial}^2 u\circ h)\partial\overline{h}\overline{\partial h}\\ &+ (\partial u\circ h)\partial\overline{\partial}h + (\overline{\partial}u\circ h)\partial\overline{\partial h}.\end{aligned}$$

Composing on the right with f, we see that

$$\begin{aligned}\left(\partial\overline{\partial}(u\circ h)\right)\circ f &= \partial^2 u(\partial h\circ f)(\overline{\partial}h\circ f) + \overline{\partial}\partial u(\partial\overline{h}\circ f)(\overline{\partial}h\circ f)\\ &+ \partial\overline{\partial}u(\partial h\circ f)(\overline{\partial h}\circ f) + \overline{\partial}^2 u(\partial\overline{h}\circ f)(\overline{\partial h}\circ f)\\ &+ \partial u(\partial\overline{\partial}h\circ f) + \overline{\partial}u(\partial\overline{\partial h}\circ f).\end{aligned}$$

Applying (3.3) and (3.4), the proposition follows. □

We wish to find an extension of $(f^\mu)^*\partial\overline{\partial}$ which is holomorphic in μ. Because the formula for $(f^\mu)^*\partial\overline{\partial}$ contains quantities such as $|\partial f^\mu|^2$ and $|\mu|^2$, simply replacing f^μ by $f_{\mu,\nu}$ does not give a holomorphic extension of $\partial\overline{\partial}$. Nor do other simple extensions, such as $(f^\mu)^*\partial\,(f^\nu)^*\overline{\partial}$. On the other hand, replacing f^μ, $\overline{f^\mu}$ and $\overline{\mu}$ by $f_{\mu,\nu}$, $\overline{f_{\nu,\mu}}$, and $\overline{\nu}$, respectively we obtain by Lemma 3.2 an operator which depends holomorphically on μ and anti-holomorphically on ν.

Definition 3.2. Given a pair of Beltrami differentials (μ,ν), let

$$\alpha_{\mu,\nu} = \frac{1}{(1-\mu\overline{\nu})\partial f_{\mu,\nu}}$$

Define a second order differential operator $\Delta_{\mu,\nu}$ on functions on $\mathbb{H}$ by the formula

$$\Delta_{\mu,\nu} = (f_{\mu,\nu} - \overline{f_{\nu,\mu}})^2(\partial\overline{\partial})_{\mu,\nu}$$

where

$$(\partial\overline{\partial})_{\mu,\nu} = \alpha_{\mu,\nu}\overline{\alpha_{\nu,\mu}}\left(-\mu\,\partial^2 + (1+\mu\overline{\nu})\,\overline{\partial}\partial - \overline{\nu}\,\overline{\partial}^2 + (\overline{\partial}_\mu\log\alpha_{\mu,\nu})\,\partial + (\partial_{\overline{\nu}}\log\overline{\alpha_{\nu,\mu}})\,\overline{\partial}\right).$$

The principal symbol of $\Delta_{\mu,\nu}$ in complex coordinates (z,ζ) on the cotangent bundle $T^*\mathbb{H}$, where $\sigma(\overline{\partial}) = i\zeta$, equals

$$\sigma_2(\Delta_{\mu,\nu})(\zeta) = -(f_{\mu,\nu} - \overline{f_{\nu,\mu}})^2\,\alpha_{\mu,\nu}\overline{\alpha_{\nu,\mu}}(\zeta-\mu\overline{\zeta})(\overline{\zeta}-\overline{\nu}\zeta)$$

Lemma 3.4. *The differential operator $\Delta_{\mu,\nu}$ is elliptic for any pair of Beltrami differentials (μ,ν).*

Proof. By (3.1), we have

$$f_{\mu,\nu}(z) - \overline{f_{\nu,\mu}(\overline{z})} = f_{\mu,\nu}(z) - f_{\mu,\nu}(\overline{z})$$

which is nowhere vanishing on $\mathbb{H}$, since $f_{\mu,\nu}$ is a homeomorphism of $\mathbb{C}$. The functions $\partial f_{\mu,\nu}$ and $\overline{\partial f_{\nu,\mu}}$ are nowhere vanishing on $\mathbb{H}$ by Lemma 3.1. We also have the bounds $\|\mu(z)\|_\infty, \|\nu(z)\|_\infty < 1$, and the lemma follows. □

The following theorem is immediate.

Theorem 3.5. *The elliptic family $\Delta_{\mu,\nu}$ is holomorphic in μ and anti-holomorphic in ν, and coincides with $(f^\mu)^*\Delta$ when $\mu = \nu$.*

The following proposition shows that $\Delta_{\mu,\nu}$ is the unique such family of operators.

Proposition 3.6. *Let $A_{\mu,\nu}$ be a family of operators on $C^\infty(\mathbb{H})$ holomorphic in μ and anti-holomorphic in ν. If $A_{\mu,\mu} = 0$ for all μ, then $A_{\mu,\nu} = 0$ for all μ, ν.*

We need an elementary lemma.

Lemma 3.7. *Let $\phi(s,t)$ be a function of complex variables s,t which is holomorphic in s and anti-holomorphic in t. Suppose $\phi(s,s) = 0$ for all s. Then $\phi(s,t) = 0$ for all s,t.*

Proof of Proposition 3.6. Fix $\psi \in C^\infty(\mathbb{H})$, $z \in \mathbb{H}$, and Beltrami differentials μ, ν. Let s,t be complex parameters. Then $\phi(s,t) = (A_{(1-s)\mu+s\nu,(1-t)\mu+t\nu}\psi)(z)$ is holomorphic in s and anti-holomorphic in t, and $\phi(s,s) = 0$ for all s. By Lemma 3.7, $\phi(s,t) = 0$ for all s,t. This shows the proposition. □

Now fix a Riemann surface X_0 and the corresponding Fuchsian group G of the covering map $\mathbb{H} \to X_0$. We show that the restriction of the family $\{\Delta_{\mu,\nu}\}$ to G-invariant Beltrami differentials $\mu, \nu \in M^G$ on $\mathbb{H}$ induces a family of elliptic differential operators on X_0.

Lemma 3.8. *If $\mu \in M^G$ and $g \in G$, $g^*(f^\mu)^*\Delta = (f^\mu)^*\Delta$.*

Proof. By the invariance of the hyperbolic metric m_0 on $\mathbb{H}$ under conformal mappings, and by the invariance of μ under G, it is clear that the pull-back metric $(f^\mu)^*m_0$ is invariant under G. So the Laplacian $(f^\mu)^*\Delta$ associated to the pull-back metric $(f^\mu)^*m_0$ is also invariant under G. □

Proposition 3.9. *For every $g \in G$, and for every $\mu, \nu \in M^G$, $g^*\Delta_{\mu,\nu} = \Delta_{\mu,\nu}$.*

Proof. Fix $g \in G$. The family of operators $g^*\Delta_{\mu,\nu} - \Delta_{\mu,\nu}$ is holomorphic in μ and anti-holomorphic in ν, and by Lemma 3.8, it vanishes for $\mu = \nu$. Therefore, by Proposition 3.6, $g^*\Delta_{\mu,\nu} - \Delta_{\mu,\nu} = 0$ for all $\mu, \nu \in M^G$. □

By Proposition 3.9 and the identification of M^G with $M(X_0)$, we have

Theorem 3.10. *There is a unique family $\{\Delta_{\mu,\nu} \mid \mu, \nu \in M(X_0)\}$ of elliptic second order differential operators on X_0 which satisfies properties (1) and (2) of Theorem 1.2.*

3.2. **Determinant of $\Delta_{\mu,\nu}$.** In this section, we consider the determinant of $\Delta_{\mu,\nu}$ and establish the property (3) in Theorem 1.2. To define the determinant of $\Delta_{\mu,\nu}$, we use the method of using complex powers of elliptic operators developed by Seeley [Se1], [Se2], although we follow Shubin [Sh] more closely. (See also [KV].)

For the Fuchsian group G of X_0, let P be the closure of a fixed fundamental domain of G. Let Q be the neighborhood of P consisting of the union of all translates of P by elements of G whose intersection with P is nonempty.

Definition 3.3. Given $0 < k < 1$ and $E > 0$, let

$$M^G_{k,E} = \{\mu \in M^G \mid \|\mu\|_\infty \le k, \|\mu\|_{C^2(Q)} \le E\}$$

where the C^2-norm is defined using the flat metric on $\mathbb{H}$.

The following theorems will be proved in Sections 5, 6.

Theorem 3.11. *Given $0 < k < 1$, $E > 0$ and $0 < \theta_0 < \pi$, there is $\epsilon > 0$ such that if $\mu, \nu \in M^G_{k,E}$ and $\|\mu - \nu\|_{C^1(Q)} < \epsilon$, then*

$$|\arg(\sigma_2(\Delta_{\mu,\nu}))| < \theta_0.$$

Theorem 3.12. *There exists a constant $C > 0$ such that for every μ, $\nu \in M^G_{k,E}$ and for any nonzero eigenvalue λ of $\Delta_{\mu,\nu}$ on $X_0 = \mathbb{H}/G$,*

$$|\lambda| \ge C - O(\|\mu - \nu\|_{C^2(Q)}).$$

Fix $0 < \theta_0 < \pi$. For the rest of section denote $\Delta_{\mu,\nu}$ by A and assume that (μ, ν) belongs to

$$N_\epsilon = \{(\mu, \nu) \mid \mu, \nu \in M^G_{k,E} \text{ and } \|\mu - \nu\|_{C^2(Q)} \le \epsilon\}$$

where $\epsilon > 0$ will be determined in the following.

3.2.1. *Determinant of $\Delta_{\mu,\nu}$.* By Theorem 3.11, we know that for sufficiently small ϵ the principal symbol $\sigma_2(A)(x, \zeta)$ does not take values in the closed conical sector

$$\Lambda = \{\lambda : \theta_0 \le \arg\lambda \le 2\pi - \theta_0\}$$

in the spectral plane $\mathbb{C}$ for any $(x, \zeta) \in T^*S \setminus S$. This condition ensures that $\mathrm{Spec}(A) \cap \Lambda$ is finite, so there is a closed sector $\Lambda_0 \subset \Lambda$ which has only zero spectrum inside.

By Theorem 3.12, for sufficiently small $\epsilon > 0$, there is $\rho > 0$ such that

$$\mathrm{Spec}(A) \cap \{z \mid |z| < \rho\} \subset \{0\}.$$

Given $\exp(i\theta) \in \Lambda_0$, let $\Gamma_{(\theta)}$ be the contour $\Gamma_{1,\theta}(\rho) \cup \Gamma_{0,\theta}(\rho) \cup \Gamma_{2,\theta}(\rho)$, where

$$\begin{aligned}
\Gamma_{1,\theta}(\rho) &= \{x \exp(i\theta) \mid x \ge \rho\}, \\
\Gamma_{0,\theta}(\rho) &= \{\rho \exp(i\phi) \mid \theta > \phi > \theta - \pi\}, \\
\Gamma_{1,\theta}(\rho) &= \{x \exp(i(\theta - \pi)) \mid \rho \le x\}.
\end{aligned}$$

Denote by R_λ the resolvent $(A - \lambda I)^{-1}$. Then for $\mathrm{Re}\, s < 0$, define

$$(A_s)_{(\theta)} = \frac{i}{2\pi} \int_{\Gamma_{(\theta)}} \lambda^s R_\lambda \, d\lambda.$$

By the symbol calculus of [Sh], A_s is trace class for $\mathrm{Re}\, s < -1$. In the following, we omit θ from the notation for $(A_s)_{(\theta)}$ and $\Gamma_{(\theta)}$.

For $s \in \mathbb{C}$, define the modified complex power $A^{s,o}$ of A by

$$A^{s,o} = A^k A_{s-k}$$

where k is an integer chosen so that $\operatorname{Re} s - k < 0$. To see that this definition does not depend on the choice of k, consider the operator

$$P_0 = \frac{i}{2\pi} \int_{|\lambda|=\rho} R_\lambda \, d\lambda.$$

Observe that $P_0^2 = P_0$, $P_0 A_s = 0$, and that P_0 commutes with A, A_s and $A^{s,o}$. Then the well-definedness of $A^{s,o}$ follows since

$$A^k A_{-k} = A_{-k} A^k = 1 - P_0.$$

The modified complex power $A^{s,o}$ has group property:

$$A^{s,o} A^{w,o} = A^{s+w,o}.$$

Following the arguments in [Sh] (pp. 94–106), we may show that the kernel $A^{-s,o}(x,y)\,dy$ of $A^{-s,o}$ can be meromorphically extended to all of $\mathbb{C}$, with simple poles contained in the set

$$\left\{ \frac{2-j}{2} \;\middle|\; j \geq 0 \right\} \setminus \{-j \mid j \geq 0\}.$$

It follows that the meromorphic function

$$\operatorname{Tr}(A^{-s,o}) = \int_M A^{-s,o}(x,x)\,dx$$

is regular at $s = 0$.

Definition 3.4. $\det'(A) = \exp(-\partial_s|_{s=0} \operatorname{Tr}(A^{-s,o}))$

As remarked by Kontsevich and Vishik in [KV], a change in the choice of contour Γ_θ changes $\partial_s|_{s=0} \operatorname{Tr} A^{-s,o}$ by an element of $2\pi i\mathbb{Z}$. After taking the exponential, the determinant $\det'(A)$ is well-defined.

We summarize our discussion in the following theorem.

Theorem 3.13. *There exists $\epsilon > 0$ such that $\det'(\Delta_{\mu,\nu})$ is defined on N_ϵ.*

3.2.2. *Holomorphy of* $\det'(\Delta_{\mu,\nu})$. Suppose A belongs to a differentiable family of operators all of which satisfy the above conditions for a fixed contour Γ. Then we have the following well-known variation formula for the determinant, which can be proved by symbol calculus of the kernel of complex powers as in [Sh].

$$d \log \det'(A) = \partial_s|_{s=0} \operatorname{Tr}(s A^{-s-1,o}\, dA) \tag{3.5}$$

In order to argue from (3.5) that $\det'(\Delta_{\mu,\nu})$ is holomorphic with respect to μ and ν, we must clarify one subtle point: the contour Γ must be chosen so that the spectrum of the operator $\Delta_{\mu,\nu}$ does not cross it as we perform the differentiation.

Fix $\mu_1, \nu_1 \in M(X_0)$ and $\delta > 0$. For complex numbers $|s|, |t| < \delta$, let

$$(\mu_s, \nu_t) = (\mu + s\mu_1,\ \nu + t\nu_1) \ \in N_\epsilon$$

and denote Δ_{μ_s,ν_t} by $A(s,t)$ and $\Delta_{\mu_s,\nu_t} - \lambda$ for $\lambda \in \Lambda$ by $A_\lambda(s,t)$.

Lemma 3.14. *If δ is sufficiently small, there exists $R > 0$ such that the resolvent $A_\lambda(s,t)^{-1}$ is bounded on*

$$\Lambda_R = \{\lambda \in \Lambda \mid |\lambda| \geq R\}.$$

Proof. Consider a parametrix $B_\lambda(s,t)$ of $A(s,t)$ and consider the equation

$$B_\lambda(s,t)A_\lambda(s,t) = I + C_\lambda(s,t),$$

where $C_\lambda(s,t)$ is a smoothing operator such that

$$(1 + |\lambda|)\|C_\lambda(s,t)\|$$

is bounded. (See [Sh] pp.85–86.) By continuity of the kernel of $C_\lambda(s,t)$ with respect to s,t, we see that $\|C_\lambda(s,t)\|$ is uniformly bounded for $|s|,|t| < \delta$, when δ is sufficiently small, and from this the existence of R. $\square$

The boundedness of the resolvent $A_\lambda(s,t)^{-1}$ is an open condition; thus, if the operator $A(0,0)$ has no eigenvalues in the bounded domain

$$\{z \in \Lambda_0 \mid \rho < |z| < R\},$$

then $A(s,t)$ has no eigenvalues in this domain either, for sufficiently small δ. Recall that the only eigenvalue of $A(s,t)$ inside the disk $\{z \mid |z| < \rho\}$ is 0, for sufficiently small δ.

In conclusion, for each $(\mu,\nu) \in N_\epsilon$ we can choose a contour Γ in such a way that the only eigenvalue of Δ_{μ_s,ν_t} inside Γ is zero, for any small variation (μ_s,ν_t) of (μ,ν) in N_ϵ. Since the determinant is independent of the choice of the contour, we have

Theorem 3.15. *The function* $\det'(\Delta_{\mu,\nu})$ *is holomorphic in the region N_ϵ, where ϵ is chosen as in Theorem 3.13.*

The property (3) in Theorem 1.2 is a direct consequence of this theorem. Note that the flat Euclidean norm $\|\cdot\|_{C^2(Q)}$ for M^G and the hyperbolic norm $\|\cdot\|_{C^2(X_0)}$ for $M(X_0)$ are equivalent since Q is a finite cover of compact X_0.

4. Estimates for quasiconformal mappings

We start by reviewing some basic facts about quasiconformal mappings due to Ahlfors and Bers [AB]. Given $p > 2$, let $C_p > 1$ be the constant associated to p by Ahlfors and Bers (see p. 386, [AB]); note that

$$\lim_{p \searrow 2} C_p = 1.$$

Fix $0 < k < 1$, and choose $p > 2$ such that $C_p < 1/k$. We abbreviate $L^p(\mathbb{C})$ to L^p. Let μ and ν be complex valued functions in $L^\infty(\mathbb{C})$ with norm $\|\mu\|_\infty, \|\nu\|_\infty \leq k$.

Definition 4.1. [AB] The normalized solution $w^\mu : \mathbb{C} \to \mathbb{C}$ of the Beltrami equation $\overline{\partial}_\mu w^\mu = 0$ is the unique continuous solution which fixes 0, 1, and ∞.

It is known that the function w^μ is a homeomorphism of the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. If $w^\nu = w^\rho \circ w^\mu$, then

$$\rho = \left(\frac{\nu - \mu}{1 - \overline{\mu}\nu}\frac{\partial w^\mu}{\overline{\partial w^\mu}}\right) \circ (w^\mu)^{-1},$$

and $(w^\mu)^{-1} = w^{\tilde{\mu}}$ where

$$\tilde{\mu} = \left(-\mu \frac{\partial w^\mu}{\overline{\partial w^\mu}}\right) \circ (w^\mu)^{-1}. \tag{4.1}$$

Note that $\|\tilde{\mu}\|_\infty = \|\mu\|_\infty$.

Denote the spherical distance in the extended complex plane by $[z_1, z_2]$. By Lemma 16 of [AB], there are positive constants $\alpha(k)$ and $c(k)$ such that

$$[w^\mu(z_1), w^\mu(z_2)] \leq c(k)\,[z_1, z_2]^{\alpha(k)}.$$

Let $D_R = \{z \in \mathbb{C} \mid |z| \leq R\}$ be the disk of radius R in $\mathbb{C}$. Since the spherical and Euclidean distances are equivalent in compact domains, we see that if $z_1, z_2 \in D_R$, then

$$|w^\mu(z_1) - w^\mu(z_2)| \leq c(k, R)\,|z_1 - z_2|^{\alpha(k)}. \tag{4.2}$$

In particular, taking $z_0 = 0$, we see that

$$\|w^\mu\|_{L^\infty(B_R)} \leq c(k, R)\,R^{\alpha(k)}. \tag{4.3}$$

We also have the following lemma. (See p. 398 of [AB].)

Lemma 4.1. *If μ and ν are Beltrami differentials on $\widehat{\mathbb{C}}$ with $\|\mu\|_\infty, \|\nu\|_\infty \leq k$, then for all $z \in \widehat{\mathbb{C}}$,*

$$[w^\mu(z), w^\nu(z)] \leq C(k)\,\|\mu - \nu\|_\infty.$$

In particular,

$$\|w^\mu - w^\nu\|_{L^\infty(D_R)} \leq C(k, R)\,\|\mu - \nu\|_\infty. \tag{4.4}$$

We will need the following interior Schauder estimates for the operators $\overline{\partial}_\mu$.

Proposition 4.2. *Fix a bounded open domain Ω in $\mathbb{C}$, a relatively compact open subset $\Omega_1 \subset\subset \Omega$, a positive integer n, and real numbers $0 < \delta < 1$, $0 < k < 1$, and $E > 0$. Let μ and ν be Beltrami differentials on $\mathbb{C}$ satisfying $\|\mu\|_\infty$, $\|\nu\|_\infty \leq k$ and $\|\mu\|_{C^{n-1,\delta}(\Omega)}$, $\|\nu\|_{C^{n-1,\delta}(\Omega)} \leq E$. Then there is a positive constant C, depending only on the above data, such that $\|w^\mu\|_{C^{n,\delta}(\Omega_1)} \leq C$ and*

$$\|w^\mu - w^\nu\|_{C^{n,\delta}(\Omega_1)} \leq C\{\|\mu - \nu\|_{C^{n-1,\delta}(\Omega)} + \|\mu - \nu\|_\infty\}.$$

Proof. As long as $\|\mu\|_{L^\infty(\Omega)}$ is bounded by $k < 1$, the operators $\overline{\partial}_\mu$ are uniformly elliptic on Ω, and we have the uniform Schauder estimates

$$\|w^\mu\|_{C^{n,\delta}(\Omega_1)} \leq C\|w^\mu\|_{C^0(\Omega)},$$

from which $\|w^\mu\|_{C^{n,\delta}(\Omega_1)} \leq C$ follows by (4.3).

Note that

$$\overline{\partial}_\mu(w^\mu - w^\nu) = (\mu - \nu)\partial w^\nu$$

This implies the uniform Schauder estimates

$$\|w^\mu - w^\nu\|_{C^{n,\delta}(\Omega_1)} \leq C\{\|w^\mu - w^\nu\|_{C^0(\Omega)} + \|(\mu - \nu)\partial w^\nu\|_{C^{n-1,\delta}(\Omega)}\},$$

and applying (4.4), the desired estimate on $\|w^\mu - w^\nu\|_{C^{n,\delta}(\Omega_1)}$ follows. $\square$

The goal of the rest of section is to verify the following theorem.

Theorem 4.3. *Let $\Omega_1 \subset\subset \Omega \subset\subset \mathbb{C}$. Suppose that $\|\mu\|_\infty \leq k$ and that $\|\partial\mu\|_{L^p(\Omega)} < \infty$. Then the normalized solution w^μ of the Beltrami equation of μ satisfies*

$$\inf_{\Omega_1} |\partial w^\mu| \geq C e^{-C\|\partial\mu\|_{L^p(\Omega)}}.$$

We will first consider the case where μ has compact support; we imitate the proof of Lemma 7 in [AB]. First, we recall some results from [AB] on the inhomogeneous Beltrami equation.

Definition 4.2. For $\sigma \in L^p$, let $w^{\mu,\sigma} : \mathbb{C} \to \mathbb{C}$ be the unique solution of the inhomogeneous Beltrami equation $\overline{\partial}_\mu w = \sigma$ such that $w(0) = 0$ and $\partial w \in L^p$.

Two properties of $w^{\mu,\sigma}$ which we will need are

$$\|\partial w^{\mu,\sigma}\|_p \leq \frac{C_p \|\sigma\|_p}{1 - kC_p} \tag{4.5}$$

and

$$|w^{\mu,\sigma}(z_1) - w^{\mu,\sigma}(z_2)| \leq \frac{c_p \|\sigma\|_p}{1 - kC_p} |z_1 - z_2|^{1-2/p}. \tag{4.6}$$

(For the definition of the constant c_p, see p. 386 of [AB].)

Lemma 4.4. *Suppose that $\|\mu\|_\infty \leq k$ and that $\partial\mu \in L^p$. If μ has support in D_R, there is a constant C, depending only on R, such that*

$$\inf_{z\in\mathbb{C}} |\partial w^\mu| \geq \frac{1}{1+k} e^{-C\|\partial\mu\|_p}.$$

Proof. Let $\lambda = w^{\mu,\partial\mu}$. By (4.5),

$$\|\overline{\partial}\lambda\|_p \leq C\|\partial\mu\|_p, \tag{4.7}$$

while by (4.6),

$$|\lambda(z_1) - \lambda(z_2)| \leq C\, \|\partial\mu\|_p |z_1 - z_2|^{1-2/p}.$$

Since $\lambda(0) = 0$,

$$|\lambda(z)| \leq C\, \|\partial\mu\|_p |z|^{1-2/p}. \tag{4.8}$$

In particular, when $|z| \leq R + 1$,

$$\|\lambda(z)\| \leq C(R+1)^{1-2/p}\|\partial\mu\|_p.$$

If $R + 1 < |z| < r$, then since $\overline{\partial}\lambda(z) = 0$ for $|z| > R$, Green's formula shows that

$$\lambda(z) = \frac{1}{2\pi i}\int_{|\zeta|=r} \frac{\lambda(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi i}\int\int_{D_R} \frac{\overline{\partial}\lambda(\zeta)}{\zeta - z} d\zeta d\overline{\zeta}.$$

Thus

$$d\lambda(z) = \frac{1}{2\pi i}\int_{|\zeta|=r} \frac{\lambda(\zeta)}{(\zeta - z)^2} d\zeta + \frac{1}{2\pi i}\int\int_{D_R} \frac{\overline{\partial}\lambda(\zeta)}{(\zeta - z)^2} d\zeta d\overline{\zeta}.$$

By (4.8),

$$\left|\frac{1}{2\pi i}\int_{|\zeta|=r}\frac{\lambda(\zeta)}{(\zeta-z)^2}d\zeta\right| \le \frac{1}{2\pi(r-|z|)^2}\int_{|\zeta|=r}|\lambda(\zeta)|\,|d\zeta| \le C\,\frac{r^{1-2/p}}{(r-|z|)^2}\|\partial\mu\|_p.$$

By (4.7),

$$\begin{aligned}\left|\frac{1}{2\pi i}\int\int_{D_R}\frac{\overline{\partial}\lambda(\zeta)}{(\zeta-z)^2}d\zeta d\overline{\zeta}\right| &\le \frac{1}{2\pi(|z|-R)^2}\int\int_{D_R}|\overline{\partial}\lambda(\zeta)|\,|d\zeta d\overline{\zeta}|\\ &\le C\frac{(\pi R^2)^{1-1/p}}{2\pi(|z|-R)^2}\|\partial\mu\|_{L^p(\Omega)}.\end{aligned}$$

Taking $r\to\infty$, we see that

$$|d\lambda(z)| \le C\frac{\|\partial\mu\|_p}{(|z|-R)^2}.$$

Let $\hat{z}=z/|z|$. It follows that

$$\begin{aligned}|\lambda(z)| &\le |\lambda((R+1)\hat{z})| + \int_{R+1}^{r}|d\lambda(s\hat{z})|\,ds\\ &\le C\|\partial\mu\|_p\left((R+1)^{1-1/p}+\int_1^\infty\frac{ds}{s^2}\right).\end{aligned}$$

In summary, we see that

$$\|\lambda\|_\infty \le C\|\partial\mu\|_p. \tag{4.9}$$

Let $\rho=e^\lambda$. Since $\overline{\partial}\rho=\partial(\mu\rho)$, there exists a C^1 function f such that $\partial f=\rho$ and $\overline{\partial}f=\mu\rho$. As remarked in [AB], f is a homeomorphism on the extended complex plane $\widehat{\mathbb{C}}$ and $f(\infty)=\infty$. Clearly, the normalized solution w^μ is

$$w^\mu(z)=\frac{f(z)-f(0)}{f(1)-f(0)},$$

hence

$$|\partial w^\mu| \ge \frac{e^{-|\lambda|}}{|f(1)-f(0)|}.$$

The numerator is bounded below by (4.9), while the denominator is bounded above using the mean value theorem:

$$\begin{aligned}|f(1)-f(0)| &\le \sup_D(|\partial f|+|\overline{\partial}f|)\\ &\le (1+k)\sup_D e^{|\lambda|} \le\ (1+k)e^{C\|\partial\mu\|_p}.\end{aligned}$$ □

Proof of Theorem 4.3. Choose an open set Ω' such that $\Omega_1\subset\subset\Omega'\subset\subset\Omega$. Let η be a C^∞ cut-off function which equals 1 on Ω' and 0 outside Ω. Let $\psi=w^{\eta\mu}\circ(w^\mu)^{-1}$. Note that $\overline{\partial}\psi=0$ on $w^\mu[\Omega']$. Thus, on Ω',

$$\partial w^{\eta\mu}=(\partial\psi\circ w^\mu)\partial w^\mu.$$

It follows that

$$|\partial w^{\mu}| = \frac{|\partial w^{\eta\mu}|}{|\partial\psi \circ w^{\mu}|}. \tag{4.10}$$

We must bound this below on Ω_1. The numerator is bounded below by Lemma 4.4.

To get an upper bound for the denominator of (4.10), note

$$\sup_{\Omega_1} |\partial\psi \circ w^{\mu}| = \sup_{w^{\mu}[\Omega_1]} |\partial\psi|.$$

Let $r = \operatorname{dist}(w^{\mu}[\Omega_1], \mathbb{C}\backslash w^{\mu}[\Omega'])$. Since ψ is holomorphic on $w^{\mu}[\Omega']$, we see that for $z \in w^{\mu}[\Omega_1]$,

$$\partial\psi(z) = \frac{1}{2\pi i}\int_{|\zeta - z| = r} \frac{\psi(\zeta)}{(\zeta - z)^2} d\zeta$$

and therefore,

$$\sup_{\Omega_1} |\partial\psi \circ w^{\mu}| \le r^{-1} \sup_{w^{\mu}[\Omega']} |\psi|.$$

But, by (4.3),

$$\sup_{w^{\mu}[\Omega']} |\psi| = \sup_{w^{\mu}[\Omega']} |w^{\eta\mu} \circ (w^{\mu})^{-1}| = \sup_{\Omega'} |w^{\eta\mu}| \le C.$$

It remains to bound r below.

Recall the definition (4.1) of the Beltrami differential $\tilde{\mu}$. By (4.2) and (4.3), if $z_1 \in w^{\mu}[\Omega_1]$ and $z_2 \in \mathbb{C} \setminus w^{\mu}[\Omega']$, there is a constant $\alpha(k) > 0$ such that

$$|w^{\tilde{\mu}}(z_1) - w^{\tilde{\mu}}(z_2)| \le C|z_1 - z_2|^{\alpha(k)}.$$

From this, we have

$$\operatorname{dist}(\Omega_1, \mathbb{C} \setminus \Omega') \le C \operatorname{dist}(w^{\mu}[\Omega_1], \mathbb{C} \setminus w^{\mu}[\Omega'])^{\alpha}.$$

So $r \ge C \operatorname{dist}(\Omega_1, \mathbb{C} \setminus \Omega)^{1/\alpha}$. □

5. Proof of Theorem 3.11

Recall that

$$\sigma_2(\Delta_{\mu,\nu}) = -(f_{\mu,\nu} - \overline{f_{\nu,\mu}})^2 \left((1 - \mu\overline{\nu})^2 \partial f_{\mu,\nu} \overline{\partial f_{\nu,\mu}}\right)^{-1} (\zeta - \mu\overline{\zeta})(\overline{\zeta} - \overline{\nu}\zeta).$$

By invariance of $\Delta_{\mu,\nu}$ under G we only need to estimate the argument of this symbol on P, and for this we will use the results of Section 4.

5.1. **Angle estimates for** $(\zeta - \mu\overline{\zeta})(\overline{\zeta} - \overline{\nu}\zeta)$**.**

Let $\varrho = \mu - \nu$. Then

$$\begin{aligned}(\zeta - \mu\overline{\zeta})(\overline{\zeta} - \overline{\nu}\zeta) &= (1 + \mu\overline{\nu})\zeta\overline{\zeta} - \mu\overline{\zeta}^2 - \overline{\nu}\zeta^2 \\ &= (1 + |\nu|^2)|\zeta|^2 - (\nu\overline{\zeta}^2 + \overline{\nu}\zeta^2) + \varrho\overline{\nu}|\zeta|^2 - \varrho\overline{\zeta}^2.\end{aligned}$$

But

$$(1 + |\nu|^2)|\zeta|^2 - (\nu\overline{\zeta}^2 + \overline{\nu}\zeta^2) \ge (1 - |\nu|)^2|\zeta|^2 \ge (1 - k)^2|\zeta|^2,$$

while on P,

$$|\varrho\overline{\nu}|\zeta|^2 - \varrho\overline{\zeta}^2| \le 2\epsilon|\zeta|^2.$$

Therefore, for sufficiently small ϵ, we have on P

$$|\arg((\zeta - \mu\overline{\zeta})(\overline{\zeta} - \overline{\nu}\zeta))| = O(\epsilon).$$

5.2. **Angle estimates for** $-(f_{\mu,\nu} - \overline{f_{\nu,\mu}})^2$**.** We estimate $\arg(-(f_{\mu,\nu} - \overline{f_{\nu,\mu}})^2)$ by means of the decomposition

$$f_{\mu,\nu} - \overline{f_{\nu,\mu}} = (f^\mu - \overline{f^\mu}) + (f_{\mu,\nu} - f^\mu) + (f^\mu - \overline{f_{\nu,\mu}}).$$

Lemma 5.1. *There is a constant $C = C(k,P) > 0$ such that if μ is a Beltrami differential such that $\|\mu\|_\infty \le k$, and $z \in P$,*

$$\operatorname{Im} f^\mu(z) > C.$$

Proof. Let $F(K)$ be the family of K-quasiconformal mappings from the $\mathbb{H}$ to itself fixing 0, 1 and ∞. In particular, $f^\mu \in F(K)$, with

$$K = \frac{1+k}{1-k},$$

By Theorem 2.1 of [Le], $F(K)$ is normal on $\mathbb{H}$, that is, every sequence of elements of $F(K)$ contains a subsequence which is locally uniformly convergent in $\mathbb{H}$. Let y be the infimum

$$y = \inf_{f \in F(K), z \in P} \operatorname{Im} f(z).$$

Choose sequence $(f_n) \in F(K)$ and $(z_n) \in P$ such that

$$\lim_{n\to\infty} \operatorname{Im} f_n(z_n) \to y.$$

Since P is compact, by passing to a subsequence, we may assume that (z_n) converges to a limit $z_\infty \in P$. Since $F(K)$ is normal, there is a subsequence which is locally uniformly convergent in $\mathbb{H}$, with continuous limit f_∞ such that $\operatorname{Im} f_\infty(z_\infty) = y$. By Theorem 2.2 of [Le], f_∞ is K-quasiconformal, hence injective. Thus, $y > 0$, since $f_\infty(z_\infty)$ is in the interior of D. □

It follows that

$$\inf_P |f^\mu - \overline{f^\mu}| \ge C.$$

By (4.4),

$$\|f^\mu - \overline{f_{\nu,\mu}}\|_{L^\infty(P)} + \|f_{\mu,\nu} - f^\mu\|_{L^\infty(P)} \le C\|\varrho\|_\infty \le C\epsilon.$$

Therefore, for sufficiently small ϵ, we have on P,

$$|\arg(-(f_{\mu,\nu} - \overline{f_{\nu,\mu}})^2)| = O(\epsilon).$$

5.3. **Angle estimates for** $1 - \mu\overline{\nu}$**.** We have

$$1 - \mu\overline{\nu} = 1 - |\nu|^2 - \varrho\overline{\nu}.$$

Since $1 - |\nu|^2 \ge 1 - k^2$ and $|\varrho\overline{\nu}| \le \epsilon k$, we see that

$$|\arg(1 - \mu\overline{\nu})| = O(\epsilon).$$

5.4. **Angle estimates for $\partial f_{\mu,\nu}\overline{\partial} f_{\mu,\nu}$.** To estimate the argument of

$$\begin{aligned}\partial f_{\mu,\nu}\overline{\partial f_{\nu,\mu}} &= (\partial f^\mu + \partial f_{\mu,\nu} - \partial f^\mu)(\overline{\partial f^\mu} + \overline{\partial f_{\nu,\mu}} - \overline{\partial f^\mu}) \\ &= |\partial f^\mu|^2 + \partial f^\mu(\overline{\partial f_{\nu,\mu}} - \overline{\partial f^\mu}) + \overline{\partial f^\mu}(\partial f_{\mu,\nu} - \partial f^\mu) \\ &+ (\partial f_{\mu,\nu} - \partial f^\mu)(\overline{\partial f_{\nu,\mu}} - \overline{\partial f^\mu}),\end{aligned}$$

we need a lower bound for $|\partial f^\mu|$ and upper bounds for ∂f^μ, $\partial f_{\mu,\nu} - \partial f^\mu$ and $\overline{\partial f_{\nu,\mu}} - \overline{\partial} f^\mu$. Theorem 4.3, applied with $\Omega_1 = P$ and $\Omega = Q$, implies that

$$\inf_P |\partial f^\mu| \geq C.$$

By Proposition 4.2, we have the estimates $\|f^\mu\|_{C^{1,\delta}(P)} < C$,

$$\|f_{\mu,\nu} - f^\mu\|_{C^{1,\delta}(P)} < C\|\mu - \nu\|_\infty,$$

and $\|f_{\nu,\mu} - f^\mu\|_{C^{1,\delta}(P)} < C(\|\mu - \nu\|_{C^1(Q)} + \|\mu - \nu\|_\infty)$. Therefore, for sufficiently small ϵ, we have on P,

$$|\arg(\partial f_{\mu,\nu}\overline{\partial f_{\nu,\mu}})| = O(\epsilon).$$

Combining the above estimates, we obtain Theorem 3.11.

6. Proof of Theorem 3.12

Before we proceed for the proof let us fix some notations. Let m_0 be the Kähler form of the standard hyperbolic metric on $\mathbb{H}$, let

$$m = (f^\mu)^* m_0 = -2i\,\frac{|\partial f^\mu|^2(1 - |\mu|^2)}{(f^\mu - \overline{f^\mu})^2} dz \wedge d\overline{z}$$

be the Kähler form of the pull-back hyperbolic metric by f^μ induced on X_0, and let Δ_m be the corresponding Laplacian. Let $\langle -, - \rangle$ be the inner product on $L^2(X_0, m)$, and let $\|\cdot\|_2$ be the L^2-norm. With respect to the frame $\{dz, d\overline{z}\}$ of the cotangent bundle $T^*X_0 \otimes \mathbb{C}$, the Hodge star operator $\star$ (with respect to m) acts on 1-forms as

$$\star\begin{pmatrix} a \\ b \end{pmatrix} = \frac{i}{(1 - |\mu|^2)^2}\begin{pmatrix} 2\overline{\mu} & -(1 + |\mu|^2) \\ |\mu|^2 + 1 & -2\mu \end{pmatrix}\begin{pmatrix} \bar{a} \\ \bar{b} \end{pmatrix}.$$

From this, it is easy to see that for $u \in C^\infty(X_0)$,

$$\begin{aligned}\|du\|_2^2 = \int_{X_0} du \wedge \star du &\geq \int_P (1 - |\mu|^2)(|\partial u|^2 + |\overline{\partial} u|^2)\,\frac{i\,dz \wedge d\overline{z}}{2} \\ &\geq (1 - k)\|du\|^2_{L^2(X_0, m_0)}.\end{aligned} \tag{6.1}$$

If ∇ is the gradient operator of the metric m, and ∇^* is its adjoint with respect to $\langle -, - \rangle$, then $\Delta_m = \nabla^*\nabla$; it follows that $\langle \Delta_m u, u \rangle = \|\nabla u\|_2^2$.

Lemma 6.1. *Let $\epsilon = \|\mu - \nu\|_{C^2(Q)}$. Write $f = O_i(\epsilon^\ell)$ to denote that f is a C^i function (or tensor) such that $\|f\|_{C^i(P)} \leq C(k, E)\epsilon^\ell$. Then we have*

$$\begin{aligned}&f_{\mu,\nu} = O_2(1), \quad (\partial f_{\mu,\nu})^{-1} = O_0(1), \quad f_{\mu,\nu} - f^\mu = O_2(\epsilon) \quad f_{\mu,\nu} - f^\nu = O_2(\epsilon), \\ &\alpha_{\mu,\nu} = O_1(1), \quad \overline{\partial}_\mu \log \alpha_{\mu,\nu} = O_0(1), \qquad m/m_0 = O_0(1);\end{aligned} \tag{1}$$

$$f_{\mu,\nu} - \overline{f_{\nu,\mu}} = f^\mu - \overline{f^\mu} + O_2(\epsilon), \qquad \alpha_{\mu,\nu} = \alpha + O_1(\epsilon) \tag{2}$$
$$\overline{\partial}_\mu \log \alpha_{\mu,\nu} = \overline{\partial}_\mu \log \alpha + O_0(\epsilon), \quad \partial_{\overline{\nu}} \log \overline{\alpha_{\nu,\mu}} = \partial_{\overline{\mu}} \log \overline{\alpha} + O_0(\epsilon);$$

$$\Delta_{\mu,\nu} = \Delta_m + O_1(\epsilon)\overline{\partial}\partial + O_1(\epsilon)\overline{\partial}^2 + O_0(\epsilon)\partial + O_0(\epsilon)\overline{\partial}. \tag{3}$$

Proof. (1) is by Proposition 4.2 and by Theorem 4.3, (2) is by straightforward calculation using (1), and (3) is by (1) and (2) and the definition of $\Delta_{\mu,\nu}$ (see Definition 3.2). □

From Lemma 6.1 (3), we may write

$$\Delta_{\mu,\nu} = \Delta_m + O_1(\epsilon)\nabla^2 + O_0(\epsilon)\nabla,$$

where $O_i(\epsilon)$ is a tensor on X_0 whose C^i-norm is bounded by ϵ. Localization (by a partition of unity) and integration by parts shows that

$$\langle u, \Delta_{\mu,\nu} u\rangle = \big(1 + O(\epsilon)\big)\|\nabla u\|^2 + O(\epsilon)\|u\|_2^2. \tag{6.2}$$

Let U be the space of constant functions on X_0, let $U^\perp$ be its orthogonal complement in $L^2(X_0, m)$, and let $\Delta^*_{\mu,\nu}$ be the adjoint of $\Delta_{\mu,\nu}$ with respect to the metric m. If $f \in C^\infty(X_0)$, $\Delta^*_{\mu,\nu} f \in U^\perp$. Therefore, every eigenfunction u of $\Delta^*_{\mu,\nu}$ with nonzero eigenvalue λ belongs to $U^\perp$. If we let

$$v = u - \frac{\int_{X_0} u\, m_0}{\int_{X_0} m_0},$$

then clearly,

$$\frac{\|dv\|_2^2}{\|v\|_2^2} \le \frac{\|du\|_2^2}{\|u\|_2^2} = \frac{\|\nabla u\|_2^2}{\|u\|_2^2}.$$

By (6.1),

$$\|dv\|^2_{L^2(X_0, m_0)} \lesssim \|dv\|_2^2.$$

Since m and m_0 are equivalent metrics, that is, $C^{-1}m_0 \le m \le C m_0$, we see that

$$\|v\|_2^2 = \int_{X_0} |v|^2 m \sim \int_{X_0} |v|^2\, m_0.$$

By the Poincaré inequality applied to v for the metric m_0 on X_0, we see that

$$0 < C \le \frac{\|\nabla u\|_2^2}{\|u\|_2^2}, \tag{6.3}$$

where the bound C depends only on k and E.

Since u is an eigenfunction of $\Delta^*_{\mu,\nu}$ with nonzero eigenvalue λ, we have by (6.2),

$$\begin{aligned} |\lambda|\, \|u\|_2^2 &= |\langle u, \Delta^*_{\mu,\nu} u\rangle| = |\langle \Delta_{\mu,\nu} u, u\rangle| \\ &\ge \big(1 - O(\epsilon)\big)\|\nabla u\|_2^2 - O(\epsilon)\|u\|_2^2. \end{aligned}$$

Therefore by the Poincaré inequality (6.3), we see that for sufficiently small ϵ,

$$|\lambda| \ge C - O(\epsilon).$$

This completes the proof of Theorem 3.12, since the spectrum of $\Delta^*_{\mu,\nu}$ is the complex conjugate of the spectrum of $\Delta_{\mu,\nu}$.

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