First Name: ______________________ Last Name: ______________________

Student-No: ______________________ Section: ______________________

Grade:

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Indefinite Integrals

1. 9 marks Each part is worth 3 marks. Please write your answers in the boxes.

(a) Calculate the indefinite integral $\int e^{-x}\sqrt{1+e^{-x}}\,dx$.

Answer: $I = -\frac{2}{3}\left(1+e^{-x}\right)^{3/2} + C$

Solution:

Method I: Let $u = 1 + e^{-x}$, so that $e^{-x}dx = -du$. Then,

$$I = -\int \sqrt{u}\,du = -\frac{2}{3}u^{3/2} + C\,.$$

Using $u = 1 + e^{-x}$ we get $I = -\frac{2}{3}\left(1+e^{-x}\right)^{3/2} + C$.

Method II: Let $u = \sqrt{1+e^{-x}}$, so that $u^2 = 1 + e^{-x}$. Differentiating implicitly gives $e^{-x}\,dx = -2u\,du$. Then,

$$I = -2\int u^2\,du = -\frac{2}{3}u^3 + C\,.$$

Using $u = \sqrt{1+e^{-x}}$ we get $I = -\frac{2}{3}\left(1+e^{-x}\right)^{3/2} + C$.

(b) Calculate the indefinite integral $\int (x+1)e^{-x}\,dx$ for $x > 0$.

Answer: $I = -(x+2)e^{-x} + C$.

Solution:

Method I: Let $u = (x+1)$ and $dv/dx = e^{-x}$. We calculate $du/dx = 1$ and $v = -e^{-x}$, so that one step of integration by parts gives

$$I = uv - \int v\frac{du}{dx}\,dx = -(x+1)e^{-x} + \int e^{-x}\,dx = -(x+1)e^{-x} - e^{-x} + C\,.$$

Method II: First, write

$$\int (x+1)e^{-x}\,dx = \int xe^{-x}\,dx + \int e^{-x}\,dx = \int xe^{-x}\,dx - e^{-x}\,.$$

Let $u = e^{-x}$, so that $e^{-x}\,dx = -du$ and $\log u = -x$. Then,

$$\int xe^{-x}\,dx = \int \log(u)\,du = u\log(u) - u + C\,.$$

Using $u = e^{-x}$ we get $\int xe^{-x}\,dx = -xe^{-x} - e^{-x} + C$ and hence

$$\int (x+1)e^{-x}\,dx = -(x+2)e^{-x} + C\,.$$

(c) (A Little Harder): Calculate the indefinite integral $\int \tan^5(x)\sec^3(x)\,dx$.

Answer: $I = \frac{\sec^7(x)}{7} - \frac{2\sec^5(x)}{5} + \frac{\sec^3(x)}{3} + C$.

Solution:

Method I: Let $u = \sec x$, so that $du = \tan(x)\sec(x)\,dx$. Then, using the identity $\tan^2(x) = \sec^2(x) - 1 = u^2 - 1$,

$$I = \int \left(u^2-1\right)^2 u^2\,du = \int \left(u^6 - 2u^4 + u^2\right)\,du = \frac{u^7}{7} - \frac{2u^5}{5} + \frac{u^3}{3} + C.$$

Using $u = \sec(x)$ we get $I = \frac{\sec^7(x)}{7} - \frac{2\sec^5(x)}{5} + \frac{\sec^3(x)}{3} + C$.

Method II: First note that

$$I = \int \frac{\sin^5(x)}{\cos^8(x)} dx.$$

Let $u = \cos x$, so that $du = -\sin(x)\,dx$. Then, using the identity $\sin^2(x) = 1 - \cos^2(x) = 1 - u^2$,

$$I = -\int \frac{(1-u^2)^2}{u^8}\,du = -\int \left(\frac{1}{u^8} - \frac{2}{u^6} + \frac{1}{u^4}\right)\,du = \frac{1}{7u^7} - \frac{2}{5u^5} + \frac{1}{3u^3} + C.$$

Using $\frac{1}{u} = \sec(x)$ we get $I = \frac{\sec^7(x)}{7} - \frac{2\sec^5(x)}{5} + \frac{\sec^3(x)}{3} + C$.

Method III: Let $u = \sec^2(x)$, so that $\sec^2(x)\tan(x)\,dx = \frac{1}{2}\,du$. We assume that $\sec(x) > 0$ so that $\sqrt{u} = \sec(x)$. The case of $\sec(x) < 0$ follows by plugging $\sqrt{u} = -\sec(x)$. Then, using the identity $\tan^2(x) = \sec^2(x) - 1 = u - 1$, we get

$$I = \frac{1}{2}\int (u-1)^2\sqrt{u}\,du = \frac{1}{2}\int \left(u^{5/2} - 2u^{3/2} + u^{1/2}\right)\,du = \frac{u^{7/2}}{7} - \frac{2u^{5/2}}{5} + \frac{u^{3/2}}{3} + C\,.$$

Using $\sqrt{u} = \sec(x)$ we get $I = \frac{\sec^7(x)}{7} - \frac{2\sec^5(x)}{5} + \frac{\sec^3(x)}{3} + C$.

Definite Integrals

2. 12 marks Each part is worth 4 marks. Please write your answers in the boxes.

(a) Calculate $I = \int_0^{\pi/8} \sin^2(2x)\, dx$.

Answer: $I = \frac{\pi}{16} - \frac{1}{8}$.

Solution: Use $\sin^2(x) = \frac{1-\cos(2x)}{2}$ to get

$$I = \int_0^{\pi/8} \frac{1-\cos(4x)}{2}\, dx = \left[\frac{x}{2} - \frac{\sin(4x)}{8}\right|_{x=0}^{\pi/8} = \frac{\pi}{16} - \frac{1}{8}.$$

(b) Calculate $I = \int_1^e x^2 \ln x\, dx$.

Answer: $I = \frac{2e^3}{9} + \frac{1}{9}$.

Solution: Let $u = \ln x$ and $dv/dx = x^2$. We calculate $du/dx = \frac{1}{x}$ and $v = \frac{x^3}{3}$, so that one step of integration by parts gives

$$I = \frac{x^3}{3} \ln x|_{x=1}^e - \int_1^e \frac{x^2}{3}\, dx = \frac{e^3}{3} - \frac{x^3}{9}|_{x=1}^e = \frac{e^3}{3} - \frac{e^3}{9} + \frac{1}{9} = \frac{2e^3}{9} + \frac{1}{9}.$$

(c) (A Little Harder): Calculate $I = \int_0^\infty e^{-x}\sin(x)\,dx$.

Answer: $I = \frac{1}{2}$.

Solution: We first recall that

$$I = \int_0^\infty e^{-x}\sin(x)\,dx = \lim_{L\to\infty}\int_0^L e^{-x}\sin(x)\,dx\,.$$

We compute the indefinite integral $\int e^{-x}\sin(x)\,dx$. Let $u = \sin(x)$ and $dv/dx = e^{-x}$. We calculate $du/dx = \cos(x)$ and $v = -e^{-x}$, so that one step of integration by parts gives

$$\int e^{-x}\sin(x)\,dx = -e^{-x}\sin(x) + \int e^{-x}\cos(x)\,dx\,.$$

Let $u = \cos(x)$ and $dv/dx = e^{-x}$. We calculate $du/dx = -\sin(x)$ and $v = -e^{-x}$, so that another step of integration by parts gives

$$\int e^{-x}\sin(x)\,dx = -e^{-x}\sin(x) - e^{-x}\cos(x) - \int e^{-x}\sin(x)\,dx$$

Moving $\int e^{-x}\sin(x)\,dx$ to the left-hand-side and dividing by 2 yields

$$\int e^{-x}\sin(x)\,dx = \frac{-e^{-x}}{2}(\sin(x)+\cos(x))\,.$$

It follows that

$$I = \lim_{L\to\infty}\left[\frac{-e^{-x}}{2}(\sin(x)+\cos(x))\right]\Big|_{x=0}^{L} = \lim_{L\to\infty}\left[\frac{1}{2} - \frac{e^{-L}(\sin(L)+\cos(L))}{2}\right] = \frac{1}{2}\,.$$

Riemann Sum, FTC, and Volumes

3. 12 marks Each part is worth 4 marks. Please write your answers in the boxes.

(a) Calculate the infinite sum

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{8i}{n^2}e^{-4i^2/n^2}$$

by first writing it as a definite integral. Then, **evaluate this integral**.

Answer: $1 - e^{-4}$

Solution: We identify $a = 0$, $b = 1$, $\Delta x = 1/n$, $x_i = i/n$, and $f(x_i) = 8x_i e^{-4x_i^2}$. This yields

$$S \equiv \lim_{n\to\infty}\sum_{i=1}^{n}\frac{8i}{n^2}e^{-4i^2/n^2} = \lim_{n\to\infty}\sum_{i=1}^{n}(\Delta x)f(x_i) = \int_0^1 8xe^{-4x^2}\,dx\,.$$

To calculate the integral we let $u = 4x^2$, which yields

$$S = -e^{-4x^2}\Big|_{x=0}^{1} = 1 - e^{-4}\,.$$

(b) For $x > 0$ define $F(x) = \int_1^x t^{-1/2}\,dt$ and $g(x) = \sqrt{F(x^2)}$. Calculate $g'(2)$.

Answer: $g'(2) = \frac{1}{\sqrt{2}}$.

Solution: We use the product rule to get $g'(x) = \frac{1}{2\sqrt{F(x^2)}}F'(x^2)2x$. Now, by FTC I, we get $F'(x^2) = (x^2)^{-1/2} = \frac{1}{x}$. This yields,

$$g'(x) = \frac{1}{\sqrt{F(x^2)}}\,. \tag{1}$$

Now, let $x = 2$ and calculate that

$$F(4) = \int_1^4 t^{-1/2}\,dt = \left[2\sqrt{t}\right|_{t=1}^{4} = 4 - 2 = 2\,.$$

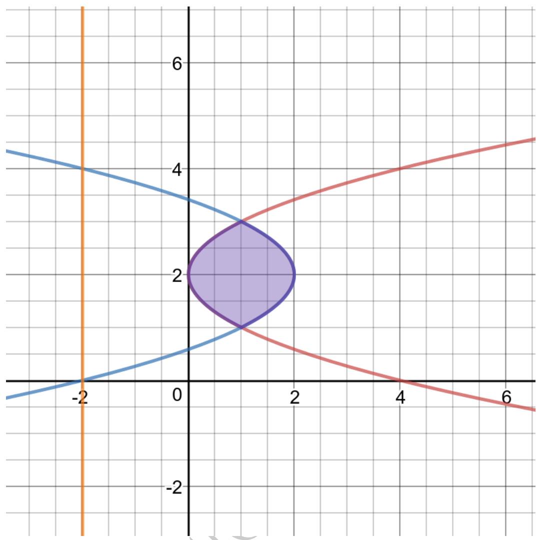
Therefore, from (1), we get

$$g'(2) = \frac{1}{\sqrt{F(4)}} = \frac{1}{\sqrt{2}}\,.$$

(c) Write a definite integral, with specified limits of integration, for the volume obtained by revolving the bounded region between $x = (y-2)^2$ and $x = 2-(y-2)^2$ about the vertical line $x = -2$. **Do not evaluate the integral**.

Answer: $V = 12\pi \int_1^3 \left[1-(y-2)^2\right]\, dy$

Solution:

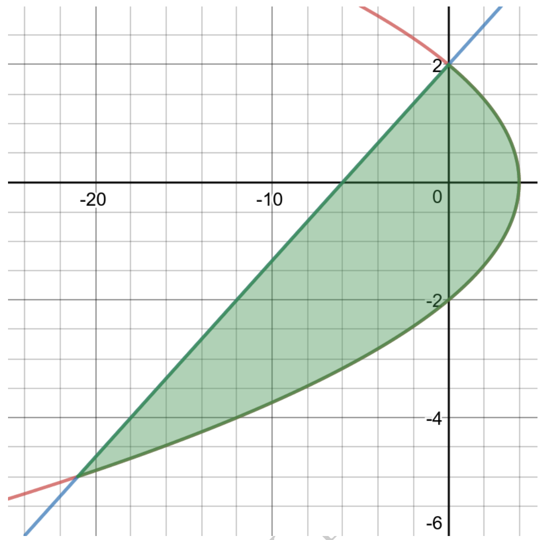


The two curves intersect when $(y-2)^2 = 2-(y-2)^2$, which yields $(y-2)^2 = 1$ so that $y-2 = \pm 1$. This gives $y = 1$ and $y = 3$ for both $x = 1$. The intersection points are in the first quadrant. Define $x_R = 2-(y-2)^2$ (right blue curve) and $x_L = (y-2)^2$ (left red curve). Then, at each y in $[1, 3]$, we have that $(x_R + 2)$ and $(x_L + 2)$ are the distances of the two curves from the axis of rotation $x = -2$ shown by the orange curve. This yields

$$\begin{aligned} V &= \pi \int_1^3 \left[(x_R+2)^2 - (x_L+2)^2\right]\, dy\,, \\ &= \pi \int_1^3 \left[\left(4-(y-2)^2\right)^2 - \left((y-2)^2+2\right)^2\right]\, dy\,, \\ &= 12\pi \int_1^3 \left[1-(y-2)^2\right]\, dy\,. \end{aligned}$$

4. (a) [2 marks] Plot the finite area enclosed by $y^2 = 4 - x$ and $x = 3y - 6$.

Solution:
The area is the region enclosed by the two curves in the plot:



(b) [4 marks] Write a definite integral with specific limits of integration that determines this area. **Do not evaluate the integral**.

Answer: $\int_{-5}^{2} [(4 - y^2) - (3y - 6)]\ dy$.

Solution: To find the intersection points we set $y^2 = 4 - x = 4 - (3y - 6)$. This yields, $y^2 + 3y - 10 = (y - 2)(y + 5) = 0$, which gives $y = -5$ and $y = 2$. We label $x_T = 4 - y^2$ (red curve) and $x_B = 3y - 6$ (blue curve), and observe that $x_T > x_B$ on $-5 < y < 2$. The area is best calculated as an integral in y, so that

$$A = \int_{-5}^{2} (x_T - x_B)\ dy = \int_{-5}^{2} \left[(4 - y^2) - (3y - 6)\right]\ dy\,.$$

Alternatively, the area can be calculated as an integral in x, so that

$$A = \int_{-20}^{0} \left(\frac{x + 6}{3} - \sqrt{4 - x}\right)\ dx + \int_{0}^{4} 2\sqrt{4 - x}\, dx\,.$$

5. A solid has as its base the region in the xy-plane between $y = 1 - x^2/36$ and the x-axis. The cross-sections of the solid perpendicular to the x-axis are squares.

(a) 4 marks Write a definite integral that determines the volume of the solid.

Solution: The intersection points with the x-axis are $x = \pm 6$. This gives, $V = \int_{-6}^{6} A(x)\,dx$ as the volume, where $A(x)$ is the cross-sectional area of the solid at position x. This cross-section is a square that has area $A(x) = [y(x)]^2$. Here we have used the fact that the area of a square with side of length b is b^2. This gives,

$$V = \int_{-6}^{6} [y(x)]^2\,dx = \int_{-6}^{6} \left[1 - \frac{x^2}{36}\right]^2 dx\,.$$

(b) 2 marks **Evaluate the integral** to find the volume of the solid.

Answer: 32/5

Solution: Since the integrand is even, we write $V = 2\int_0^6 \left[1 - \frac{x^2}{36}\right]^2 dx$. Now put $x = 6u$, so that $dx = 6du$, and so

$$V = 12\int_0^1 (1-u^2)^2\,du = 12\int_0^1 \left(1 - 2u^2 + u^4\right)\,du = 12\left(1 - \frac{2}{3} + \frac{1}{5}\right) = \frac{32}{5}\,.$$