UNIVERSITY OF TORONTO
DEPARTMENT OF MATHEMATICS
MAT 235 Y — CALCULUS II
FALL–WINTER 2007–2008
ASSIGNMENT #6, DUE ON March 13
COVER PAGE

INSTRUCTIONS: Submit this page as your cover page. Use different pages for each of the assigned problems. Staple all your papers and clearly write your name and student number in each of the pages. Show and explain all the steps in each of your solutions. Marks will be deduced for messy or unclear solutions. Late submissions will not be accepted.

NAME: ______________________________ , ____________________
(family name, please print) (given name)

STUDENT NUMBER: ____________________________

SECTION: (please choose one)

SECTION	DAY/TIME	ROOM	INSTRUCTOR	MY SECTION
L–0101	MWF/9	SS–2135	Y. Kim	
L–0201	MWF/2	SS–1073	R. Pujol	
L–0301	MWF/2	RW–110	B. Hovinen	
L–501	R/6-9	LM–161	F. Recio	

PLEASE DO NOT WRITE HERE

Question 1 (out of 5)	Question 2 (out of 3)	Question 3 (out of 4)	Question 4 (out of 3)	TOTAL (out of 15)

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PROBLEMS

DO NOT SUBMIT YOUR SOLUTIONS WITHOUT THE COVER PAGE. READ THE INSTRUCTIONS WRITTEN ON THAT PAGE.

1. (Gravitational attraction of a large ball) Let B be a very large metal ball with the radius R (imagine the Earth!) that has mass density depending only on the distance from its center. Let M be the total mass of B. Let a point object P with unit mass is located at the distance D from the center of B and assume that D is greater than R: in other words, P is outside the ball B (imagine a satellite in the sky). Note that by Newton's law of gravitation, for two point objects P_1 and P_2 having mass m_1 and m_2 respectively, there is the gravitational force attracts the two point objects with the magnitude of force equal to $G\frac{m_1 m_2}{d^2}$, where d denotes the distance between the two object and G is the gravitational constant. In the following you are asked to prove that the gravitational force $\vec{F}$ of B on P has the magnitude

$$(*) : \qquad |\vec{F}| = \frac{GM}{D^2}.$$

Note that the ball B is large so you cannot consider it as a point even approximately! (This exercise is one of the fundamental computations in Astrophysics, where the stars or planets are often considered as point masses in the description of their dynamics. To verify this result, Isaac Newton had to postpone publishing his *Principia*.)

(a) (3 marks). It is often convenient to use a potential function of the force $\vec{F}$, i.e. a function φ such that $\vec{F} = -\nabla\varphi$. Consider a shell of B, i.e. the sphere S_ρ of radius ρ ($0 \le \rho \le R$), whose center is the same as the ball B. Suppose this shell S_ρ has an infinitesimal width $d\rho$. On this surface S_ρ the mass density is constant. Let $f(\rho)$ denote this density (it is the function of only ρ). Thus the total mass dM of S_ρ is

$$dM = f(\rho) \cdot \text{ the surface area of } S_\rho \cdot d\rho.$$

The infinitesimal gravitational potential $d\varphi$ at P of the gravitational force of the shell S_ρ can be computed by

$$d\varphi = G\, f(\rho)\, d\rho \iint_{S_\rho} \frac{1}{\delta}\, dS,$$

where δ denotes the distance of a point on the shell S_ρ to the point P. Show that

$$d\varphi = \frac{G}{D} dM.$$

Hint: You may want to use the law of cosines:

$$c^2 = a^2 - 2ab\cos C + b^2,$$

where a,b,c are the side lengths of a triangle with the corresponding (opposite) angles A, B and C.

(b) (1 mark). Use (a) to compute the potential φ at P of the gravitational force of the ball B. Here one can use the superposition principle of the gravitational potential that $\varphi = \int d\varphi$, i.e. the potential φ of the ball is the sum of the potentials of all the infinitesimal shells.

(c) (1 mark). Use (b) to deduce $(*)$.

2. (Cycloid and line integrals)
Let C be the cycloid in the plane given by the vector-valued function

$$\mathbf{r}(t) = (t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j} \ , \quad 0 \le t \le 2\pi.$$

(a) (1 mark). Find the area of the region enclosed by C and the x-axis.

(b) (2 marks). Let $\mathbf{F}(x, y) = (-y + ye^{xy})\mathbf{i} + (x + xe^{xy})\mathbf{j}$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

3. (Tubular shape). (For this exercise, you may want to practice with the torus example as in #60 in p.1081 in the textbook.) Let C be the cycloid given by the vector-valued function

$$\mathbf{r}(t) = \langle 0, t - \sin t, 1 - \cos t \rangle, \quad 0 \le t \le 2\pi$$

in the three-dimensional space $\mathbf{R}^3$. Consider the unit disk D in the xy-plane in $\mathbf{R}^3$, i.e.

$$D = \{(x, y, 0) \mid x^2 + y^2 \le 1\}.$$

Let this disk move in the space in the following way: the center moves along the cycloid C for $0 \le t \le 2\pi$, and at each moment the disk is perpendicular to the unit tangent vector of the cycloid. The trajectory of this moving disk is a solid tubular region T around C. Let S denote the boundary surface of this solid region except the initial and final disks, i.e. S is like a twisted tube.

(a) (1 mark). Give parametric equations for S and for T.

(b) (1 mark). Use (a) to write down a double integral for the surface area of S. But, do not compute this area.

(c) (1 mark). Use (a) to write down a triple integral for the volume of T. But, do not compute this volume.

(d) (1 mark). Use (a) to evaluate the surface integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S},$$

where $\mathbf{F}(x, y, z) =< 0, 0, x >$ and the surface S is oriented with outward normal vector.

4. (Line integrals of closed paths)
Let

$$\mathbf{F}(x, y) = \frac{-y}{x^2 + y^2}\mathbf{i} + \frac{x}{x^2 + y^2}\mathbf{j}.$$

(a) (2 mark). Let C be the counter-clockwisely oriented curve given by the polar equation $r = 2 + \cos\theta, \ 0 \le \theta \le 2\pi$. Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$. Here, $\oint$ denotes the line integral along a closed path.
<u>Hint:</u> One may want to use Green's theorem.

(b) (1 marks). Let C be the counter-clockwisely oriented curve given by the polar equation $r = 2 + \cos(7\,\theta), \ 0 \le \theta \le 10\pi$. Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$.