First Name: ______________________ Last Name: ______________________

Student-No: ______________________ Section: ______________________

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## Indefinite Integrals

1. 9 marks Each part is worth 3 marks. Please write your answers in the boxes.

   (a) Calculate the indefinite integral $\int \frac{\sin(x)}{\sqrt{\cos(x)}}\, dx$ for $0 < x < \pi/2$.

   Answer: $I = -2\sqrt{\cos(x)} + C$

   **Solution:** Let $u = \cos(x)$, so that $\sin(x)dx = -du$. Then,

   $$I = -\int u^{-1/2}\, du = -2\sqrt{u} + C\,.$$

   Using $u = \cos(x)$ we get $I = -2\sqrt{\cos(x)} + C$.

   (b) Calculate the indefinite integral $\int \frac{x+1}{x^2+3x}\, dx$ for $x > 0$.

   Answer: $\frac{\ln(|x|)}{3} + \frac{2\ln(|x+3|)}{3} + C$

   **Solution:** The denominator $x^2 + 3x$ factorizes as $x(x+3)$. Thus, the partial fraction decomposition is $\dfrac{x+1}{x(x+3)} = \dfrac{A}{x} + \dfrac{B}{x+3}$.
   Multiplying everything by $x(x+3)$ we get $x + 1 = A(x+3) + Bx$, and by plugging in the values $x = 0$ and $x = -3$ we obtain $A = \dfrac{1}{3}$ and $B = \dfrac{2}{3}$. Then

   $$\int \frac{x+1}{x^2+3x}\, dx = \int \left(\frac{1}{3x} + \frac{2}{3(x+3)}\right) dx = \frac{\ln(|x|)}{3} + \frac{2\ln(|x+3|)}{3} + C.$$

(c) (A Little Harder): Calculate the indefinite integral $\int x^2 e^{-x}\,dx$.

Answer: $e^{-x}(-x^2 - 2x - 2) + C$

**Solution:** Let $u = x^2$ and $dv/dx = e^{-x}$. We calculate $du/dx = 2x$ and $v = -e^{-x}$, so that one step of integration by parts gives

$$I = uv - \int v \frac{du}{dx}\,dx = -x^2 e^{-x} + \int 2xe^{-x}\,dx\,.$$

Now we apply integration by parts again to $J = \int 2xe^{-x}\,dx$ choosing $u = 2x$ and $dv/dx = e^{-x}$, obtaining

$$J = uv - \int v \frac{du}{dx}\,dx = -2xe^{-x} + \int 2e^{-x}\,dx = e^{-x}(-2x - 2) + C\,.$$

Plugging this into our first equation for $I$ we get

$$I = -x^2 e^{-x} + J = e^{-x}(-x^2 - 2x - 2) + C\,.$$

## Definite Integrals

2. 12 marks Each part is worth 4 marks. Please write your answers in the boxes.

(a) Calculate $\int_0^{\pi/2} \cos^3(x)\,dx$.

Answer: $\frac{2}{3}$

**Solution:** Since the power of cosine is odd, we hold on to one copy of it and turn the rest into sines. We get

$$\int_0^{\pi/2} \cos^3(x)\,dx = \int_0^{\pi/2} \cos(x)(1-\sin^2(x))\,dx.$$

Now we can use the substitution $u = \sin(x)$. We have $du/dx = \cos(x)$, and that $x = 0$ and $x = \pi/2$ map to $u = 0$ and $u = 1$. This yields,

$$\int_0^{\pi/2} \cos(x)(1-\sin^2(x))\,dx = \int_0^1 (1-u^2)\,du = \left[u - \frac{u^3}{3}\right]_0^1 = \frac{2}{3}.$$



(b) Calculate $\int_0^3 \frac{9x^2}{x^2+9}\,dx$.

Answer: $27 - \frac{27\pi}{4}$

**Solution:** We first note that $\dfrac{x^2}{x^2+9} = 1 - \dfrac{9}{x^2+9}$. Then

$$\int_0^3 \frac{9x^2}{x^2+9}\,dx = 9\int_0^3 \left(1 - \frac{9}{x^2+9}\right)dx = 9 \cdot 3 - 81\int_0^3 \frac{1}{x^2+9}\,dx.$$

To solve the second integral we use the substitution $x = 3u$, so that

$$\int_0^3 \frac{1}{x^2+9} = \int_0^1 \frac{3}{9(u^2+1)}\,du = \left[\frac{\arctan(u)}{3}\right]_0^1 = \frac{\pi}{12} - 0.$$

Plugging this back into the first equation we get

$$\int_0^3 \frac{9x^2}{x^2+9}\,dx = 27 - \frac{27\pi}{4}.$$

(c) (A Little Harder): Calculate $\int_1^{e^2} \frac{\ln x}{x^2}\,dx$.

Answer: $1 - \frac{3}{e^2}$

**Solution:** We use integration by parts, picking $dv/dx = \dfrac{1}{x^2}$ and $u = \ln x$. We compute $v = -\dfrac{1}{x}$ and $du/dx = \frac{1}{x}$. Thus applying the IBP formula we get

$$I = uv - \int v\frac{du}{dx}\,dx = \left[-\frac{\ln x}{x}\right]_1^{e^2} + \int \frac{1}{x^2}\,dx = -\frac{2}{e^2} + \left[-\frac{1}{x}\right]_1^{e^2} = 1 - \frac{3}{e^2}.$$

## Riemann Sum, FTC, and Volumes

3. [12 marks] Each part is worth 4 marks. Please write your answers in the boxes.

(a) Calculate the infinite sum

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3i^2}{n^3}\sqrt{1+\frac{i^3}{n^3}}$$

by first writing it as a definite integral. Then, **evaluate this integral**.

Answer: $\frac{2}{3}\left(2\sqrt{2}-1\right)$ .

**Solution:** We try to pick $\Delta x = \frac{1}{n}, a = 0, b = 1, x_i = \frac{i}{n}$, so we have to write the summand in the form $\Delta x f\left(\frac{i}{n}\right)$. By collecting a $\frac{1}{n}$ in the expression we get $\frac{3}{n}\left(\frac{i}{n}\right)^2\sqrt{1+\left(\frac{i}{n}\right)^3}$ so we have $f(x) = 3x^2\sqrt{1+x^3}$, and thus

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3i^2}{n^3}\sqrt{1+\frac{i^3}{n^3}} = \lim_{n\to\infty}\sum_{i=1}^{n}\Delta x f(x_i) = \int_0^1 3x^2\sqrt{1+x^3}\,dx.$$

Using the substitution $u = 1 + x^3$ we get

$$\int_0^1 3x^2\sqrt{1+x^3}\,dx = \int_1^2 \sqrt{u}\,du = \frac{2}{3}\left[u^{3/2}\right]_1^2 = \frac{2}{3}\left(2\sqrt{2}-1\right).$$



(b) For $x \geq 0$ define $F(x)$ and $g(x)$ by $F(x) = \int_0^x \cos^2(t)\,dt$ and $g(x) = xF(x^2)$. Calculate $g'(\sqrt{\pi})$.

Answer: $\frac{5}{2}\pi$

**Solution:** By the product rule, chain rule, and FTC I we have

$$g'(x) = F(x^2) + 2x^2F'(x^2) = F(x^2) + 2x^2\cos^2(x^2).$$

Setting $x = \sqrt{\pi}$, we get $g'(\sqrt{\pi}) = F(\pi) + 2\pi\cos^2(\pi) = F(\pi) + 2\pi$. Now,

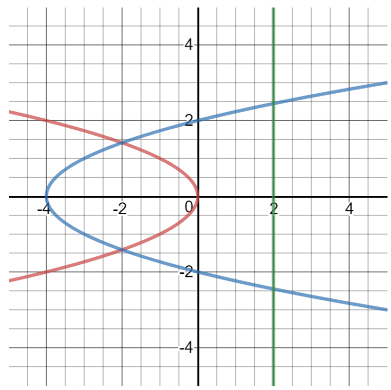
$$F(\pi) = \int_0^\pi \cos^2(t)\,dt = \int_0^\pi \frac{1+\cos(2t)}{2}\,dt = \frac{1}{2}\left[t + \frac{\sin(2t)}{2}\right]_0^\pi = \frac{\pi}{2}.$$

So we conclude that $g'(\sqrt{\pi}) = \frac{\pi}{2} + 2\pi = \frac{5}{2}\pi$.

(c) Write a definite integral, with specified limits of integration, for the volume obtained by revolving the bounded region between $x = -y^2$ and $x = -4+y^2$ about the vertical line $x = 2$. **Do not evaluate the integral**.

Answer: $V = \pi \int_{-\sqrt{2}}^{\sqrt{2}} ((6-y^2)^2 - (2+y^2)^2)\, dy$.

**Solution:** The plot is as shown:




The red curve is $x = -y^2$, the blue curve is $x = -4 + y^2$ and the green line is the axis of rotation $x = 2$. The red and blue curves meet where $-y^2 = -4 + y^2$, which gives us $y = \pm\sqrt{2}$ and $x = -2$.

A slice of the rotational solid at height $y$ will be a circular crown with inner radius $r_y = 2 + y^2$ (the distance between $x = 2$ and $x = -y^2$) and outer radius $R_y = 6 - y^2$ (the distance between $x = 2$ and $x = -4 + y^2$).

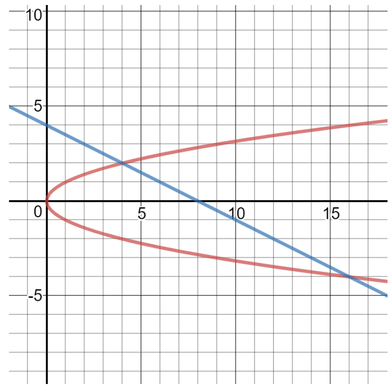
Thus the area of the slice will be

$$A_y = \pi(R_y^2 - r_y^2) = \pi((6-y^2)^2 - (2+y^2)^2)\,.$$

This gives the volume $V = \pi \int_{-\sqrt{2}}^{\sqrt{2}} ((6-y^2)^2 - (2+y^2)^2)\, dy$.

4. (a) [2 marks] Plot the finite area enclosed by $y^2 = x$ and $x = 8 - 2y$.

**Solution:** The area is the region enclosed between the blue and red curves:




(b) [4 marks] Write a definite integral with specific limits of integration that determines this area. **Do not evaluate the integral**.

Answer: $A = \int_{-4}^{2} (8 - 2y - y^2)\, dy$



**Solution:** The two curves meet when $y^2 = 8-2y$, which has solutions $y = 2, -4$. Seeing as how for both curves $x$ is expressed as a function of $y$, we choose to write the area as an integral in the variable $y$, with $x_B(y) = y^2$ and $x_T(y) = 8 - 2y$. By evaluating at $y = 0$ we see that $x_T(y) \geq x_B(y)$ on the interval $[-4, 2]$. Then we get the integral

$$\int_{-4}^{2} [x_T(y) - x_B(y)]\, dy = \int_{-4}^{2} \left(8 - 2y - y^2\right) dy\,.$$

Alternatively, as an integral in $x$, the area is

$$A = 2\int_0^4 \sqrt{x}\, dx + \int_4^{16} \left(4 - \frac{x}{2} + \sqrt{x}\right) dx\,.$$

5. A solid has as its base the region in the $xy$-plane between $y = 1 - x^2/9$ and the $x$-axis. The cross-sections of the solid perpendicular to the $x$-axis are semi-circles with the diameter of the semi-circle in the base.

(a) 4 marks Write a definite integral that determines the volume of the solid.

Answer: $V = \frac{\pi}{8}\int_{-3}^{3}\left(1 - x^2/9\right)^2\,dx$

**Solution:** The points of intersection with the $x$-axis are given by $x^2/9 = 1$, so we get $x = \pm 3$.

A vertical slice of the solid at $x$ will be a half circle whose diameter is $1 - x^2/9$, and thus will have area $A_x = \frac{\pi}{2}\left(\frac{1}{2}(1 - x^2/9)\right)^2$. Then the volume is given by

$$V = \int_{-3}^{3} A_x\,dx = \int_{-3}^{3}\frac{\pi}{8}(1 - x^2/9)^2\,dx.$$

(b) 2 marks **Evaluate the integral** to find the volume of the solid.

Answer: $\frac{2}{5}\pi$



**Solution:** We simplify by using the substitution $u = x/3$ so that

$$V = \int_{-3}^{3}\frac{\pi}{8}(1 - x^2/9)^2\,dx = \frac{3\pi}{8}\int_{-1}^{1}(1 - u^2)^2\,du.$$

Now we note that the function is even, so that

$$V = \frac{3\pi}{8}\int_{-1}^{1}(1 - u^2)^2\,du = \frac{3\pi}{4}\int_{0}^{1}(1 - u^2)^2\,du.$$

Finally we expand the formula and compute the integral:

$$V = \frac{3\pi}{4}\int_{0}^{1}(1 - u^2)^2\,du = \frac{3\pi}{4}\int_{0}^{1}\left(u^4 - 2u^2 + 1\right)\,du = \frac{3\pi}{4}\left[\frac{u^5}{5} - \frac{2u^3}{3} + u\right]_0^1.$$

We calculate that

$$V = \frac{3\pi}{4}\left(\frac{1}{5} - \frac{2}{3} + 1\right) = \frac{2}{5}\pi\,.$$