UNIVERSITY OF TORONTO
DEPARTMENT OF MATHEMATICS
MAT 235 Y — CALCULUS II
FALL–WINTER 2007–2008
ASSIGNMENT #4, DUE ON January 10
COVER PAGE

INSTRUCTIONS: Submit this page as your cover page. Use different pages for each of the assigned problems. Staple all your papers and clearly write your name and student number in each of the pages. Show and explain all the steps in each of your solutions. Marks will be deduced for messy or unclear solutions. Late submissions will not be accepted.

NAME: ________________________________ , ______________________
(family name, please print) (given name)

STUDENT NUMBER: ___________________________

SECTION: (please choose one)

SECTION	DAY/TIME	ROOM	INSTRUCTOR	MY SECTION
L–0101	MWF/9	SS–2135	Y. Kim	
L–0201	MWF/2	SS–1073	R. Pujol	
L–0301	MWF/2	RW–110	B. Hovinen	
L–501	R/6-9	LM–161	F. Recio	

PLEASE DO NOT WRITE HERE

Question 1 (out of 3)	Question 2 (out of 3)	Question 3 (out of 3)	Question 4 (out of 3)	Question 5 (out of 3)	TOTAL (out of 15)

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PROBLEMS

<u>DO NOT SUBMIT YOUR SOLUTIONS WITHOUT THE COVER PAGE.</u>
<u>READ THE INSTRUCTIONS WRITTEN ON THAT PAGE.</u>
<u>THERE ARE FIVE PROBLEMS WITH SEVERAL SUBPROBLEMS.</u>

1. (The gradient of a function) Consider a mountain totally covered with snow which has the height $M(x, y) = 10000 - 2x^2 - y^2$ at a point (x, y) on a map. Here the boundary of the mountain is the set $\{(x, y) \in \mathbf{R}^2 \mid 10000 - 2x^2 - y^2 = 0\}$. Suppose at the map location $(1, 1)$ (so the actual location on the mountain is $(1, 1, 9997)$) a big ice ball starts rolling down. It is a free motion which is determined totally by the slope of the surface. For example, the initial direction of the motion is completely determined by the slope of the mountain at the initial point $(1, 1)$. If you want to find the ice ball at the boundary of the mountain after certain time, where should you go to? Determine the (x, y)-coordinates of the location.

<u>Hint</u>: You want to find the path of the ice ball on the map and want to find the point where the path intersects the boundary of the mountain. (You only need to know the shape of path, not the actual motion in time.) For the path, first set up some differential equation by trying to relate the velocity of the path with the function $M(x, y)$. At certain moment in your calculation, you may want to use the following fact:

$$\text{If } \frac{d}{ds}x(s) = Cx(s),\ C \text{ a constant, then } x(s) = x(0)e^{C\,s}.$$

2. (Minimum of a function) Let $A(0, 0)$, $B(6, 0)$, $C(2, 3)$ be three points in the xy-plane.

(a) Minimize the average distance squared from the three points A, B and C. More precisely, let $F(x, y) = d_A(x, y)^2 + d_B(x, y)^2 + d_C(x, y)^2$, where $d_A(x, y)$, $d_B(x, y)$, $d_C(x, y)$ denote the distance from (x, y) to the points A, B, and C, respectively. Find the point (x_m, y_m) where F attains its minimum.

(b) Minimize the average distance squared from the three points A, B and C, where the point (x, y) is restricted to the domain D defined as follows:

$$D := \{(x, y) \in \mathbf{R}^2 \mid (x - 6)^2 + (y - 3)^2 \leq 4\ ,\ x \leq 6, \text{ and } y \leq 3\}.$$

<u>Hint</u>: You may want to use Lagrange multiplier method for the curved boundary of D.

3. (Minimum of a function) Let $A(-1, 1)$, $B(1, 2)$ be two points in the xy-plane. Let L denote the x-axis. Minimize the average distance from A, B, and the line L. More precisely, let $d_A(x, y)$, $d_B(x, y)$, and $d_L(x, y)$ denote the distance (not squared) from (x, y) to the points A, B, and the line L respectively. Find the minimum point (x_m, y_m) of the function $\delta(x, y) = d_A(x, y) + d_B(x, y) + d_L(x, y)$ following the outline below.

(a) For each functions d_A, d_B, d_L, find the gradient and the set of *singular points*, i.e the set of points where the gradient is not defined. Give an interpretation of the direction and the magnitude of the gradient vector for each d_A, d_B and d_L.

(b) Use (a) to find the critical points of $\delta(x,y)$, i.e. the point where $\nabla\delta(x,y)=0$.

(c) Check the values of $\delta(x,y)$ at its singular points. Compare them with the value (or values) of the critical points. Make conclusion.

4.
(a) **(local minimum, local maximum, and saddle point)** For the function $f(x,y) = x^4 - xy + y^4$, find all the critical points (the points where the gradient vanishes). Determine for each critical point whether it is a local minimum, a local maximum or a saddle point.

(b) **(absolute maximum and minimum)** Find the absolute maximum and minimum values of f in (a) if there are any. Justify your answer.

(c) **(Least squares)** What numbers x, y come closest to satisfying the three equations $x-y=1$, $2x+y=-1$, $x+2y=1$? Square and add the errors, $(x-y-1)^2+A+B$. Then minimize. (Find appropriate A and B, and then finish the minimization.)

5. (Lagrange Multipliers) For (a) & (b) use Lagrange multipliers to find the maximum and minimum values of f subject to the given constraints.

(a) $f(x,y)=\frac{1}{x}+\frac{1}{y}$. Constraint: $\frac{1}{x^2}+\frac{1}{y^2}=1$.

(b) $f(x,y)=x^2+2y^2+3z^2$. Constraint: $x+y+z=1$ and $x-y+2z=2$.

(c) A package in the shape of a rectangular box can be mailed by the Toronto-Imaginary Post Service if the sum of its length and girth (the perimeter of a cross-section perpendicular to the length) is at most 108 cm. Find the dimensions of the package with largest volume that can be mailed.