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On the Point-Particle (Newtonian) Limit of the Non-Linear Hartree Equation

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Abstract: We consider the nonlinear Hartree equation describing the dynamics of weakly interacting non-relativistic Bosons. We show that a *nonlinear* Møller wave operator describing the scattering of a soliton and a wave can be defined. We also consider the dynamics of a soliton in a slowly varying background potential $W(\varepsilon x)$. We prove that the soliton decomposes into a soliton plus a scattering wave (radiation) up to times of order ε^{-1}. To leading order, the center of the soliton follows the trajectory of a classical particle in the potential $W(\varepsilon x)$.

1. Introduction and Summary of Main Results

The problem of identifying classical regimes of quantum mechanics is a long standing problem of quantum theory. For simple systems it was first studied by Schrödinger in 1926; see [1]. In this paper, we explore a classical regime for a class of systems of identical, non-relativistic bosons, e.g., bosonic atoms such as ^{7}Li, with very weak two-body interactions described by a potential $-\kappa\Phi$ of van der Waals or Newtonian type satisfying certain regularity properties described below. These bosons move under the influence of an external potential λV, where V is a smooth, positive function on physical space $\mathbb{R}^3$ and $\lambda \geq 0$. The potential λV describes e.g. a trap confining the bosons.

Let κ denote the strength of the two-body interaction between two bosons as compared to their average kinetic energy, (e.g. in the sense that Φ is small as compared to the kinetic energy operator of two bosons, in the sense of Kato and Rellich, [2]). We are interested in understanding the dynamics of a *"condensate"* of $N = \mathcal{O}\left(\kappa^{-1}\right)$ bosons in the *"mean-field regime"*, where κ is very small. By a "condensate" we mean a state of the system with the property that all except for $o(N)$ bosons are in the *same* one-particle state described by a wave function $\psi(x)$, $x \in \mathbb{R}^3$. N-particle states of this kind are also called *coherent states*.

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Let $\psi_0 = \psi_0(x)$, $x \in \mathbb{R}^3$, denote the initial one-particle wave function of a coherent state of the system at time $t = 0$. In the *mean-field limit*,

$$\kappa \to 0,\ N \to \infty, \quad \text{with } \kappa \cdot N =: \nu = \text{const.}, \tag{1.1}$$

the quantum-mechanical time evolution of a condensate of bosons has the property that it maps the initial coherent state with a one-particle wave function ψ_0 to a coherent state at a later time t with a one-particle wave function ψ_t. As proven by K. Hepp [3] (see also [4] for some refinements and extensions), the one-particle wave function ψ_t of the condensate turns out to be a solution of the (non-linear) *Hartree equation*, Eq. (1.2) below.

If the two-body interactions are dominantly attractive, as for ^{7}Li atoms, and, given κ, the number of bosons is large enough (i.e., $N > N_{\text{crit.}}(\kappa)$, or $\nu > \nu_{\text{crit.}}$), the system has *bound states*. In other words, the bosons may condense into a tightly bound, spatially sharply localized cluster. In the mean-field regime, such bound states appear to be (weakly) well approximated by coherent states with a one-particle wave function corresponding to a non-trivial local minimum of the *Hartree energy functional*.

Turning on a very slowly varying external potential,

$$\lambda V(x) := W(\varepsilon x), \tag{1.2}$$

where W is a smooth, positive function, and ε is much smaller than the diameter of a bound state of N bosons when $\lambda = 0$, one expects that the position, $r(t) \in \mathbb{R}^3$, of the center of mass of that bound state closely follows a solution of Newton's equations of motion,

$$\dot{r}(t) = v(t),\, \dot{v}(t) = -\varepsilon\,(\nabla W)\,(\varepsilon r\,(t))\,, \tag{1.3}$$

for times t with $|t| < \mathcal{O}\left(\varepsilon^{-1}\right)$.

It is in this precise sense that the quantum system of bosons described above approaches a classical regime in the mean-field limit.

For attractive two-body interactions, the Hartree equation describing the dynamics of a condensate (coherent state) in the mean-field limit has a *self-focussing* non-linearity. As a consequence, it has non-trivial *"solitary wave solutions"* looking like approximate δ-functions, for ν sufficiently large. These solitary wave solutions are precisely the one-particle wave functions of coherent bound states in the mean-field limit.

Our main objective in this paper is to study slow motion of solitons of the Hartree equation. We propose to show that, under the influence of a slowly varying external potential $W(\varepsilon x)$, the center of mass position, $r(t)$, of a solitary-wave solution of the self-focussing Hartree equation remains close to a solution of Newton's equations of motion stated above, for all times t with $|t| < \mathcal{O}\left(\varepsilon^{-1}\right)$. (We do, however, not prove rigorous results on the precise way in which a system of identical bosons approaches its mean-field limit; but see [3–5].)

Our main results on the self-focussing Hartree equations have been announced in [6], where the reader can find additional background material and motivation coming from physics.

In order to be able to describe our main results concisely, we introduce some notation and recall some known results on the Hartree equation.

Let $H^1(\mathbb{R}^n)$ denote the Sobolev space,

$$H^1(\mathbb{R}^n) = \left\{\psi(x),\ x \in \mathbb{R}^n \;\middle|\; \|\nabla\psi\|_2 + \|\psi\|_2 < \infty\right\}, \tag{1.4}$$

where ψ denotes a measurable complex function on $\mathbb{R}^n$, $\nabla\psi$ denotes its gradient, and $\|(\cdot)\|_2$ denotes the L^2-norm. We study properties of solutions of the *Hartree equation*

$$i\partial_t\psi_t = -\frac{1}{2}\triangle\psi_t + \lambda V\psi_t - \nu\left(\Phi * |\psi_t|^2\right)\psi_t. \tag{1.5}$$

In Eq. (1.5),

$$\psi_t(x) = \psi(x,t), \quad x \in \mathbb{R}^n, t \in \mathbb{R},$$

is a time (t)-dependent, complex-valued scalar function on physical space $\mathbb{R}^n$ belonging to the Sobolev space $H^1(\mathbb{R}^n)$, for each time t; $\triangle$ denotes the scalar Laplacian, $\lambda V(x),\ \lambda \in \mathbb{R}$, is an external potential, with V a smooth, bounded, positive function on $\mathbb{R}^n$, and $-\Phi(x)$ is a radially symmetric two-body potential, with $\Phi \in L^p(\mathbb{R}^n, d^n x) + L^\infty(\mathbb{R}^n),\ p \geq \frac{n}{2}$; furthermore $*$ denotes convolution. We shall use the following standard notation:

For an arbitrary measurable function ψ on $\mathbb{R}^n$,

$$\int \psi := \int_{\mathbb{R}^n} \psi(x)\, d^n x, \tag{1.6}$$

$$\|\psi\|_p := \left(\int |\psi|^p\right)^{1/p} \tag{1.7}$$

is the norm on the space $L^p = L^p(\mathbb{R}^n, d^n x),\ 1 \leq p < \infty$,

$$\|\psi\|_{H^1} := \|\nabla\psi\|_2 + \|\psi\|_2 \tag{1.8}$$

is the norm on $H^1 = H^1(\mathbb{R}^n)$, and

$$(\psi * \chi)(x) := \int_{\mathbb{R}^n} \psi(x-y)\chi(y)d^n y \tag{1.9}$$

denotes the convolution of ψ with another such function χ.

There are two important functionals on Sobolev space H^1 which are conserved under the flow $\psi := \psi_0 \mapsto \psi_t, \psi \in H^1$, determined by the Hartree equation (1.5). The first one is the L^2-norm of ψ

$$\mathcal{N}\left(\bar{\psi}, \psi\right) := \int |\psi|^2 = \|\psi\|_2^2 \tag{1.10}$$

and the second one is the *Hamilton* (or *energy*) *functional*

$$\begin{aligned}\mathcal{H}\left(\bar{\psi}, \psi\right) :=\ & \frac{1}{4}\int |\nabla\psi|^2 + \frac{\lambda}{2}\int V|\psi|^2 \\ & -\frac{1}{4}\int\left(\Phi * |\psi|^2\right)|\psi|^2 .\end{aligned} \tag{1.11}$$

We note that if Φ is a non-negative function belonging to $L^p + L^\infty,\ p \geq \frac{n}{2}$, then, for an arbitrary $\delta > 0$, there exists a finite constant $C(\delta)$ such that

$$0 \leq \int\left(\Phi * |\psi|^2\right)|\psi|^2 \leq \delta\mathcal{N}\left(\bar{\psi}, \psi\right)\|\nabla\psi\|_2^2 + C(\delta)\,\mathcal{N}\left(\bar{\psi}, \psi\right)^2, \tag{1.12}$$

see e.g. [7]. Thus, for an arbitrary, but fixed value of $\mathcal{N}(\bar{\psi}, \psi)$, and for arbitrary λ, $|\lambda| < \infty$, the Hamilton functional $\mathcal{H}(\bar{\psi}, \psi)$ is bounded from below.

Under the assumptions that $\lambda V(x)$ has a minimum at $x = x_*$, $|x_*| < \infty$, that $\Phi(x) \geq 0$ and that the value, N, of the functional $\mathcal{N}(\bar{\psi}, \psi)$ is large enough, one can show (see Sect. 3) that the Hamilton functional $\mathcal{H}(\bar{\psi}, \psi)$ restricted to the sphere

$$\mathcal{S}_N := \left\{ \psi \mid \psi \in H^1,\ \mathcal{N}\left(\bar{\psi}, \psi\right) = N \right\} \tag{1.13}$$

in Sobolev space reaches its minimum on a positive function $Q_N \in \mathcal{S}_N$ concentrated near x_* and decaying exponentially fast in $|x|$, as $|x| \to \infty$. This result still holds when $\lambda = 0$ (i.e., for a vanishing external potential); but if Q_N is a minimizer of $\mathcal{H}\ \big|_{\mathcal{S}_N}$ then so is $Q_{N,a}$, where $Q_{N,a}(x) := Q_N(x - a)$, for arbitrary $a \in \mathbb{R}^N$. This is a consequence of the *translation invariance* of $\mathcal{H}$, for $\lambda = 0$.

A minimizer, Q_N of $\mathcal{H}\ \big|_{\mathcal{S}_N}$ is a solution of the non-linear eigenvalue equation

$$-\frac{1}{2} \triangle Q + \lambda V Q - \left(\Phi * Q^2\right) Q = E Q, \tag{1.14}$$

for some real number E, with

$$\mathcal{N}\left(\bar{Q}, Q\right) = N\ .$$

Then

$$\psi(x, t) = Q_N(x) e^{-iEt}$$

is a stationary solution of the Hartree equation (1.5). Multiplying Eq. (1.14) by $Q := Q_N$ and integrating, we find that

$$E = \frac{1}{2N} \int (\nabla Q_N)^2 + \frac{\lambda}{N} \int V Q_N^2 - \frac{1}{N} \int \left(\Phi * Q_N^2\right) Q_N^2\ . \tag{1.15}$$

One should notice that EN is *not* the value of the energy functional $\mathcal{H}(\bar{\psi}, \psi)\ \big|_{\mathcal{S}_N}$ evaluated on the minimizer $\psi = Q_N$, because one is minimizing $\mathcal{H}(\bar{\psi}, \psi)$ in the presence of a constraint, namely $\mathcal{N}(\bar{\psi}, \psi) = N$.

Let $Q_N^{(0)}$ be a minimizer of the Hamilton functional $\mathcal{H}(\bar{\psi}, \psi)\ \big|_{\mathcal{S}_N}$, with $\lambda = 0$, centered at $x = 0$; ($Q_N^{(0)}$ is known to exist and to be non-trivial, for N large enough). We set $\lambda = 1$ and choose

$$V(x) \equiv V^{(\varepsilon)}(x) := W(\varepsilon x), \tag{1.16}$$

where W is a fixed, smooth, bounded, positive function on $\mathbb{R}^n$, and $\varepsilon > 0$ is a parameter. Our main concern, in this paper, is to construct *local* (in time t) *solutions* of the Hartree equation (1.5), with $\lambda = 1$ and $V = V^{(\varepsilon)}$ as in (1.16), of the form

$$\psi(x, t) = \left[Q_N^{(0)}\left(x - r(t)\right) + h_\varepsilon\left(x - r(t), t\right) \right] e^{i\theta(x,t)}, \tag{1.17}$$

where h_ε is a *small, dispersive correction* to the *solitary wave* described by $Q_N^{(0)}(x - r(t))e^{i\theta(x,t)}$, with

$$\|h_\varepsilon(\cdot, t)\|_{H^1} \lesssim 0(\varepsilon^{3/2}), \tag{1.18}$$

$\theta(x, t)$ is a time-dependent phase,

$$\theta(x, t) = v(t)\cdot x - Et + \vartheta_0(t), \tag{1.19}$$

where $v(t) = \frac{dr(t)}{dt}$ is the velocity of the solitary wave, and $\vartheta_0(t)$ is *independent* of x, *for all times* t *with* $|t| \lesssim 0(\varepsilon^{-1})$, and provided the soliton trajectory $(r(t), v(t))$ solves appropriate equations of motion. It will be shown that $(r(t), v(t))$ must solve the *Newtonian equations of motion*

$$\begin{aligned} \dot{r}(t) &= v(t), \\ \dot{v}(t) &= -\varepsilon(\nabla W)(\varepsilon r(t)) + a(t), \end{aligned} \tag{1.20}$$

where $a(t)$ is a *"friction force"*, with

$$|a(t)| \lesssim 0(\varepsilon^2), \tag{1.21}$$

for $|t| \lesssim 0(\varepsilon^{-1})$. The friction force $a(t)$ will be determined more precisely in Sect. 3.

Neglecting the friction force $a(t)$, Eqs. (1.20) are Newton's equations of motion for a point particle of mass N moving in an external acceleration field of strength ε with potential $V^{(\varepsilon)}$. Thus, for the velocity $v(t)$ of this particle to deviate substantially from the initial condition $v(0) = v_0$, the time t must be $0(\varepsilon^{-1})$. For times t, with $|t| \lesssim 0(\varepsilon^{-1})$, the friction force $a(t)$ has a negligibly small effect, for small ε.

A solution of the Hartree equation (1.5) of the form (1.17), with properties (1.18) through (1.21), for times t with $|t| \lesssim 0(\varepsilon^{-1})$, describes the motion of an extended particle in a shallow potential well $V^{(\varepsilon)}$ interacting weakly with a dispersive medium of infinitely many degrees of freedom with which it can exchange mass and energy. The point-particle limit in which Newton's laws of motion become exact is the limit $\varepsilon \to 0$. For $\varepsilon > 0$, the interactions between the extended particle and the dispersive medium can lead to phenomena such as mass accretion, loss of mass and energy from the particle into dispersive waves, and friction, for times t large on a scale of ε^{-1}. The intuitive picture is one of a bound cluster of "dust" describing an extended particle, which exhibits Newtonian motion with friction. The friction is caused by the loss of some "dust" originally bound to the particle. This loss of "dust" is only observed when the motion of the particle is *not inertial* (i.e., accelerated or decelerated) and is described by dispersive waves satisfying a wave equation which is essentially the linearization of Eq. (1.5) around a solitary wave described by $Q^{(0)}_{N(t)}(x - r(t))e^{i\theta(x,t)}$. For very large times, the trajectory of the extended particle is expected either to approach an inertial motion diverging to spatial infinity (if $W(x) \to \text{const}, \text{ as } |x| \to \infty$ and if the initial mass and velocity of the particle were large enough), or to approach a local minimum of W where the particle will come to rest. This dissipative behavior of the particle motion is an example of the general phenomenon of "dissipation through radiation". Some simple results on the large-time asymptotics of solutions of the Hartree equation (1.5) (existence of wave operators) are proven in Sect. 4. But it is fair to say that we do not yet have a good mathematical understanding of large-time behavior of solutions of Eq. (1.5). For some earlier results on scattering for the Hartree and nonlinear Schrödinger equation, see, e.g., [7,8] and references given there.

Our analysis of solutions of the Hartree equation (1.5) of the form described in (1.17), with properties (1.18) through (1.21), is based on a *key assumption*, which is, implicitly, an assumption on the two-body potential $-\Phi$ that will not be made explicit in this paper: Let

$$(f, g) := \int \bar{f} g$$

denote the usual scalar product on L^2, and let $\mathcal{H}''$ denote the Hessian of the Hamilton functional $\mathcal{H}(\bar{\psi}, \psi)$, with $\lambda = 0$, at $\psi = Q_N^{(0)}$. Furthermore, let $\mathcal{H}''_{\text{real}}$ denote the restriction of $\mathcal{H}''$ on real-valued functions, and extend it to a complex-linear operator. It will be shown in Sect. 3 that $\mathcal{H}''_{\text{real}}$ is given by an unbounded, selfadjoint operator on L^2 defining a quadratic form on H^1 which is bounded from below. It is not hard to see that

$$\big(Q, (\mathcal{H}''_{\text{real}} - E)Q\big) = \varepsilon_0(Q, Q) < 0, \quad \text{for } Q := Q_N^{(0)}, \tag{1.22}$$

where $E = E_N$ and

$$\varepsilon_0 = -\frac{2}{N} \int \left(\Phi * Q^2\right) Q^2 . \tag{1.23}$$

Actually, $\mathcal{H}''_{\text{real}} - E$ has only one negative eigenvalue. Since $\mathcal{H}$ is translation-invariant, it follows that $\nabla Q := \{\partial_1 Q, \dots, \partial_n Q\}$, $\partial_j := \frac{\partial}{\partial x^j}$, $j = 1, \dots, n$, are n non-vanishing, linearly independent zero-modes for $\mathcal{H}''_{\text{real}} - E$ orthogonal to Q, i.e.,

$$(\mathcal{H}''_{\text{real}} - E)\partial_j Q = 0, \text{ and } \big(\partial_j Q, Q\big) = 0, \tag{1.24}$$

for all $j = 1, \dots, n$. Thus 0 is an at least n-fold degenerate eigenvalue of $\mathcal{H}''_{\text{real}} - E$. Since Q is a minimizer of $\mathcal{H}(\bar{\psi}, \psi)\big|_{\mathcal{S}_N}$, there is no spectrum of $\mathcal{H}''_{\text{real}} - E$ in the interval $(\varepsilon_0, 0)$. Furthermore, it is easy to see that the spectrum of $\mathcal{H}''_{\text{real}} - E$ in the interval $[0, -E)$, where

$$E = \frac{1}{N}\left(\frac{1}{2}\int (\nabla Q)^2 - \int \left(\Phi * Q^2\right) Q^2\right) < 0 \tag{1.25}$$

is *pure-point*, while, on the half-line $[-E, \infty)$, it is continuous. Thus, there is a gap, $\varepsilon_2 > 0$, between 0 and the rest of the spectrum of $\mathcal{H}''_{\text{real}} - E$ in $[0, \infty)$; see Sect. 3 for details.

Our *key assumption* is that the *multiplicity of the eigenvalue* 0 of $\mathcal{H}''_{\text{real}} - E$ is *precisely* equal to n. This implies that

$$\big(h, (\mathcal{H}''_{\text{real}} - E)h\big) \geq \varepsilon_2(h, h), \quad \varepsilon_2 > 0, \tag{1.26}$$

for all functions $h \in H^1$ with $h \perp \{Q, \nabla Q\}$ in the L^2-scalar product $(\cdot, \cdot)$.

We are now prepared to summarize the contents of this paper and to state our main results in the form of theorems.

In Sect. 2, we recall the Hamiltonian nature of the Hartree equation (1.5) on the phase space H^1. We exhibit continuous symmetries of the Hamilton functional that give rise to Eq. (1.5) and derive the corresponding conservation laws. We show that the Hartree equation can also be viewed as the *Euler–Lagrange equation* derived from an *action functional*. The Lagrangian formulation of the non-linear Hartree equation is useful to study the formal *point-particle limit* (the $\varepsilon \to 0$ limit in (1.16) through (1.21)). This limit is discussed, in general terms but without mathematical proofs, in Sect. 2, using ideas similar to those in [9] in an analysis of vortex motion in the Ginzburg-Landau equation, which is based on an effective-action formalism. We also discuss some expected features of the non-linear Hartree dynamics in the large-time limit.

Our first main result is proven in Sect. 3.

Theorem 1.1. *Suppose that assumption (1.26) holds for all minimizers $Q = Q_N^{(0)}$, with N in an open neighborhood of some $N_0 > 0$. We also assume that*

$$\Phi(x) \textit{ is radial}, \quad \|\Phi\|_{W^{2,1}(\mathbb{R}^3)\cap W^{2,\infty}(\mathbb{R}^3)} \leq C_\Phi \tag{1.27}$$

for some constant C_Φ. Then there is a positive constant C_0 such that, for an arbitrary $T < \infty$, there is an $\varepsilon_0 > 0$ with the property that, for any $0 < \varepsilon \leq \varepsilon_0$ and any initial condition of the form

$$\psi(x, 0) = \psi_0(x) = \left[Q\,(x - r_0) + h_{\varepsilon,0}(x)\right] e^{iv_0x}, \tag{1.28}$$

with $Q = Q_{N_0}$ and $\|h_{\varepsilon,0}\|_{H^1} \leq C_0\varepsilon^{3/2}$, the Hartree equation, Eq. (1.5), with $\lambda = 1$ and $V(x) = W(\varepsilon x)$ as in (1.16), has a solution of the form (1.17), for all times t with $|t| < T\varepsilon^{-1}$, with the following properties:

1. *The phase $\theta(x, t)$ is as in (1.19);*
2. *the trajectory $(r(t), v(t))$ of the extended-particle solution (1.17) is a solution of the equations of motion (1.20) with initial conditions $r(t) = r_0,\ v(t) = v_0$, for a friction force $a(t)$ bounded by*

$$|a(t)| \leq C_1\varepsilon^2;$$

3. *the dispersive correction h_ε satisfies*

$$\|h_\varepsilon(\cdot, t)\|_{H^1} \ \leq \ C_2\varepsilon^{3/2},$$

for some finite constants C_1, C_2 depending on T.

This result makes the point-particle limit ($\varepsilon \to 0$) of the Hartree equation (1.5) precise for initial conditions describing a single extended particle (solitary wave) moving in a shallow potential well, $W(\varepsilon x)$, and perturbed by a small amount of radiation (described by h_ε). It is a special case of the more general situation considered in Sect. 3. A more detailed discussion and the proof of Theorem 1.1 form the contents of Sect. 3.

The results just described raise the issue of *asymptotic properties* of the dynamics determined by the Hartree equation, as time t tends to $\pm\infty$. In Sect. 4, we establish a result on the scattering of small-amplitude waves off a single solitary wave. For simplicity, we suppose that physical space is three-dimensional, $n = 3$, (but our methods can be applied whenever $n \geq 3$), we set $\lambda = 0$, and we choose Φ to be a non-negative, bounded function of rapid decrease, as $|x| \to \infty$. We consider an *"asymptotic profile"* described by

$$\psi_{as}(x, t) = Q\,(x - r_0 - v_0t)\, e^{i\left(x\cdot v_0 - \left[\frac{1}{2}v_0^2 + E\right]t\right)} + h_{as}(x, t), \tag{1.29}$$

where h_{as} is a solution of the free-particle Schrödinger equation

$$i\partial_t h_{as}(x, t) = -\frac{1}{2}\,\Delta h_{as}(x, t), \tag{1.30}$$

with initial condition $h_{as}(x,0) =: h_{as,0}(x)$ belonging to and being sufficiently small in the space $H^4(\mathbb{R}^3) \cap W^{3,1}\left(\mathbb{R}^3, \left(1+|x|^2\right) d^3x\right)$ and such that the Fourier transform, $\hat{h}_{as,0}(k)$, vanishes at $k = v_0$. In (1.29), $Q = Q^{(0)}_{N_0} \in \mathcal{S}_{N_0}$ is a solution of Eq. (1.14), with $\lambda = 0$, and it is assumed that inequality (1.26) is satisfied for $Q = Q^{(0)}_N \in \mathcal{S}_N$, for all N in a small neighborhood of $N_0 > 0$.

Theorem 1.2. *For an asymptotic profile $\psi_{as}(x,t)$ as described in* (1.29), (1.30), *and under the hypotheses stated above, there are solutions, $\psi_\pm(x,t)$, of the Hartree equation* (1.5) *(for $\lambda = 0$) such that*

$$\psi_\pm(x,t) \longrightarrow \psi_{as}(x,t), \quad \text{as } t \to \pm\infty, \tag{1.31}$$

in $H^2(\mathbb{R}^3)$. Their difference is of order $O(t^{-1})$.

Thus the non-linear Møller wave maps $\Omega_\pm : \psi_{as} \longrightarrow \psi_\pm$ exist as symplectic maps on asymptotic profiles of the form (1.29), (1.30). We emphasize that the effect of the scattering wave on the location and the phase of the soliton has to be tracked precisely for *all* time. The stability of the soliton is quite simple and can be obtained purely from energy consideration. A review can be found in Sect. 3 (see also Weinstein [14]). Therefore, the key points of Theorem 1.2 are its two precise assertions: 1. The location of the soliton is almost "linear." 2. The scattering wave behaves like an ordinary dispersive wave, (described by $h_{as}(x,t)$), plus a small correction. The condition on the Fourier transform of $h_{as,0}$ is a technical one and we expect to remove it later on. Our result constitutes the first step toward scattering theory.

The proof of Theorem 1.2 is the contents of the final section, Sect. 4, of this paper.

2. The Hartree Equation as a Hamiltonian System with Infinitely Many Degrees of Freedom, and Its Point-Particle Limit

In the introduction, we have described results indicating how the Hartree equation (1.2) captures the dynamics of a system of very many non-relativistic bosons with very weak two-body interactions in a condensate state. This regime has been called the "mean-field limit". Actually, the mean-field limit is equivalent, mathematically, to the *classical limit* in which the value of Planck's constant, $\hbar$, is sent to 0. We are accustomed to expect (actually in general erroneously) that the unitary dynamics of a quantum-mechanical system reduces to the Hamiltonian dynamics of a corresponding classical system, in the classical limit. In the examples studied in this paper, this expectation is justified.

2.1. The Hamiltonian nature of the Hartree equation. The *phase space*, Γ, for the Hartree equation (1.5) is the Sobolev (energy) space $H^1(\mathbb{R}^n)$ defined in (1.4). We use $\psi(x)$ and its complex conjugate $\bar{\psi}(x)$, $x \in \mathbb{R}^n$, as complex coordinates for Γ. The symplectic 2-form on Γ is given by $\frac{i}{2}\, d\psi \wedge d\bar{\psi}$. It leads to the following *Poisson brackets:*

$$\{\psi(x), \psi(y)\} = \left\{\bar{\psi}(x), \bar{\psi}(y)\right\} = 0, \tag{2.1}$$

$$\left\{\psi(x), \bar{\psi}(y)\right\} = 2i\delta(x-y)\,. \tag{2.2}$$

The Hamilton functional, $\mathcal{H}(\bar{\psi}, \psi)$, leading to the Hartree equation (1.5) is given by

$$\mathcal{H}(\bar{\psi}, \psi) = \frac{1}{4} \int \left[|\nabla\psi|^2 + 2\lambda V|\psi|^2 - \left(\Phi * |\psi|^2\right) |\psi|^2 \right] . \tag{2.3}$$

For $\Phi \in L^p + L^\infty$, $p \geq \frac{n}{2}$, $\mathcal{H}$ is well defined on Γ and bounded below on the spheres

$$\mathcal{S}_N = \left\{ \psi \mid \psi \in \Gamma,\ \mathcal{N}\left(\bar{\psi}, \psi\right) = N < \infty \right\}, \tag{2.4}$$

where

$$\mathcal{N}\left(\bar{\psi}, \psi\right) = \int |\psi|^2 \ ; \tag{2.5}$$

see inequality (1.12). Hamilton's equations of motion for ψ are given by

$$\begin{aligned} \dot{\psi}_t(x) &= \left\{ \mathcal{H}\left(\bar{\psi}_t, \psi_t\right),\ \psi_t(x) \right\} \\ &= i \left[\frac{1}{2} \triangle\psi_t(x) - \lambda V(x)\psi_t(x) \right. \\ &\qquad \left. + \left(\Phi * |\psi_t|^2\right)(x)\psi_t(x) \right], \end{aligned} \tag{2.6}$$

which is precisely the Hartree equation (1.5).

From (2.3) we infer the following *symmetries* and corresponding *conservation laws*.

(1) *Gauge invariance of the first kind*. The phase transformations

$$\psi(x) \mapsto e^{i\theta}\psi(x),\ \ \bar{\psi}(x) \mapsto e^{-i\theta}\bar{\psi}(x) \tag{2.7}$$

leave $\mathcal{H}(\bar{\psi}, \psi)$ invariant. These transformations describe the symplectic flow generated by the Hamiltonian vector field corresponding to the function $\frac{1}{2}\ \mathcal{N}(\bar{\psi}, \psi)$. Since they are a symmetry of $\mathcal{H}(\bar{\psi}, \psi)$, it follows that

$$\{\mathcal{H}, \mathcal{N}\} = 0, \tag{2.8}$$

and hence $\mathcal{N}$ is conserved, and the spheres $\mathcal{S}_N$ defined in (2.4) are invariant under the time evolution $\psi \mapsto \psi_t$ described by (2.6).

(2) *Galilei invariance, for* $\lambda = 0$. We shall assume henceforth that Φ is *rotation-invariant*. If the external potential λV vanishes then arbitrary Galilei transformations are symmetries of $\mathcal{H}$.

Space translations, $x \to x + a$, are represented on Γ by

$$\psi(x) \mapsto \psi_a(x) := \psi(x - a),\ \ a \in \mathbb{R}^n,$$

and are generated by the *momentum functional*

$$\mathcal{P}(\bar{\psi}, \psi) := \frac{i}{2} \int \bar{\psi}\nabla\psi \ . \tag{2.9}$$

They clearly leave $\mathcal{H}(\bar{\psi}, \psi)$ invariant, hence $\mathcal{P}$ is conserved under the time evolution and

$$\{\mathcal{H}, \mathcal{P}\} = 0 \ . \tag{2.10}$$

Rotations, R_{ab}, in the (ab)-plane of $\mathbb{R}^n$, $1 \le a < b \le n$, are represented on Γ by

$$\psi(x) \mapsto \psi_{R_{ab}}(x) := \psi\left(R_{ab}^{-1} x\right) .$$

They are generated by the angular momentum functionals

$$\mathcal{L}_{ab}(\bar{\psi}, \psi) := \frac{i}{2} \int \bar{\psi} \left(x^a \partial_b - x^b \partial_a\right) \psi, \tag{2.11}$$

with $\partial_b = \partial/\partial x^b$. Since Φ has been assumed to be rotation-invariant, rotations leave $\mathcal{H}(\bar{\psi}, \psi)$ invariant, hence the functionals $\mathcal{L}_{ab}$ are conserved under the time evolution and Poisson-commute with $\mathcal{H}$, for all (ab).

Finally, boosts (velocity transformations), $x \to x - vt,\ v \in \mathbb{R}^n$, t denotes time, are represented on time-dependent trajectories, $\psi_t(x)$, in Γ by

$$\psi_t(x) \mapsto \psi_t(v; x) := \psi_t(x - vt) e^{i\left(v \cdot x - \frac{v^2}{2} t\right)} . \tag{2.12}$$

They do not leave $\mathcal{H}$ invariant, but one easily checks that if $\psi_t(x)$ is a solution of Hamilton's equations of motion (2.6) then so is $\psi_t(v; x)$, for arbitrary $v \in \mathbb{R}^n$. The conserved quantity corresponding to (2.12) is given by

$$\mathcal{M}_v\left(\bar{\psi}_t, \psi_t\right) := \int \bar{\psi}_t v \cdot (x + it\nabla)\, \psi_t . \tag{2.13}$$

It follows that the "centre of mass motion" of a solution ψ_t of (2.6) is *inertial*.

We conclude this section by noting that, as usual, Hamilton's equations of motion (2.6) can also be viewed as Euler–Lagrange equations derived from an action principle. The action functional is defined on a space of continuously differentiable (in time) trajectories in phase space Γ. It is given by

$$S\left(\bar{\psi}, \psi\right) := \int_{t_1}^{t_2} dt \left[\frac{i}{2} \int \bar{\psi}_t \dot{\psi}_t - \mathcal{H}\left(\bar{\psi}_t, \psi_t\right)\right] . \tag{2.14}$$

The Hartree equation (2.6) is obtained from the action functional $S(\bar{\psi}, \psi)$ by variation with respect to $\bar{\psi}$, i.e., it is equivalent to the equation

$$\delta S\left(\bar{\psi}, \psi\right) / \delta \bar{\psi}_t(x) = 0, \tag{2.15}$$

under the boundary conditions that

$$\delta \psi_{t_i}(x) = 0, \quad i = 1, 2 . \tag{2.16}$$

Global existence and uniqueness of solutions of the equations of motion (2.6), for $\Phi \in L^p + L^\infty,\ p \ge \frac{n}{2}$, is proven in [7] and refs. given there.

2.2. Stationary solutions of the Hartree equations, for fixed values of $\mathcal{N}$, $\mathcal{P}$ and $\mathcal{L}_{ab}$. In this section, we consider stationary solutions of the non-linear Hartree equations (2.6), assuming that $\lambda V = 0$ and that Φ is rotation-invariant. Since the L^2-norm $\mathcal{N}(\bar{\psi}, \psi)$, the momentum functional $\mathcal{P}(\bar{\psi}, \psi)$ and the angular momentum functionals $\mathcal{L}_{ab}(\bar{\psi}, \psi)$ are conserved, we may put them to fixed values, N, P and L_{ab}, respectively. In order to find stationary solutions of (2.6), with $\mathcal{N}(\bar{\psi}, \psi) = N$, $\mathcal{P}(\bar{\psi}, \psi) = \pi$ and $\mathcal{L}_{ab}(\bar{\psi}, \psi) = \lambda_{ab}$, we may look for critical points of the generalized energy functional

$$\begin{aligned} \mathcal{E}\left(\bar{\psi}, \psi; E, P, L^{ab}\right) &:= \mathcal{H}\left(\bar{\psi}, \psi\right) + \frac{E}{2}\left(N - \mathcal{N}\left(\bar{\psi}, \psi\right)\right) \\ &+ P\cdot\left(\pi - \mathcal{P}\left(\bar{\psi}, \psi\right)\right) + \sum_{a<b} L^{ab}\left(\lambda_{ab} - \mathcal{L}_{ab}\left(\bar{\psi}, \psi\right)\right), \end{aligned} \tag{2.17}$$

where E, P and L^{ab} are *Lagrange multipliers*. By varying $\mathcal{E}\left(\bar{\psi}, \psi; E, P, L^{ab}\right)$ with respect to $\bar{\psi}$, ψ, E, P and L^{ab}, we find the equations

$$\begin{aligned} &-\frac{1}{2}\triangle\psi - \left(\Phi * |\psi|^2\right)\psi - E\psi \\ &\quad - iP\cdot\nabla\psi - i\sum_{a<b} L^{ab}\left(x^a\partial_b - x^n\partial_a\right)\psi = 0, \end{aligned} \tag{2.18}$$

(variation with respect to $\bar{\psi}$), and

$$\mathcal{N}\left(\bar{\psi}, \psi\right) = N \quad \text{(variation with respect to } E\text{)}, \tag{2.19}$$

$$\mathcal{P}\left(\bar{\psi}, \psi\right) = \pi \quad \text{(variation with respect to } P\text{)}, \tag{2.20}$$

and

$$\mathcal{L}_{ab}\left(\bar{\psi}, \psi\right) = \lambda_{ab} \quad \text{(variation with respect to } L^{ab}\text{)}, \tag{2.21}$$

$1 \le a < b \le n$.

Not much is known about the general solution of Eqs. (2.18) through (2.21). But, for the purposes of this paper, the following solutions are particularly important: We look for a *rotation-invariant* absolute minimum, $Q_N^{(0)}$, of the Hamilton functional $\mathcal{H}(\bar{\psi}, \psi)$ restricted to the sphere $\mathcal{S}_N$, which has zero momentum. Equations (2.18) through (2.21) then simplify to

$$-\frac{1}{2}\triangle\psi - \left(\Phi * |\psi|^2\right)\psi = E\psi, \tag{2.22}$$

$$\mathcal{N}\left(\bar{\psi}, \psi\right) = N, \tag{2.23}$$

and the solution, $\psi = Q_N^{(0)}$, must satisfy

$$\mathcal{P}\left(\bar{Q}_N^{(0)}, Q_N^{(0)}\right) = \frac{i}{2}\left(Q_N^{(0)}, \nabla Q_N^{(0)}\right) = 0 \tag{2.24}$$

and

$$\left(x^a\partial_b - x^b\partial_a\right)Q_N^{(0)} = 0, \quad \text{for all } a < b\,. \tag{2.25}$$

Equation (3.22) is identical to Eq. (1.14), for $\lambda V = 0$, and E is given by

$$E = \frac{1}{2N}\int \left(\nabla Q_N^{(0)}\right)^2 - \frac{1}{N}\int \left(\Phi * Q_N^{(0)2}\right) Q_N^{(0)2}, \tag{2.26}$$

see Eq. (1.15), which is strictly negative, for a non-trivial minimizer $Q_N^{(0)}$.

Lemma 2.1. *For a positive, rotation-invariant potential $\Phi \in L^p + L^\infty,\ p \geq \frac{n}{2}$, with $\Phi(x) \to 0$, as $|x| \to \infty$, there exists a constant $N_* = N_*(\Phi)$, with $0 \leq N_* < \infty$, such that, for $N > N_*$, (2.23) has a non-trivial solution $\psi = Q_N^{(0)}$, with $\mathcal{N}\left(\bar{Q}_N^{(0)}, Q_N^{(0)}\right) = N$, corresponding to a local minimum of $\mathcal{H}\left(\bar{\psi}, \psi\right)\big|_{\mathcal{S}_N}$. The phase of $Q_N^{(0)}$ can be chosen such that $Q_N^{(0)} > 0$. The non-linear eigenvalue E is given by (2.26) and is strictly negative, for $N > N_*$. The function $Q_N^{(0)}(x)$ is smooth and decays exponentially, as $|x| \to \infty$, with decay rate $\sqrt{-E}$.*

Remarks. (i) From the theory of quantum-mechanical bound states we infer that, in $n = 1, 2$ dimensions, $N_* = 0$, while, for $n \geq 3$, N_* is strictly positive if Φ is integrable, but vanishes for potentials of very long range, such as the Coulomb potential; see [10].

(ii) Given a solution, $Q_N^{(0)}$, of (3.22), the function

$$\psi_t(v; x) := Q_N^{(0)}(x - r - vt)e^{i\left(v\cdot x - \left[\frac{1}{2}\ v^2 + E\right]t\right)} \tag{2.27}$$

solves the Hartree equation (2.6), with $\lambda V = 0$, for arbitrary $r \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$. This follows from the Galilei invariance of the theory. For ψ_t as in (2.27),

$$\mathcal{P}\left(\bar{\psi}_t, \psi_t\right) = Nv\ . \tag{2.28}$$

Equation (2.6) also has wave-like solutions with $\mathcal{P} \neq 0$, (e.g. $\psi_t(x) = \psi_0 \exp i\,(k \cdot x - E\,(k, \psi_0)\,t)$, which has infinite energy and momentum). It would be of interest to also study square-integrable, stationary rotating soliton solutions of (2.6) with $\mathcal{L}_{ab} \neq 0$.

(iii) It is straightforward to extend Lemma 2.1 to systems where $\lambda V \neq 0$. Such generalizations are of particular interest when V has *symmetries*. Then minimizers, $Q_N^{(0)}$, of $\mathcal{H}(\bar{\psi}, \psi)\big|_{\mathcal{S}_N}$ tend to *break* the symmetries of λV if N is large enough.

(iv) Let $\mathcal{H}''$ denote the Hessian of $\mathcal{H}(\bar{\psi}, \psi)$ at $\psi = Q_N^{(0)}$, $(\lambda = 0)$. In our proofs of Theorems 1.1 and 1.2 (see Sects. 3 and 4), we shall always assume that assumption (1.26) holds, for all N in an open neighborhood of some $N_0 > N_*$.

Since the proof of Lemma 2.1 is standard, it is omitted. The interesting analytical issues arise in the problems described in Remarks (iii) and (iv). They deserve further study.

2.3. A heuristic discussion of the point-particle limit of the Hartree equation. In this section we start from the results reviewed in the last section (see Lemma 2.1) to study the point-particle (Newtonian) limit of the Hartree equation. In this limit the Hartree equation reduces to the Newtonian mechanics of point-particles interacting through two-body potential forces. We use ideas closely related to those proposed in [9] in an analysis of vortex motion in the plane, as described by the Ginzburg–Landau equations.

Let λV and Φ be as in Eqs. (1.5), (2.6). We set $\lambda = 1$ and consider a family of external potentials of the form

$$V(x) \equiv V^{(\varepsilon)}(x) := W(\varepsilon x), \tag{2.29}$$

where W is some smooth, positive function on $\mathbb{R}^n$, and $\varepsilon > 0$ is a parameter. Furthermore, the two-body potential, $-\Phi$, is chosen to be

$$\Phi(x) = \Phi_s(x) + \Phi_\ell(\varepsilon x), \tag{2.30}$$

where $\Phi_s(x)$ is a rotation-invariant, smooth function decaying rapidly in $\rho := |x|$, as $\rho \to \infty$, and with the properties that

$$\frac{d\Phi_s(\rho)}{d\rho} < 0, \quad \text{for} \quad \rho > 0, \tag{2.31}$$

and that the key gap assumption (1.26) stated in Sect. 1 holds for $\Phi = \Phi_s$. The perturbing potential Φ_ℓ is rotation-invariant and smooth and may be of long range, e.g.

$$\left|\Phi_\ell(\rho)\right| \sim \rho^{2-n}, \quad \text{as} \quad \rho \to \infty, \tag{2.32}$$

for $n \geq 3$, which is the behavior of the Coulomb- and of Newton's gravitational potential. For simplicity, we assume that $\left|d\Phi_\ell(\rho)\big/d\rho\right|$ is uniformly bounded in ρ.

We pick k positive integers $N_1, \dots, N_k$, with $N_j > N_*(\Phi_s)$, for all j. For $\lambda V = 0$ and $N > N_*(\Phi_s)$, we define

$$\delta_N := \sqrt{N^{-1} \int d^n x \, Q_N^{(0)}(x)^2 x^2}, \tag{2.33}$$

where $Q_N^{(0)}$ is a rotation-invariant minimizer of the functional $\mathcal{H}(\bar{\psi}, \psi)\big|_{\mathcal{S}_N}$, as described in Lemma 2.1.

We consider an initial condition, $\psi_0(x)$, for the Hartree equation (2.6) describing a configuration of k far-separated "solitons", $Q_{N_j}^{(0)}(x - r_j)$, $r_j \in \mathbb{R}^n$, $j = 1, \dots, k$, (perturbed by a small-amplitude wave), with the following properties: Each soliton $Q_{N_j}^{(0)}(x)$ is a rotation-invariant solution of Eq. (3.22), with $\Phi = \Phi_s$ and $N = N_j$, minimizing $\mathcal{H}(\bar{\psi}, \psi)\big|_{\mathcal{S}_N}$ (for $\lambda = 0$, $\Phi = \Phi_s$). Furthermore

$$\left(\max_{j=1,\dots,k} \delta_{N_j}\right) \Big/ \left(\min_{1 \leq i < j \leq k} \left|r_i - r_j\right|\right) \leq \varepsilon, \tag{2.34}$$

where ε is the parameter introduced in (2.29), (2.30).

Our goal is to construct a solution, ψ_t, of the Hartree equation (2.6) of the form

$$\psi_t(x) = \sum_{j=1}^{k} Q^{(0)}_{N_j(t)}\left(x - r_j(t)\right) e^{i\theta_j(x,t)} + h_\varepsilon(x,t), \tag{2.35}$$

where $r_j(0) = r_j$, as in (2.34), and $\dot r_j(0) = v_j \in \mathbb{R}^n$, $j = 1, \dots, k$, with the following properties: There is a positive constant T such that, for all times t with $|t| < \frac{T}{\varepsilon}$,

(a) $\left|\left|\left|h_\varepsilon(\cdot,t)\right|\right|\right| \sim o(\varepsilon)$,
for an appropriately chosen norm $\left|\left|\left| (\cdot) \right|\right|\right|$,

(b) $\theta_j(x,t) = \dot r_j(t)\cdot\left[x - r_j(t)\right] + \vartheta_j(t)$,

where $\vartheta_j(t)$ is independent of x, and

(c) $\left|\dot N_j(t)\right| = o(\varepsilon)$.

The trajectories $r_1(t), \dots, r_k(t)$ and the phases $\vartheta_1(t), \dots, \vartheta_k(t)$ will turn out to satisfy equations of motion which can be derived from the Hartree equation. In this section we do not present a mathematical proof of the claim that solutions of the Hartree equation (2.6) of the form (2.35) with properties (a)–(c) exist; (but see Sect. 3). We merely verify that a function $\psi_t(x)$ of the form (2.35) with properties (a)–(c) approaches a critical point of the action functional $S(\bar\psi, \psi)$ introduced in (2.14), as $\varepsilon \to 0$, *provided* the trajectories $r_j(t)$ satisfy certain Newtonian equations of motion and the phases $\vartheta_j(t)$ are suitably chosen ($j = 1, \dots, k$). Since critical points of $S(\bar\psi, \psi)$ satisfy the Hartree equation (2.6), this makes it plausible that solutions of (2.6) of the form (2.35) with properties (a) – (c) exist. This claim is proven in Sect. 3 for $k = 1$.

Our heuristic analysis is based on the following simple facts:

(1) For $i \neq j$,
$$\int d^n x\, Q^{(0)}_{N_i}(x - r_i)\, Q^{(0)}_{N_j}\left(x - r_j\right) \to 0,$$
exponentially fast, as $|r_i - r_j| = 0(\varepsilon^{-1}) \to \infty$. This follows from Lemma 2.1.

(2) $\left(Q^{(0)}_{N_i(t)}, h_\varepsilon(\cdot,t)\right) = o(\varepsilon)$, for $|t| \le \dfrac{T}{\varepsilon}$,
as $\varepsilon \to 0$, for all $i = 1, \dots, k$; see (2.35) and property (a).

(3) $\left(Q^{(0)}_{N_i}, \nabla Q^{(0)}_{N_i}\right) = 0$, for all i,
by translation invariance (see Eq. (1.24)).

(4) For $y := x - r_i(t)$,
$$\int d^n y \left|Q^{(0)}_{N_i(t)}(y)\right|^2 y = 0, \quad \text{for all } i,$$
by rotation invariance.

(5) $\dot N_i(t) = 2\left(Q^{(0)}_{N_i(t)}, Q^{(0)}_{\dot N_i(t)}\right)$, for all i,
because $N_i = \left(Q^{(0)}_{N_i}, Q^{(0)}_{N_i}\right)$.

Using that

$$\begin{aligned}\frac{\partial}{\partial t}&\left[Q^{(0)}_{N_j(t)}\left(x-r_j(t)\right)e^{i\theta_j(x,t)}\right]\\ &=\left[Q^{(0)}_{\dot N_j(t)}\left(x-r_j(t)\right)-\dot r_j(t)\cdot\nabla Q^{(0)}_{N_j(t)}\left(x-r_j(t)\right)\right.\\ &\quad\left.+\,i\dot\theta_j(x,t)Q^{(0)}_{N_j(t)}\left(x-r_j(t)\right)\right]e^{i\theta_j(x,t)},\end{aligned}\tag{2.36}$$

with

$$\dot\theta_j(x,t)=\ddot r_j(t)\left[x-r_j(t)\right]-\dot r_j(t)^2+\dot\vartheta_j(t),\tag{2.37}$$

and

$$\nabla\theta_j(x,t)=\dot r_j(t),\tag{2.38}$$

we find that, for $\psi_t(x)$ as in (2.35), the action functional $S(\bar\psi,\psi)$ introduced in (2.14), with $-\frac{T}{\varepsilon}\ \le t_1<t_2\le\frac{T}{\varepsilon}$, is given by

$$\begin{aligned}S\left(\bar\psi,\psi\right)=\ &\frac{1}{2}\int_{t_1}^{t_2}dt\sum_{j=1}^{k}\left[\frac{i}{2}\,\dot N_j-\int\left|Q^{(0)}_{N_j}\left(x-r_j\right)\right|^2\ddot r_j\cdot\left(x-r_j\right)\right.\\ &+N_j\dot r_j^2-N_j\dot\vartheta_j-\frac{1}{2}\int\left|\nabla Q^{(0)}_{N_j}\right|^2-\frac{N_j}{2}\,\dot r_j^2\\ &-N_jW\left(\varepsilon r_j\right)+\frac{1}{2}\int\left(\Phi*\left|Q^{(0)}_{N_j}\right|^2\right)\left|Q^{(0)}_{N_j}\right|^2\\ &\left.+\frac{1}{2}\sum_{i:i\neq j}N_iN_j\Phi_\ell\left(\varepsilon\left(r_i-r_j\right)\right)+s_\varepsilon\right],\end{aligned}\tag{2.39}$$

where s_ε is an error term $\sim o(\varepsilon)$. In the first term on the R.S. of (2.39) we have used (5), the second term proportional to $\ddot r_j$ vanishes by (4), in the third and fourth term we have used (2.37), in the sixth term we have used (2.38), and various cross terms vanish because of (3) or only contribute to the error term because of (1) and (2). We have also used that

$$\int d^nx\,W(\varepsilon x)\left|Q^{(0)}_{N_j}\left(x-r_j\right)\right|^2=N_jW\left(\varepsilon r_j\right)+o(\varepsilon)\ ;$$

and that, for $i\neq j$,

$$\int d^nx\int d^ny\left|Q^{(0)}_{N_i}\left(x-r_i\right)\right|^2\Phi\left(x-y\right)\left|Q^{(0)}_{N_j}\left(y-r_j\right)\right|^2=N_iN_j\Phi_\ell\left(r_i-r_j\right)+o(\varepsilon),$$

by (4) and because $\Phi_s(x)$ decays rapidly in $|x|$. Thus

$$\begin{aligned}S\left(\bar\psi,\psi\right)=\ &\frac{1}{2}\,S_{\text{Newton}}\left(\left\{r_j,\ N_j\right\}_{j=1,\dots,k}\right)\\ &+\frac{1}{2}\int_{t_1}^{t_2}dt\sum_{j=1}^{k}\left[\frac{i}{2}\,\dot N_j-N_j\dot\vartheta_j-2\mathcal{H}\left(Q^{(0)}_{N_j},Q^{(0)}_{N_j}\right)+s_\varepsilon\right],\end{aligned}\tag{2.40}$$

where

$$S_{\text{Newton}}\left(\left\{r_j, N_j\right\}_{j=1,\dots,k}\right) = \int_{t_1}^{t_2} dt \sum_{j=1}^{k} \left[\frac{N_j}{2}\,\dot r_j^2 - N_j W\left(\varepsilon r_j\right) + \frac{1}{2}\sum_{i:i\neq j} N_i N_j \Phi_\ell\left(\varepsilon\left(r_i - r_j\right)\right)\right] \tag{2.41}$$

is the usual Hamiltonian action for k point particles with masses $N_1, \dots, N_k$ in an external *acceleration field* with potential $W(\varepsilon\cdot)$ and interacting through two-body forces with potential $N_i N_j \Phi_\ell\left(\varepsilon\left(r_i - r_j\right)\right)$.

In order to guarantee that the ansatz (2.35) yields a solution of the Hartree equation (2.6) with properties (a), (b) and (c), we must require that the *variation of the action* $S\left(\bar\psi, \psi\right)$ calculated in (2.40), (2.41) *with respect to the variational parameters* r_j, N_j, ϑ_j, $j = 1, \dots, k$, and h_ε *vanish!* To write down the variational equations, we observe that the second term on the R.S. of (2.40) is *independent* of $r_1, \dots, r_k$, except for the error term s_ε, which is $o(\varepsilon)$. Thus, varying $S\left(\bar\psi, \psi\right)$ with respect to $r_1, \dots, r_k$ yields *Newton's equations of motion*

$$\ddot r_j = -\,\varepsilon\,(\nabla W)\left(\varepsilon r_j\right) + \frac{\varepsilon}{2}\sum_{i:i\neq j} N_i\,(\nabla\Phi_\ell)\left(\varepsilon\left(r_j - r_i\right)\right) + a_j, \tag{2.42}$$

where a_j comes from the error term s_ε, and $|a_j(t)| \sim o(\varepsilon)$, for $|t| \le \frac{T}{\varepsilon}$; $j = 1, \dots, k$. Variation with respect to $N_1, \dots, N_k$ yields the equations

$$\dot\vartheta_j = \frac{1}{2}\,\dot r_j^2 - W\left(\varepsilon r_j\right) + \sum_{i:i\neq j} N_i \Phi_\ell\left(\varepsilon\left(r_i - r_j\right)\right) - \frac{\partial}{\partial N_j}\mathcal{H}\left(Q^{(0)}_{N_j}, Q^{(0)}_{N_j}\right) + o(\varepsilon)\,. \tag{2.43}$$

It is easy to see that

$$\begin{aligned}\frac{\partial}{\partial N_j}\,\mathcal{H}\left(Q^{(0)}_{N_j}, Q^{(0)}_{N_j}\right) &= \frac{1}{2N_j}\int\left(\nabla Q^{(0)}_{N_j}\right)^2 - \frac{1}{N_j}\int\left(\Phi * Q^{(0)^2}_{N_j}\right) Q^{(0)^2}_{N_j}\\ &= E_j - N_j\Phi_\ell(0) + o(\varepsilon),\end{aligned} \tag{2.44}$$

see Eq. (2.26). Hence, for $|t| \le \frac{T}{\varepsilon}$,

$$\dot\vartheta_j = \frac{1}{2}\,\dot r_j^2 - W\left(\varepsilon r_j\right) + \sum_{i=1}^{k} N_i \Phi_\ell\left(\varepsilon\left(r_i - r_j\right)\right) - E_j + o(\varepsilon)\,. \tag{2.45}$$

Variation with respect to $\vartheta_1, \ldots, \vartheta_k$ yields the equations

$$\dot{N}_j = o(\varepsilon), \tag{2.46}$$

(approximate conservation of masses of particles), and, finally, variation with respect to h_ε yields an equation of motion of the form

$$\frac{\partial}{\partial t} h_\varepsilon(x,t) = X\left(h_\varepsilon, \left\{r_j, N_j, \vartheta_j\right\}_{j=1}^{k}\right)(x,t), \tag{2.47}$$

with $|||X||| \sim o(\varepsilon)$, for $|t| \leq \frac{T}{\varepsilon}$, where $|||\,(\cdot)\,|||$ is an appropriately chosen norm.

At a heuristic level, eqs. (2.42) and (2.46) show very clearly that the limit $\varepsilon \to 0$ corresponds to the point-particle limit in which the masses, $N_1, \ldots, N_k$, of the particles ("solitons") are constant and their trajectories are solutions of Newton's equations of motion, on time scales of $0(\varepsilon^{-1})$.

It is interesting and useful to work out explicit expressions for all the terms of $o(\varepsilon)$ in Eqs. (2.42), (2.45), (2.46) and (2.47), in order to understand more about the *corrections* to the *Newtonian point-particle limit* and to get a handle on phenomena like radiation loss and dissipation through emission of small-amplitude dispersive radiation. But, since our discussion in this section is at a formal level, let's not! In the special case where $k = 1$, the terms of size $o(\varepsilon)$ are analyzed in Sect. 3.

The analysis of the correction term s_ε in expression (2.40) for the action functional and of the properties of solutions of Eq. (2.47) is crucial in attempting to understand the *long-time behavior* of solutions of the Hartree equation (2.6). In the introduction, we have drawn attention to results of Soffer and Weinstein [8], see also [12], concerning "non-linear Rayleigh scattering" for small-amplitude solutions of the non-linear Schrödinger- or Hartree equations with a suitable external potential λV. One would like to extend their results in the direction of a theory of *non-linear resonances* (metastable states) and gain understanding of the phenomenon of *"approach to a groundstate"*. Of particular interest are situations where the Hamilton functional $\mathcal{H}(\bar{\psi}, \psi)$, see (2.3), restricted to a sphere $\mathcal{S}_N$ in phase space has several *distinct* local minima, for N large enough. This happens when λV has several minima separated by large barriers and $-\Phi$ is the potential of an attractive force. One would then like to understand the shape of the *"basins of attraction"* in phase space of the local minima of $\mathcal{H}(\bar{\psi}, \psi)\big|_{\mathcal{S}_N}$: The forward (backward) basin of attraction of a family of local minima of $\mathcal{H}(\bar{\psi}, \psi)\big|_{\mathcal{S}_N}$ parametrized by N consists of all initial conditions in phase space which approach an element of this family plus dispersive radiation decaying to 0 at the free dispersion rate, as $t \to +\infty$ ($t \to -\infty$). This is the phenomenon of "approach to a groundstate".

More ambitiously, one might try to construct a *"centre manifold"* of asymptotically attracting configurations of solitons to which solutions of the Hartree equation with initial conditions sufficiently close to the centre manifold converge locally in space, as $|t| \to \infty$. See [12] for some preliminary results.

Let us consider an example: We choose an initial condition for the Hartree equation describing two far-separated solitons at positions r_1, r_2 and with initial velocities v_1, v_2. We suppose that $\lambda V = 0$ and that $-\Phi_\ell$ is purely attractive and of short range. The "masses" N_1, N_2 of the solitons and the initial conditions r_1, v_1 and r_2, v_2 are chosen such that the two solitons form a *bound state*, i.e., that

$$\frac{N_1}{2}\, v_1^2 + \frac{N_2}{2}\, v_2^2 - N_1 N_2 \Phi_\ell\left(\varepsilon\left(r_1 - r_2\right)\right) < 0\,. \tag{2.48}$$

One would then like to calculate the power, $P_R(t)$, of emission of dispersive radiation through a sphere of radius $R \gg \max\ (|r_1|, |r_2|)$. Moreover, one would like to show that, as $t \to \pm\infty$, a typical configuration of two solitons satisfying (2.48) collapse to a single soliton moving through space at a constant velocity. This phenomenon would describe the "radiative collapse of a binary system".

More generally, it would be interesting to understand how, at intermediate times, small inhomogeneities in the initial conditions for solutions of the Hartree equation grow to form a structure of rotating bodies (solitons) perturbed by outgoing, dispersive radiation, before it eventually approaches a number of far separated solitons escaping from each other. [In studying such problems, one finds out that the Hartree equation not only "knows" about Newton's equations of motion, it also "knows" about the Euler equations for the motion of rigid bodies.]

The problems described here are problems on the *scattering theory* for the Hartree equation. If $-\Phi$ is attractive, i.e., for a self-focussing non-linearity, scattering theory is bound to be very subtle, involving infinitely many "scattering channels", and is beyond the reach of our methods; (see, however, Sect. 4 for some preliminary results, and [7] for the case where $-\Phi$ is repulsive).

3. Proof of Theorem 1.1

In this section, we prove the first main result (Theorem 1.1) of this paper.

3.1. Stability of soliton solutions of Hartree equations. We first review the stability of the soliton solutions to the Hartree equation without external potential, i.e., for $\lambda = 0$. The equation is

$$i\partial_t \psi = 2\frac{\partial \mathcal{H}}{\partial \bar{\psi}} = -\frac{1}{2}\Delta\psi - (\Phi * |\psi|^2)\psi, \tag{3.1}$$

where $\frac{\partial \mathcal{H}}{\partial \bar{\psi}}$ ($\mathcal{H} = \mathcal{H}^{(\lambda=0)}$, see (1.11)) is the first variation of the energy functional w.r.t. $\bar{\psi}$. Recall that Q is a minimizer of $\mathcal{H}$ under the constraint $\mathcal{N}(\bar{\psi}, \psi) := \|\psi\|^2 = N$, for some N fixed, and thus Q satisfies the equation

$$-\frac{1}{2}\Delta Q - (\Phi * |Q|^2)Q = EQ, \tag{3.2}$$

for some non-linear eigenvalue (Lagrange multiplier) E. Suppose the function ψ can be written in the form $\psi = (Q + h)e^{-iEt}$. Then the linearized equation satisfied by h takes the form

$$i\partial_t h = Lh, \tag{3.3}$$

where

$$Lh = -\frac{1}{2}\Delta h - Eh - (\Phi * Q^2)h - Q(\Phi * (Q(h + \bar{h}))). \tag{3.4}$$

Due to the appearance of $\bar{h}$ on the right side of (3.4), L is not a complex-linear operator. It is therefore convenient to separate the last equation into real and imaginary parts

$$Lh = L_+A + iL_-B, \qquad h = A + iB, \quad (A \text{ and } B \text{ real}), \tag{3.5}$$

where

$$L_- = -\frac{1}{2}\Delta - E - \Phi * Q^2,$$
$$L_+ = L_- - 2Q[\Phi * (Q\cdot)], \qquad (L_+A = L_-A - 2Q[\Phi * (QA)]).$$

In matrix form,

$$\frac{\partial}{\partial t}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 & L_- \\ -L_+ & 0 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} =: \mathcal{L}\begin{pmatrix} A \\ B \end{pmatrix}. \tag{3.6}$$

($\mathcal{L}$ is the matrix form of $-iL$; it determines a linear Hamiltonian vector field.)

The operators L_- and L_+ also appear naturally in the second variation of the energy functional $\mathcal{H}$. Writing $\psi = u + iv$, we have by explicit computation

$$\begin{aligned}\mathcal{H}(Q+h) = \mathcal{H}(Q) &+ \int dx \frac{\partial \mathcal{H}}{\partial u}\Big|_Q A + \int dx \frac{\partial \mathcal{H}}{\partial v}\Big|_Q B \\ &+ \frac{1}{2}\Big[\int dx A \frac{\partial^2 \mathcal{H}}{\partial u \partial u}\Big|_Q A + 2\int dx A \frac{\partial^2 \mathcal{H}}{\partial u \partial v}\Big|_Q B \\ &+ \int dx B \frac{\partial^2 \mathcal{H}}{\partial v \partial v}\Big|_Q B\Big] + O(h^3)\end{aligned}$$

where

$$\frac{\partial \mathcal{H}}{\partial u}\Big|_Q = \left(\frac{\partial \mathcal{H}}{\partial u}\right)_{\psi=\bar{\psi}=Q}.$$

Notice that $\mathcal{H}$ has no cross terms in u and v, except in the nonlinear term depending only on $|\psi|^2$. Since Q is real, we have that

$$\frac{\partial \mathcal{H}}{\partial v}\Big|_Q = 0\,.$$

Thus the first order term is just

$$\int dx \frac{\partial \mathcal{H}}{\partial u}\Big|_Q A = 2\int dx \frac{\partial \mathcal{H}}{\partial \bar{\psi}}\Big|_Q A = E\int dx\, QA,$$

where we have used Eq. (3.2). Similarly,

$$\frac{\partial^2 \mathcal{H}}{\partial u \partial v}\Big|_Q = 0,$$

and the second order term is just

$$\begin{aligned}&\frac{1}{2}\Big[\int dx A \frac{\partial^2 \mathcal{H}}{\partial u \partial u}\Big|_Q A + \int dx B \frac{\partial^2 \mathcal{H}}{\partial v \partial v}\Big|_Q B\Big] \\ &= \int dx A L_+ A + \int dx B L_- B + E\int dx (A^2 + B^2)\,.\end{aligned}$$

(Observe that $\mathcal{H}''_{\text{real}} - E = L_+$.) We have thus proved that

$$\mathcal{H}(Q+h) = \mathcal{H}(Q) + E[(Q,A) + \|h\|^2] + \text{Re}(Lh, h) + O(h^3), \tag{3.7}$$

where $(f, g) = \int \bar{f} g dx$ is the standard L_2 scalar product and

$$\mathrm{Re}(Lh, h) = \int dx A L_+ A + \int dx B L_- B.$$

Let $Q_\varepsilon \equiv Q_{N+\varepsilon}$ be the (real) minimizer centered at the origin, with $\|Q_\varepsilon\|^2 = N + \varepsilon$. Let $h_\varepsilon = Q_\varepsilon - Q$. Then

$$\varepsilon = \|Q + h_\varepsilon\|^2 - \|Q\|^2 = 2 \int Q h_\varepsilon + \int h_\varepsilon^2 = 2 \int Q h_\varepsilon + O(\varepsilon^2).$$

We define $\mathcal{E}(N)$ as the minimal energy subject to the constraint $\|\psi\|^2 = N$:

$$\mathcal{E}(N) = \inf_{\|\psi\|^2 = N} \mathcal{H}(\psi).$$

The last two equations and (3.7) then yield the standard relation

$$\frac{\partial \mathcal{E}(N)}{\partial N} = E/2. \tag{3.8}$$

For an arbitrary h with $\mathrm{Re} h \perp Q$, Eq. (3.7) yields

$$\left[\int dx A L_+ A + \int dx B L_- B \right] = \mathcal{H}(Q + h) - \mathcal{H}(Q) - E\|h\|^2 + O(h^3). \tag{3.9}$$

Since $\mathcal{H}(Q + h) \geq \mathcal{E}(\|Q + h\|^2) = \mathcal{E}(N + \|h\|^2)$, (because $\mathrm{Re} h \perp Q$, $\|Q + h\|^2 = \|Q\|^2 + \|h\|^2$), we obtain from Eq. (3.8)

$$\mathcal{H}(Q + h) - \mathcal{H}(Q) - E\|h\|^2 \geq O(h^3).$$

This proves that

$$\int dx A L_+ A \geq 0, \qquad \int dx B L_- B \geq 0,$$

for all $A \perp Q$ and arbitrary B. Thus $L_- \geq 0$, and L_+ has at most one negative eigenvalue. From the explicit form of L_- and L_+ we conclude that

$$L_- Q = 0, \qquad L_+ \nabla Q = 0, \qquad L_-(xQ) = -\nabla Q. \tag{3.10}$$

Since Q is positive and $L_- \geq 0$, its null space is the span of Q, i.e.,

$$L_- \geq 0, \qquad N(L_-) = \mathrm{span}_{\mathbb{R}}\{Q\}.$$

From the explicit form of L_+ we have that

$$(Q, L_+ Q) =: \varepsilon_0 \cdot (Q, Q) < 0, \tag{3.11}$$

where

$$\varepsilon_0 = -2(N(Q))^{-1} \int Q^2 \left(\Phi * Q^2 \right) < 0.$$

Thus L_+ has exactly one negative eigenvalue. The continuous spectra of L_- and L_+ can easily be shown to be the half-line $[-E, \infty)$. Since $L_+ \nabla Q = 0$, 0 is an at least n-fold degenerate eigenvalue of L_+. A key assumption in our analysis is that the whole null space of $L_+ = \mathcal{H}''_{\text{real}} - E$ is spanned by ∇Q, i.e.,

$$\boxed{N(L_+) = \text{span}_{\mathbb{R}}\{\nabla Q\}.} \tag{3.12}$$

Since the continuous spectrum of L_- and of L_+ is the half-line $[-E, \infty)$, 0 is an isolated point. Hence there is a positive number δ such that

$$(h, L_+ h) \geq \delta(h, h),$$

if h is orthogonal to the span of ∇Q and to the ground state of L_+. In particular, the number of eigenvalues strictly below δ is exactly $n + 1$. We have proved the following lemma.

Lemma 3.1. *Assume that* (3.12) *holds. Then the null spaces of L_- and L_+ are given by $N(L_-) = \text{span}_{\mathbb{R}}\{Q\}$, $N(L_+) = \text{span}_{\mathbb{R}}\{\nabla Q\}$. Furthermore, there is a constant $\varepsilon_2 > 0$ such that* (*a*) $(g, L_- g) \geq \varepsilon_2(g, g)$ *if* $g \perp Q$. (*b*) $(f, L_+ f) \geq \varepsilon_2(f, f)$ *if* $f \perp \text{span}_{\mathbb{R}}\{Q, \nabla Q\}$.

If we assume that

$$\|Q + h\|^2 = \|Q\|^2$$

the term with the factor E in (3.7) vanishes, because

$$2(Q, A) = -\|h\|^2,$$

and we have that

$$\mathcal{H}(Q + h) = \mathcal{H}(Q) + \left[\int dx A L_+ A + \int dx B L_- B\right] + O(h^3). \tag{3.13}$$

Thus if $h = A + iB$, with

$$A \perp \text{span}_{\mathbb{R}}\{\nabla Q\}, \quad B \perp Q, \quad \|Q + h\|^2 = \|Q\|^2, \tag{3.14}$$

then we can write $A = A_1 + cQ$, with $(A_1, Q) = 0$, for some c of order $\|h\|^2$, $(c(Q, Q) = (A, Q) = -(h, h)/2)$. Since $(Q, \nabla Q) = 0$, we have that $(\nabla Q, A_1) = 0$, provided that $(A, \nabla Q) = 0$. Therefore, under assumption (3.14), we can rewrite (3.7) as

$$\mathcal{H}(Q + h) - \mathcal{H}(Q) = \big[(A, L_+ A) + (B, L_- B)\big] + O(h^3) \tag{3.15}$$

$$= \big[(A_1, L_+ A_1) + (B, L_- B)\big] + O(h^3). \tag{3.16}$$

Since $A_1 \perp \text{span}_{\mathbb{R}}\{Q, \nabla Q\}$, we can apply Lemma 3.1 to conclude that $(A_1, L_+ A_1) + (B, L_- B) \geq \varepsilon_2(\|A_1\|^2 + \|B\|^2)$. Since the difference between $\|A_1\|^2$ and $\|A\|^2$ is of higher order, we obtain

$$\mathcal{H}(Q + h) - \mathcal{H}(Q) \geq \varepsilon_2 \|h\|^2 + O(h^3). \tag{3.17}$$

The last equation implies the *global (modulational) stability* of the soliton solution under small perturbations. To see this, suppose the initial data is of the form $Q + h_0$, with $\|Q + h_0\|^2 = \|Q\|^2$; (the last condition always holds, since we can choose a Q with the mass of the initial value). At a later time t, we can find r and θ such that

$$\psi_t(x - r)e^{-i\theta} = h(x) + Q(x)$$

with the mass of the correction, $\|h\|^2$, minimized. One can easily check that h satisfies condition (3.14). By inequality (3.17), $\|h\|^2$ is bounded from above by the left hand side, which is conserved under the time evolution.

3.2. Dynamical linearization of the Hartree equation around solitons. We now return to the Hartree equation (1.5) with external potential $\lambda V(x) = W(\varepsilon x)$. Since our time scale is of order $t \sim \varepsilon^{-1}$, the change in the external potential during the evolution on this time scale may not be small. Thus the argument in the last section no longer applies. We shall show that, nevertheless, the soliton solution is stable on this time scale, and we shall track the motion of the soliton precisely.

The Hartree equation (1.5) is

$$i\partial_t \psi = -\frac{1}{2}\Delta\psi + W(\varepsilon x)\psi - (\Phi * |\psi|^2)\psi =: H(\psi)\psi. \tag{3.18}$$

The solutions we are interested in are of the form

$$\psi(x,t) = [Q\,(x - r(t)) + h_\varepsilon\,(x - r(t), t)]\, e^{i\theta(x,t)}, \tag{3.19}$$

for $\varepsilon > 0$ small enough, where $Q(x) = Q^{(\varepsilon=0)}(x)$ is a minimizer of the energy functional $\mathcal{H}$, and $h_\varepsilon(x - r(t), t)$ is a small correction term which tends to 0, as $\varepsilon \to 0$; $\theta(x,t)$ is a time-dependent phase of the form

$$\theta(x,t) = v(t)\cdot(x - r(t)) - Et + \theta_1(t).$$

Also, we expect that, to leading order, the velocity $v(t)$ and the location $r(t)$ of the soliton are given by

$$\dot r(t) = v(t)\,, \qquad \dot v(t) = -\varepsilon\,(\nabla W)\,(\varepsilon r(t))\,.$$

For the time being, there is no canonical way to determine corrections to these equations, as the decomposition (3.19) is not unique. We require v, r, and θ_1 to obey the following equations:

$$\begin{aligned}
\dot r(t) &= v(t),\\
\dot v(t) &= -\varepsilon(\nabla W)(\varepsilon r(t)) + a(t),\\
\dot\theta_1(t) &= \tfrac{1}{2}v^2(t) - W(\varepsilon r(t)) + \omega(t),
\end{aligned}$$

where the (vector) acceleration correction $a(t)$ and the (scalar) "angular velocity" correction $\omega(t)$ are of higher order in ε and will be used for fine adjustment, later on. Their initial values will be discussed in Subsect. 4.5.1, when we adjust the initial datum $h_{\varepsilon,0}$.

We now derive the equations for a, ω and h. Let $\xi(x,t) = Q(y)e^{i\theta}$, $y = x - r(t)$. By explicit computation,

$$\begin{aligned}
\xi^{-1}\{i\partial_t - H(\xi)\}\,\xi &= \xi^{-1}\left[i\partial_t\xi + \frac{1}{2}\Delta\xi\right] - W(\varepsilon x) + \Phi * |\xi|^2\\
&= \left[-\dot\theta_1(t) - \partial_t[v(t)(x - r(t)] - \tfrac{1}{2}v^2(t)\right] + \left[i\frac{\nabla Q}{Q}(v(t) - \dot r(t))\right]\\
&\quad - W(\varepsilon x) + \left[E + \frac{\Delta Q}{2Q} + \Phi * |\xi|^2\right].
\end{aligned}$$

We expand the potential W around the point $r(t)$:

$$W(\varepsilon x) = W(\varepsilon r(t)) + \nabla W(\varepsilon r(t))\varepsilon(x - r(t)) + \Omega_0(x,t),$$

where the remainder $\Omega_0(x,t)$ is real and, by the mean value theorem,

$$|\Omega_0(x,t)| \le C\varepsilon^2 |y|^2, \tag{3.20}$$

where $C = C(W)$ depends on W. Recalling the equation for r, v and (3.2), we then have

$$\xi^{-1}\{i\partial_t - H(\xi)\}\,\xi = -\Omega\xi, \tag{3.21}$$

where

$$\Omega = -W(\varepsilon r) + W(\varepsilon x) + \dot{v}y + \omega = \Omega_0(x,t) + a(t)y + \omega(t). \tag{3.22}$$

Next, we consider $h(y,t) = h_\varepsilon(x - r(t), t)$. Substituting $\psi = (Q+h)e^{i\theta}$ into Eq. (3.18) and canceling $e^{i\theta}$ we get

$$\begin{aligned} i\,\{(Q+h)(i\partial_t\theta) - \dot{r}\cdot\nabla(Q+h) + \partial_t h\} \\ = \left\{-\frac{1}{2}\Delta(Q+h) - iv\cdot\nabla(Q+h) + \frac{1}{2}v^2(Q+h)\right\} \\ + W(\varepsilon x)(Q+h) - (\Phi * |Q+h|^2)(Q+h), \end{aligned}$$

where Q and h are taken at $(y,t) = (x - r(t), t)$, that is, $Q = Q(x - r(t)) = Q(y)$ and $h = h(x - r(t), t) = h(y,t)$. Using $\dot{r}(t) = v(t)$, Eq. (3.2), and

$$\partial_t\theta = \dot{v}\cdot x + \left\{-\frac{1}{2}v^2 - \dot{v}r - W(\varepsilon r) + \omega(t)\right\} - E = -W(\varepsilon x) + \Omega - \frac{1}{2}v^2 - E,$$

we obtain

$$i\partial_t h = -\frac{1}{2}\Delta h - Eh + \Omega(Q+h) - \left\{(\Phi * |Q+h|^2)(Q+h) - (\Phi * Q^2)Q\right\}. \tag{3.23}$$

Treating Ωh as an error term, we can rewrite this equation as

$$\partial_t h = -iLh + G, \tag{3.24}$$

where the operator L is given by (3.4), and the nonlinear part is

$$\begin{aligned} G = -\,i\Omega(Q+h) - iF(h), \\ \text{with } F(h) = -\left\{(\Phi * |h|^2)(Q+h) + (\Phi * [Q(h+\bar{h})])h\right\}. \end{aligned} \tag{3.25}$$

In matrix form,

$$\frac{\partial}{\partial t}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 & L_- \\ -L_+ & 0 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} + \begin{pmatrix} \mathrm{Re}G \\ \mathrm{Im}G \end{pmatrix}. \tag{3.26}$$

We observe that, except for Ω_0 which is part of Ω (and thus appears in G), all quantities in this system are evaluated at (y,t).

3.3. Properties of the linearized flow. We have shown that the linear part in the dynamical linearization of the nonlinear Hartree equation results in the standard linear evolution (3.3) with matrix form given in (3.6). We notice that L_+ and L_-, the real and imaginary part of L, can be reinterpreted as complex-linear operators which turn out to be self-adjoint in the usual L^2 space. The operator

$$\mathcal{L} = \begin{pmatrix} 0 & L_- \\ -L_+ & 0 \end{pmatrix}, \qquad \text{with } \mathcal{L}^* = \begin{pmatrix} 0 & -L_+ \\ L_- & 0 \end{pmatrix}$$

acting on $H^1 \times H^1$ is, however, not symmetric. Although our functions A and B are real, we shall view L_- and L_+ as self-adjoint operators on the Sobolev space H^1 of complex-valued functions. The operator $\mathcal{L}$ is, however, not self-adjoint and thus does not have a spectral decomposition. A standard procedure is to decompose the space $H^1 \times H^1$ into a direct sum of its generalized null space,

$$S := N_g(\mathcal{L}) = \{v : \mathcal{L}^n v = 0 \text{ for some } n\},$$

and the orthogonal compliment of the generalized null space of its adjoint, i.e., the space $M = N_g(\mathcal{L}^*)^\perp$. It is simple to check that both spaces, $S = N_g(\mathcal{L})$ and $M = N_g(\mathcal{L}^*)^\perp$, are invariant under $\mathcal{L}$. Note that the decomposition $H^1 \times H^1 = M \oplus S$ is, however, *not* an orthogonal decomposition.

Following M. I. Weinstein [13], we want to establish the following picture:

1. $H^1 \times H^1 = M \oplus S$.
2. The $H^1 \times H^1$-norm on M remains uniformly bounded under the linearized flow for all time.
3. The dynamics on S can be computed explicitly.

We use P_M and P_S to denote (*non*-orthogonal) projections with respect to the decomposition $M \oplus S$. We first establish some spectral properties of L_+ and L_-.

3.3.1. Generalized null space. We first determine the generalized null space $S = N_g(\mathcal{L})$. We recall Lemma 3.1 and the equations

$$L_- Q = 0, \quad L_+ \nabla Q = 0, \quad L_- x Q = -\nabla Q.$$

Since $Q \perp \operatorname{span}_{\mathbb{R}}\{\nabla Q\} = N(L_+)$ and L_+ is self-adjoint, there exists a solution, Γ_1, of the equation $L_+\Gamma_1 = Q$. We may assume $\Gamma_1 \perp \nabla Q$ by subtracting its projection on the ∇Q-direction. If $\Gamma_1 \perp Q$, then $(\Gamma_1, Q) = (\Gamma_1, L_+\Gamma_1) > \varepsilon_1(\Gamma_1, \Gamma_1)$, by Lemma 3.1. This contradiction shows $(\Gamma_1, Q) \neq 0$. Now we let $\Gamma = \Gamma_1 + b\nabla Q$ with $b = 2(\Gamma_1, xQ)$. Then $(\Gamma, Q) = (\Gamma_1, Q) \neq 0$, and $(\Gamma, xQ) = 0$. To summarize, we have found a Γ such that

$$L_+\Gamma = Q, \qquad (\Gamma, xQ) = 0, \qquad (\Gamma, Q) \neq 0. \tag{3.27}$$

We require $(\Gamma, xQ) = 0$, in order to construct a dual basis on S in Proposition 3.2 below.

To determine the generalized null space, we need to solve all solutions of the equation $\mathcal{L}^n \binom{u}{v} = \binom{0}{0}$ for some n. If $n = 2k$ is even, it is equivalent to $(L_-L_+)^k u = 0$ and $(L_+L_-)^k v = 0$. If $n = 2k+1$ is odd, it is equivalent to $L_+(L_-L_+)^k u = 0$ and $L_-(L_+L_-)^k v = 0$. We have solved the solutions for the case $n = 1$ above: It is the span of $\binom{0}{Q}$ and $\binom{\nabla Q}{0}$.

We next consider the case $n = 2$. The null space of L_+L_- is

$$N(L_+L_-) = {L_-}^{-1}N(L_+) = N(L_-) \oplus \mathrm{span}_{\mathbb{R}}\{xQ\} = \mathrm{span}_{\mathbb{R}}\{Q, xQ\}. \tag{3.28}$$

Similarly,

$$N(L_-L_+) = N(L_+) \oplus \mathrm{span}_{\mathbb{R}}\{\Gamma\} = \mathrm{span}_{\mathbb{R}}\{\nabla Q, \Gamma\}. \tag{3.29}$$

For the case $n = 3$, we have

$$N(L_-L_+L_-) = {L_-}^{-1}N(L_-L_+) = {L_-}^{-1}\mathrm{span}_{\mathbb{R}}\{\nabla Q, \Gamma\}\ .$$

Since $N(L_-) = \mathrm{span}_{\mathbb{R}}\{Q\}$ and $(Q, \Gamma) \neq 0$, Γ is not in the range of L_-. Thus

$${L_-}^{-1}\mathrm{span}_{\mathbb{R}}\{\nabla Q, \Gamma\} = {L_-}^{-1}\mathrm{span}_{\mathbb{R}}\{\nabla Q\} = N(L_-L_+).$$

This proves that $N(L_-L_+L_-) = N(L_+L_-)$. Similarly, $N(L_+L_-L_+) = N(L_-L_+)$. Therefore, if $\mathcal{L}^n \binom{u}{v} = \binom{0}{0}$ for some $n \geq 2$, then $\mathcal{L}^2 \binom{u}{v} = \binom{0}{0}$. Thus we have found a basis for $N_g(\mathcal{L})$. We also have similar statements for $N_g(\mathcal{L}^*)$. Summarizing, we have proved

Proposition 3.2.

$$\begin{aligned} S = N_g(\mathcal{L}) &= \mathrm{span}_{\mathbb{R}}\{\tbinom{0}{Q}, \tbinom{\nabla Q}{0}, \tbinom{0}{xQ}, \tbinom{\Gamma}{0}\}, \\ N_g(\mathcal{L}^*) &= \mathrm{span}_{\mathbb{R}}\{\tbinom{0}{\Gamma}, \tbinom{xQ}{0}, \tbinom{0}{\nabla Q}, \tbinom{Q}{0}\}. \end{aligned} \tag{3.30}$$

Notice that these vectors are dual bases and we have ordered them correspondingly. In particular, for an arbitrary function g we have

$$\begin{aligned} P_S(g) = {} & \kappa_1(\mathrm{Im}g, \Gamma)\tbinom{0}{Q} + \kappa_2(\mathrm{Re}g, xQ)\tbinom{\nabla Q}{0} \\ & + \kappa_2(\mathrm{Im}g, \nabla Q)\tbinom{0}{xQ} + \kappa_1(\mathrm{Re}g, Q)\tbinom{\Gamma}{0}, \end{aligned}$$

where $\kappa_1 = 1/(Q, \Gamma)$ and $\kappa_2 = 1/(x_jQ, \partial_jQ) = -2$. Also note that we have

$$\mathcal{L}\tbinom{0}{Q} = \tbinom{0}{0}, \quad \mathcal{L}\tbinom{\nabla Q}{0} = \tbinom{0}{0}, \quad \mathcal{L}\tbinom{0}{yQ} = -\tbinom{\nabla Q}{0}, \quad \mathcal{L}\tbinom{\Gamma}{0} = -\tbinom{0}{Q}. \tag{3.31}$$

Let $g(t)$ be a solution to the linear evolution (3.3) and denote the projection onto S by

$$P_Sg(t) = \alpha(t)\tbinom{0}{Q} + \beta(t)\tbinom{\nabla Q}{0} + \gamma(t)\tbinom{0}{xQ} + \delta(t)\tbinom{\Gamma}{0}.$$

Then by (3.31) the equations for the coefficients $(\alpha(t), \beta(t), \gamma(t), \delta(t))$ (note $\beta(t)$ and $\gamma(t)$ are vector functions) are given by

$$\begin{aligned} \tbinom{0}{Q}: &\quad \dot{\alpha} = -\delta, \\ \tbinom{\nabla Q}{0}: &\quad \dot{\beta} = -\gamma, \\ \tbinom{0}{yQ}: &\quad \dot{\gamma} = 0, \\ \tbinom{\Gamma}{0}: &\quad \dot{\delta} = 0. \end{aligned}$$

3.3.2. The flow on M. We have decomposed the space $H^1 \times H^1$ into a direct sum of the generalized null space $S = N_g(\mathcal{L})$ and $M = N_g(\mathcal{L}^*)^\perp$. The generalized null spaces for L and L^* are given by Proposition 3.2. Thus, M is the space

$$M = \{\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) : u \perp \mathrm{span}_{\mathbb{R}}\{Q, xQ\}, v \perp \mathrm{span}_{\mathbb{R}}\{\nabla Q, \Gamma\}\}.$$

Since all functions in the space $S = N_g(\mathcal{L})$ and $M^\perp = N_g(\mathcal{L}^*)$ are smooth, the projections P_S and P_M are bounded in any H^k space.

Our first aim is to prove

Lemma 3.3 (H^1-norm on M).

1. *If $g \in M$, then* $\mathrm{Re}(Lg, g)$ *is non-negative and comparable to* $\|g\|^2_{H^1}$.
2. *If $g(t) = e^{-itL}\phi$ and $0 \neq \phi \in M$, then $\|g(t)\|_{H^1}$ is uniformly bounded below and above.*

To prove this lemma, we first show that, for all vectors $\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) \in M$,

$$C^{-1}\|u\|^2_{L^2} \le (u, L_+u), \qquad C^{-1}\|v\|^2_{L^2} \le (v, L_-v)\ , \tag{3.32}$$

for some constant C, as follows from Lemma 3.1. In fact, it is sufficient to assume that $u \perp \mathrm{span}_{\mathbb{R}}\{Q, xQ\}$ and $v \perp \mathrm{span}_{\mathbb{R}}\{\Gamma\}$. (As will become clear, we only use $(\Gamma, Q) \neq 0$ and $(xQ, \nabla Q) \neq 0$ in this argument.)

For the v-part, if $(v, Q) = 0$, the claim follows from Lemma 3.1. Hence we may assume $tv = Q + w$ for some $t \neq 0$, $w \perp Q$. By assumption $0 = (\Gamma, tv) = (\Gamma, Q + w)$, hence $|(\Gamma, Q)| = |(\Gamma, w)| \le \|\Gamma\|_2\, \|w\|_2$. Therefore we have $\|w\|_2 \ge c_3 > 0$ and

$$\frac{(v, L_-v)}{(v, v)} = \frac{(w, L_-w)}{\|Q\|_2^2 + \|w\|_2^2} \ge \frac{\varepsilon_2\, \|w\|_2^2}{\|Q\|_2^2 + \|w\|_2^2} \ge \frac{\varepsilon_2 c_3^2}{\|Q\|_2^2 + c_3^2}.$$

For the u-part, if $(u, \nabla Q) = 0$, the claim follows from Lemma 3.1. Hence we may assume $u = b\nabla Q + w$ for some vector $b \neq 0$ and some $w \perp Q, \nabla Q$. By assumption, $0 = b(xQ, u) = (bxQ, b\nabla Q + w) = C|b|^2 + (bxQ, w)$, with $C \neq 0$. Hence $\|w\|_2 > C|b|$ and

$$\frac{(u, L_+u)}{(u, u)} = \frac{(w, L_+w)}{Cb^2 + \|w\|_2^2} \ge \frac{\varepsilon_2\, \|w\|_2^2}{Cb^2 + \|w\|_2^2} \ge C\varepsilon_2,$$

by a similar estimate. Hence (3.32) is proved.

Now, since $\|\nabla u\|^2$ is bounded by (u, L_+u) and $\|u\|^2$, (and hence by (u, L_+u), see (3.32)), we can replace the norm on the left hand side of (3.32) by the H_1-norm; (here the H_1-norm is the sum of the L^2-norm plus the L^2-norm of the derivative). Therefore we have proved that, for $\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) \in M$,

$$\begin{aligned} C^{-1}(u, u)_{H^1} &\le (u, L_+u) \le C(u, u)_{H^1}\ , \\ C^{-1}(v, v)_{H^1} &\le (v, L_-v) \le C(v, v)_{H^1}\ . \end{aligned} \tag{3.33}$$

The upper bounds on (u, L_+u) and (v, L_-v) are obvious. Hence the first part of the lemma is proved.

The second part follows from the first part and the next lemma, which states that the quantity $(u, L_+u) + (v, L_-v)$, which is equivalent to the H^1-norm on M, is actually conserved by the linear flow (3.3). We note that

$$(u, L_+u) + (v, L_-v) = \mathrm{Re}(Lg, g) = \mathrm{Im}(\mathcal{L}f, g).$$

Lemma 3.4. *Recall that $\mathcal{L} = -iL$, and $iL \neq Li$, (see (4.5)).*

1. $\mathrm{Re}(Lf, g) = \mathrm{Im}(\mathcal{L}f, g) = \mathrm{Im}(\mathcal{L}g, f) = -\mathrm{Im}(f, \mathcal{L}g)$.
2. *If $g(t) = e^{-itL}\phi$, then $\mathrm{Im}(\mathcal{L}^k g, g)$ is constant for any integer $k \geq 0$.*
3. *For any $g(t)$ with $\partial_t g = \mathcal{L}g + G$, one has*

$$\frac{d}{dt}\mathrm{Im}(\mathcal{L}g, g) = 2\mathrm{Im}(\mathcal{L}g, G).$$

Proof. All these assertions can be checked by simple computations. We only prove the last one in the following.

$$\begin{aligned}\frac{d}{dt}\mathrm{Im}(\mathcal{L}g, g) &= \mathrm{Im}(\mathcal{L}^2 g + \mathcal{L}G, g) + \mathrm{Im}(\mathcal{L}g, \mathcal{L}g + G) \\ &= \mathrm{Im}(\mathcal{L}G, g) + \mathrm{Im}(\mathcal{L}g, G) = 2\mathrm{Im}(\mathcal{L}g, G). \qquad \square\end{aligned}$$

The following two lemmas will be used to prove inequality (3.61) below. (Note H^k denotes the Sobolev space $W^{k,2}$.)

Lemma 3.5. (a) *For any $m \geq 1$, e^{-itL} is a bounded map from $M \cap H^m$ into itself. Explicitly, for any $\phi \in M \cap H^m$,*

$$\left\| e^{-itL}\phi \right\|_{H^m} \leq C_m \left\| \phi \right\|_{H^m}.$$

(b) *$\mathcal{L} = -iL$, restricted to M, has an inverse which is bounded from $M \cap L^2$ to $M \cap H^2$.*

Proof. Proof of (a): Let $g(t) = e^{-itL}$. The case $m = 1$ is Lemma 3.3, part 2. If $m \geq 3$ is odd, we have that

$$\begin{aligned}\|g(t)\|_{H^m}^2 &\leq C \left|\mathrm{Im}(\mathcal{L}^m g, g)\right| + C \|g(t)\|_{H^{m-2}}^2 \\ &\leq C \left|\mathrm{Im}(\mathcal{L}^m \phi, \phi)\right| + C \|\phi\|_{H^{m-2}}^2 \leq C \|\phi\|_{H^m}^2.\end{aligned}$$

The second inequality uses Lemma 3.4, part 2. (Note: If m is even, $\mathrm{Im}(\mathcal{L}^m g, g) = 0$, and the first inequality fails.) The general case follows from interpolation.
Proof of (b): For $\binom{u}{v} \in M$ we seek $\binom{x}{y} \in M$ such that $\mathcal{L}\binom{x}{y} = \binom{u}{v}$, i.e., $L_- y = u$ and $L_+ x = -v$. Notice that $u \perp Q$ and $v \perp \nabla Q$, and the null spaces of the self-adjoint operators L_- and L_+ are spanned by Q and ∇Q respectively. Since 0 is an isolated eigenvalue of L_- and L_+, it follows that L_-^{-1} and L_+^{-1} are bounded operators on the orthogonal complements of the null spaces. This proves that $\mathcal{L}$ has a bounded inverse on $M \cap L^2$.

To prove the bound, write $w = \binom{u}{v} \in M \cap H^2$. By (3.33)

$$\|u\|_{W^{1,2}}^2 \leq C(u, L_+ u) \leq \frac{1}{2}\|u\|_2^2 + C\|L_+ u\|_2^2.$$

Hence $\|u\|_{W^{1,2}} \leq C\|L_+ u\|_2$. Similarly $\|v\|_{W^{1,2}} \leq C\|L_- v\|_2$. Furthermore, write $L_+ = -\frac{1}{2}\Delta + V$. (The explicit form of V can easily be read from the definition of L_+.)

$$\|\Delta u\|_2 = 2\|L_+ u - Vu\|_2 \leq 2\|L_+ u\|_2 + C(V)\|u\|_2 \leq C\|L_+ u\|_2.$$

Hence $\|u\|_{W^{2,2}} \leq C\|L_+ u\|_2$. Similarly $\|v\|_{W^{2,2}} \leq C\|L_- v\|_2$. We conclude that $\|w\|_{W^{2,2}} \leq C\|\mathcal{L}w\|_2$. The lemma follows by a duality argument. $\square$

Let

$$X_k = H^k \cap L^2 \left\{(1+|y|^{2k})dy\right\} \tag{3.34}$$

denote the subspace of H^k of functions with prescribed decay at infinity.

Lemma 3.6 (Finite propagation speed). *For any integer $k \geq 0$, for any real $m \geq 1$, and for $\phi \in M \cap X_k \cap H^{k+m}$,*

$$\left\| y^m e^{t\mathcal{L}} \phi \right\|_{H^k} \leq C \left\| y^m \phi \right\|_{X_k} + C(1+|t|^m) \left\|\phi\right\|_{H^{k+m}} . \tag{3.35}$$

The constant C depends on k and m.

Remark. For the free Schrödinger equation, one need not assume that $\phi \in M$, since Lemma 3.5 (a) always holds.

Proof. Let α be any multi-index with $|\alpha| = k$. Let $g(t) = e^{-itL}\phi$ and $v(t) = \nabla_y^\alpha g(t)$. We have

$$\partial_t v = \mathcal{L} v + \Lambda g, \quad \text{with } \Lambda = [\nabla^\alpha, \mathcal{L}].$$

Hence,

$$\begin{aligned} \frac{d}{dt} \int y^{2m}|v|^2 &= 2\mathrm{Re} \int y^{2m} \bar{v} v_t \\ &\leq C \int y^{2m-1}|v||\nabla v| dy + C \left\|v\right\|_2^2 + C \int y^{2m}|v||\Lambda g|. \end{aligned}$$

Since Λ is a localized operator involving derivatives only up to $(k-1)^{\text{st}}$ order, (Λ vanishes for $k = 0$), the last term is bounded by $\left\|v\right\|_2 \left\|g\right\|_{H^{k-1}} \leq C \left\|\phi\right\|_{H^k}^2$. Hence, by interpolation,

$$\begin{aligned} \left| \frac{d}{dt} \left\| y^m v \right\|_2^2 \right| &\leq C \left\| y^m v \right\|_2 \cdot \left\| y^{m-1} \nabla v \right\|_2 + C \left\|\phi\right\|_{H^k}^2 \\ &\leq C \left\| y^m v \right\|_2^{2(1-1/2m)} \cdot \left\|v\right\|_{H^m}^{1/m} + C \left\|\phi\right\|_{H^k}^2 . \end{aligned}$$

Let $f(t) = \left\|y^m v\right\|_2^2$ and $N = C \left\|\phi\right\|_{H^{k+m}}^2$. By Hölder's inequality,

$$\left|f'\right| \leq f^{1-1/2m} N^{1/2m} + N \leq \frac{f}{1+t} + C(1+t)^{2m-1} N,$$

hence $f(t) \leq Cf(0) + C(1+t)^{2m} N$, which proves the claim. $\square$

We will need the case $k = 1$ when we prove Lemma 3.8.

3.4. The fine adjustment. We first recall the conclusion of dynamical linearization. We decompose the function ψ into the sum

$$\psi(x,t) = [Q\,(x-r(t)) + h_\varepsilon\,(x-r(t),t)]\,e^{i\theta(x,t)},$$

where $\theta(x,t)$ is a time-dependent phase of the form

$$\theta(x,t) = v(t)\cdot(x-r(t)) - Et + \theta_1(t),$$

with

$$\begin{aligned}
\dot r(t) &= v(t),\\
\dot v(t) &= -\nabla W(\varepsilon r(t))\varepsilon + a(t),\\
\dot\theta_1(t) &= \frac{1}{2}v^2(t) - W(\varepsilon r(t)) + \omega(t).
\end{aligned}$$

Here the (vector) acceleration correction $a(t)$ and the (scalar) angular velocity correction $\omega(t)$ are of smaller orders, and we shall determine their values in this subsection. The main correction h satisfies the equation

$$\frac{\partial}{\partial t}h = \mathcal{L}h + G, \qquad h(0) = h_{\varepsilon,0}, \tag{3.36}$$

with

$$\begin{aligned}
G &= -i\Omega(Q+h) - iF(h),\\
\Omega &= \Omega_0 + ay + \omega,\\
\Omega_0 &= W(\varepsilon x) - W(\varepsilon r) - \varepsilon y\nabla W(\varepsilon r),\\
F &= -(\Phi * |h|^2)(Q+h) - (\Phi * [Q(h+\bar h)])h.
\end{aligned}$$

We decompose $h(t)$ into a sum of its components in S and M: $h(t) = h_S(t) + h_M(t)$. The component in S is a sum of the basis vectors (3.30)

$$h_S(t) = \alpha(t)\begin{pmatrix}0\\Q\end{pmatrix} + \beta(t)\begin{pmatrix}\nabla Q\\0\end{pmatrix} + \gamma(t)\begin{pmatrix}0\\yQ\end{pmatrix} + \delta(t)\begin{pmatrix}\Gamma\\0\end{pmatrix}.$$

We now consider projections of Eq. (3.36) onto S and M. Taking inner products with the dual basis, (see Proposition 3.2), we obtain the equations on S:

$$\begin{pmatrix}0\\Q\end{pmatrix}: \quad \dot\alpha = -\delta + \kappa_1(\mathrm{Im}G,\Gamma), \tag{3.37}$$

$$\begin{pmatrix}\nabla Q\\0\end{pmatrix}: \quad \dot\beta = -\gamma + \kappa_2(\mathrm{Re}G, yQ), \tag{3.38}$$

$$\begin{pmatrix}0\\yQ\end{pmatrix}: \quad \dot\gamma = \kappa_2(\mathrm{Im}G, \nabla Q), \tag{3.39}$$

$$\begin{pmatrix}\Gamma\\0\end{pmatrix}: \quad \dot\delta = \kappa_1(\mathrm{Re}G, Q). \tag{3.40}$$

The equation on M is

$$\frac{\partial}{\partial t}h_M = \mathcal{L}h_M + P_M G. \tag{3.41}$$

Notice that $\Omega_0 = W(\varepsilon x) - W(\varepsilon r) - \varepsilon y\nabla W(\varepsilon r)$ is determined by $r(t)$, which solves

$$\ddot r = -\varepsilon\nabla W(\varepsilon r) + a.$$

This system is not closed, yet, since we still need to determine a and ω, which are used for the fine adjustment. Observe that a and ω appear explicitly in the equation on S only through ImG, that is, ayQ and ωQ. These two terms appear in (3.37) and (3.39), the equations for α and γ. Our strategy is to choose a and ω so that $\dot{\alpha} = 0$ and $\dot{\gamma} = 0$. Then $h_S(t)$ has at most linear growth.

It is important to understand the orders of these quantities. Assume that $h \leq o(\varepsilon)$. Since the force G contains an external input $\Omega_0 Q \sim \varepsilon^2$, G is of the form $O(h^2) + \varepsilon^2$. The equation for h_M, i.e., (3.41), is thus of the form

$$f' \leq f^2 + c^2\varepsilon^2 \ , \quad c > 1 \ , \tag{3.42}$$

(and we have assumed that we can take care of the linear part). The solutions of this equation can blow up at $t = (c\varepsilon)^{-1}$. Explicitly, if $f(0) = 0$ then

$$f(t) = c\varepsilon \tan(c\varepsilon t).$$

A more careful examination shows that, due to a cancellation property when integrating in time (which is due to an oscillatory behaviour in time), one can show that

$$h(t) \sim \varepsilon^{3/2}.$$

Based on this observation, we will prove that

$$a(t) \sim \varepsilon^2, \quad \omega(t) \sim \varepsilon^2, \quad P_S(h) \sim \varepsilon^3, \quad P_M(h) \sim \varepsilon^{3/2}. \tag{3.43}$$

In the following subsections, we will prove the existence of $h(y,t)$ by proving a priori bounds and using its local existence. It is also possible to prove existence by a contraction mapping argument, as we will do in Sect. 4 for the wave operator.

3.5. Initial value and equations on S.

3.5.1. Initial value. Recall that the initial datum is given by

$$\psi_0(x) = \left[Q\,(x) + h_{\varepsilon,0}\,(x) \right] e^{i v_0 x}.$$

The coordinates of the initial value $h_{\varepsilon,0}$ in the S direction can be calculated:

$$\begin{aligned}
\left(\begin{smallmatrix} 0 \\ Q \end{smallmatrix} \right) : &\quad \alpha(0) = \kappa_1(\mathrm{Im} h_{\varepsilon,0}, \Gamma), \\
\left(\begin{smallmatrix} \nabla Q \\ 0 \end{smallmatrix} \right) : &\quad \beta(0) = \kappa_2(\mathrm{Re} h_{\varepsilon,0}, yQ), \\
\left(\begin{smallmatrix} 0 \\ yQ \end{smallmatrix} \right) : &\quad \gamma(0) = \kappa_2(\mathrm{Im} h_{\varepsilon,0}, \nabla Q), \\
\left(\begin{smallmatrix} \Gamma \\ 0 \end{smallmatrix} \right) : &\quad \delta(0) = \kappa_1(\mathrm{Re} h_{\varepsilon,0}, Q).
\end{aligned}$$

By our assumption on $h_{\varepsilon,0}$, these initial values are of order $\varepsilon^{3/2}$, which is too large for our purpose. They can be made smaller by introducing suitable normalization conventions.

We first replace Q by $Q^* = Q_{\|\psi_0\|^2}$, with $\|Q^*\|_{L^2} = \|\psi_0\|_{L^2}$. We then define h_1 by the equation

$$\psi_0(x) = \left[Q\,(x) + h_{\varepsilon,0}\,(x) \right] e^{i v_0 x} = \left[Q^*\,(x) + h_1\,(x) \right] e^{i v_0 x}.$$

From the assumption $\|Q^*\|_2 = \|\psi_0\|_2$, we have

$$2(Q^*, \mathrm{Re} h_1) = -\|h_1\|^2.$$

Next, we want to choose r^*, v^*, θ^*, and write

$$\psi_0(x) = \left[Q^*\left(x - r^*\right) + h^*\left(x - r^*\right)\right] e^{iv^*(x-r^*)+i\theta^*}, \tag{3.44}$$

so that $P_S h^*$ is essentially zero. Notice that h^* is determined once we have chosen r^*, v^* and θ^*: As a function of $y = x - r^*$,

$$h^*(y) = \left[Q^*\left(y + r^*\right) + h_1\left(y + r^*\right)\right] e^{i[v_0(y+r^*)-v^*y-\theta^*]} - Q^*(y).$$

The leading term of h^* is given by (we will choose $r^* \sim 0$, $v^* \sim v_0$ and $\theta^* \sim v_0 r^*$)

$$h^*(y) \sim h_1(y) + Q^*(y)\left[i(v_0 - v^*)y + i(v_0 r^* - \theta^*)\right] + r^* \cdot \nabla Q^*(y).$$

We can now calculate the initial value of h^* along the S direction (w.r.t. Q^*) as before. The conclusion is

$$\begin{aligned}
\left(\begin{smallmatrix}0\\ Q\end{smallmatrix}\right):&\quad \alpha^* \sim \kappa_1\left(\mathrm{Im} h_1, \Gamma^*\right) \quad +\kappa_1(v_0 r^* - \theta^*)\left(Q^*, \Gamma^*\right),\\
\left(\begin{smallmatrix}\nabla Q\\ 0\end{smallmatrix}\right):&\quad \beta^* \sim \kappa_2\left(\mathrm{Re} h_1, yQ^*\right) +\kappa_2 \textstyle\sum_k r_k^*\left(\nabla_k Q^*, yQ^*\right),\\
\left(\begin{smallmatrix}0\\ yQ\end{smallmatrix}\right):&\quad \gamma^* \sim \kappa_2\left(\mathrm{Im} h_1, \nabla Q^*\right) + \kappa_2(v_0 - v^*)\left(Q^*, \nabla Q^*\right),\\
\left(\begin{smallmatrix}\Gamma\\ 0\end{smallmatrix}\right):&\quad \delta^* \sim \kappa_1\left(\mathrm{Re} h_1, Q^*\right).
\end{aligned}$$

Since the initial value $h_{\varepsilon,0}$ is of order $\varepsilon^{3/2}$, h_1 is of the same order and we can choose $v_0 r^* - \theta^*$, r^* and $v_0 - v^*$ of order $\varepsilon^{3/2}$ such that α^*, β^* and γ^* vanish to leading order. It is easy to check that the next order is bounded by ε^3. Furthermore, δ^* is of order ε^3 as well, thanks to the relation $2(Q^*, \mathrm{Re} h_1) = -\|h_1\|^2$.

In the remaining part of this section, we will prove Theorem 1.1 with ψ_0 of the form (3.44), and $P_S h^* \sim \varepsilon^3$. The initial values of $r(0)$, $v(0)$, and $\theta(0)$ are defined correspondingly. Notice that, by the assumption of the Theorem, (3.12) is satisfied by Q^*. After this case is proved, the statement in the theorem, with $Q = Q_{N_0}$, can be obtained by defining h as

$$\begin{aligned}
h(y,t) &= \psi(x,t)e^{-i\theta} - Q_{N_0}(y)\\
&= (\psi e^{-i\theta} - Q^*) + (Q^* - Q_{N_0})\\
&= h^*(y,t) + (Q^* - Q_{N_0}) = O(\varepsilon^{3/2}).
\end{aligned}$$

From now on, we may and will drop the superscript * and assume

$$\left\| P_S h_{\varepsilon,0} \right\| \le C c_0 \varepsilon^3, \tag{3.45}$$

where ε is sufficiently small: $\varepsilon \le \varepsilon_{-1}$, with ε_{-1} and C depending only on the initial setting ($\mathcal{H}$, N_0, Q,...) but not on W or T. Equation (3.45) will be used in (3.52) below.

We note that the smallness of c_0 is only used to find a suitable $h^*(0)$. It is no longer needed in the future and hence c_0 is independent of T and W. Also note that we may assume $h_{\varepsilon,0} \le c_0 \varepsilon^{1+\sigma}$ for $\sigma \in (0, 1/2]$. Then we replace $\varepsilon^{3/2}$, in the above argument, by $\varepsilon^{1+\sigma}$, and we get a similar conclusion, with (3.45) replaced by $\left\| P_S h_{\varepsilon,0} \right\| \le C c_0 \varepsilon^{2+2\sigma}$.

3.5.2. Equations on S. From now on, C denotes a constant which may depend on the quantities (Φ, Q...), but <u>not</u> on W or T.

Recall that we want to set $\dot\alpha = 0$ and $\dot\gamma = 0$ in (3.37) and (3.39), which yield equations for a and ω. From the definition of G and the inner product relations $(xQ, \Gamma) = 0$ and $\kappa_1 = 1/(Q, \Gamma)$, we have

$$\kappa_1(\mathrm{Im}G, \Gamma) = -\omega - \kappa_1(G_2, \Gamma),$$

where

$$G_2 = \Omega_0 Q + \Omega \mathrm{Re} h + \mathrm{Re} F(h). \tag{3.46}$$

Similarly, from $(Q, \nabla Q) = 0$ and $\kappa_2 = 1/(x_j Q, \partial_j Q)$, we have

$$\kappa_2(\mathrm{Im}G, \nabla Q) = -a - \kappa_2(G_2, \nabla Q).$$

Therefore, in order to have $\dot\alpha = 0$ and $\dot\gamma = 0$, it suffices to set

$$\begin{aligned} \omega &= -\delta - \kappa_1(G_2, \Gamma), \\ a &= -\kappa_2(G_2, \nabla Q). \end{aligned} \tag{3.47}$$

With this choice of a and ω, we have $\alpha(t) = \alpha(0)$, $\gamma(t) = \gamma(0)$; $\beta(t)$ and $\delta(t)$ are defined by solving the ODEs (3.38) and (3.40), i.e.,

$$\delta(t) = \int_0^t \kappa_1(\mathrm{Re}G(s), Q)ds + \delta(0), \tag{3.48}$$

$$\beta(t) = \int_0^t \kappa_2(\mathrm{Re}G(s), yQ)ds - \gamma(0)t + \beta(0). \tag{3.49}$$

qLet

$$C_w = 1 + \|W\|_{W^{3,\infty}}.$$

Then $|\Omega_0(x,t)| \le CC_w\varepsilon^2|y|^2$, (cf. (3.20)). Define

$$\zeta(t) := |a(t)| + |\omega(t)| + \varepsilon^{1/2}\|h(t)\|_{H^1} + C_w\varepsilon^2.$$

(We would like to have that $\zeta(t) = O(\varepsilon^2)$ for $0 \le t \le T\varepsilon^{-1}$.) In the following we work in the time range $[0, t_1]$ where

$$\boxed{\zeta(t) \le C_*\varepsilon^2, \quad \text{with } \varepsilon \le \varepsilon_0 \le (C_* + T + 100)^{-2}} \tag{3.50}$$

holds. Here $C_* > C_w$ is a (large) constant to be determined later. Equation (3.50) is true for $t = 0$ if C_* is sufficiently large with respect to c_0. Moreover, if $\zeta(s) < C_*\varepsilon^2$ for some $s < T\varepsilon^{-1}$, then (3.50) holds for a small time interval $[s, s + \delta s]$ by a local wellposedness result. Our goal is to show that Eq. (3.50) holds for $0 \le t \le T\varepsilon^{-1}$, by requiring ε_0 sufficiently small. Our strategy is to show that, indeed, $\zeta(t) \le \frac{1}{2}C_*\varepsilon^2$ if (3.50) holds. A local wellposedness result then guarantees that (3.50) holds for the whole time range.

The quantities ω and a are defined in terms of G_2 in (3.47), and recall the definition of G_2, Eq. (3.46). Note that $\Omega_0 Q$ is the leading term in G_2. In their definitions, the main term comes from $\Omega_0 Q$, and we have

$$|(\Omega_0 Q, \Gamma)| + |(\Omega_0 Q, \nabla Q)| \le CC_w\varepsilon^2.$$

Also

$$
\begin{aligned}
|(\Omega(t)\mathrm{Re}h(t), \Gamma)| + |(\Omega(t)\mathrm{Re}h(t), \nabla Q)| &\le C(C_w\varepsilon^2 + |a(t)| + |\omega(t)|) \, \|h\|_2 \\
&\le C\varepsilon^{-1/2}\zeta(t)^2.
\end{aligned}
$$

From the assumption (1.27) on Φ, we have for a general $\phi \in H^1$,

$$
\left\| \left(\Phi * |\phi|^2\right) \phi \right\|_{H^1} \le \left\| \Phi * |\phi|^2 \right\|_{L^\infty} \cdot \|\phi\|_{H^1} + \left\| (\nabla\Phi) * |\phi|^2 \right\|_{L^\infty} \cdot \|\phi\|_{L^2} .
$$

From the Young inequality, we have

$$
\left\| \Phi * |\phi|^2 \right\|_{L^\infty} \le \|\Phi\|_{L^\infty} \left\| |\phi|^2 \right\|_{L^1} = \|\Phi\|_{L^\infty} \|\phi\|_{L^2}^2 .
$$

Similarly, we can bound the term with Φ replaced by $\nabla\Phi$. Thus we have proved that

$$
\|F(\phi)\|_{H^1} \le C \, \|\phi\|_{H^1}^2 + C \, \|\phi\|_{H^1}^3
$$

for some constant depending on Φ. Hence we can bound $(F(h(t)), \Gamma)$ and $(F(h(t)), \nabla Q)$ by

$$
|(F(h(t)), \Gamma)| + |(F(h(t)), \nabla Q)| \le C \, \|h\|_{H^1}^2 + C \, \|h\|_{H^1}^3 \le C\varepsilon^{-1}\zeta(t)^2.
$$

Under assumption (3.50), we have thus proved that

$$
|\omega(t)| + |a(t)| \le CC_w\varepsilon^2 + C\varepsilon^{-1}\zeta(t)^2. \tag{3.51}
$$

To estimate β and δ, we note that

$$
\mathrm{Re}G = (\Omega_0 + ay + \omega)\mathrm{Im}h + \mathrm{Im}F(h).
$$

Since we are only interested in the inner products of $\mathrm{Re}G$ with Q or yQ, and Q has exponential decay, we can treat y to be of order one. Thus we have the bound

$$
|\beta(t)| + |\delta(t)| \le Cc_0(T+1)\varepsilon^3 + \int_0^t ds\varepsilon^{-1}\zeta(s)^2, \tag{3.52}
$$

where we have used (3.45) and $\varepsilon t \le T$.

3.6. Modified linear operator on M. It is important to observe that Ω is not bounded. In fact,

$$
\begin{aligned}
\Omega =& W(\varepsilon x) - W(\varepsilon r) - \varepsilon y \nabla W(\varepsilon r) + ay + \omega = O\left(\varepsilon^2(y^2+1)\right) && (3.53)\\
&= O\left(1 + \varepsilon|y|\right), && (3.54)
\end{aligned}
$$

depending on whether we use Taylor expansion. In either case Ω is not bounded. This makes the term $-i\Omega h$ in the nonlinear term G hard to control, although the term $-i\Omega Q$ stays fine since Q is localized. By a finite propagation speed estimate we will see that Ω is of order 1. However, $-i\Omega h$ still cannot be considered an error term. To overcome this difficulty, we will include this term in the linear operator.

Explicitly, Eq. (3.41) for h_M can be rewritten as

$$\partial_t h_M = (\mathcal{L} + P_M \tfrac{1}{i}\Omega)h_M + P_M \widetilde{G}, \tag{3.55}$$
$$\widetilde{G} = -i\Omega(Q + h_S) - iF(h).$$

Hence we must consider the solution propagator $\mathsf{P}(s,t)$ which solves the following problem: If $u(t) = \mathsf{P}(s,t)\phi$, then u is a solution of the equation

$$\partial_t u(t) = (\mathcal{L} + P_M \tfrac{1}{i}\Omega)u(t), \quad u(s) = \phi.$$

We note that the operator $\mathcal{L} + P_M \frac{1}{i}\Omega$ leaves M and S invariant; but we will primarily consider $\mathsf{P}(s,t)$ on M.

Now the equation for h_M can be written as

$$h_M(t) = \int_0^t \mathsf{P}(s,t) P_M \widetilde{G}(s) ds + \mathsf{P}(0,t) h_M(0). \tag{3.56}$$

We decompose $P_M\widetilde{G}$ into the sum of a main part, $\phi(s)$, and a remainder, $P_M G_3(s)$, where

$$\phi(s) = P_M(-i\Omega(s)Q) = P_M(-i\Omega_0(s)Q), \quad G_3 = -i\Omega h_S - iF(h).$$

The following lemma provides a basic estimate on the propagator $\mathsf{P}(s,t)$.

Lemma 3.7. *Assume* (3.50) *is true for* $0 \le t \le T\varepsilon^{-1}$. *For* $\phi \in M$,

$$\|\mathsf{P}(s,t)\phi\|_{H^1} \le C_{10}\,\|\phi\|_{H^1}$$

for $0 \le s \le t \le T\varepsilon^{-1}$, *where* $C_{10} = e^{CC_wT}$ *is independent of* ε.

We shall prove this lemma in the next subsection. Assuming this lemma and recalling that $G_3(s)$ is of order $h^2 + h^3$, we can bound the contribution of $G_3(s)$ to h_M by

$$\left\| \int_0^t \mathsf{P}(s,t) P_M G_3(s) ds \right\|_{H^1} \le CC_{10} \int_0^t \varepsilon^{-1}\zeta(s)^2 ds.$$

The key observation is the following lemma, which takes into account cancellations in the time integration.

Lemma 3.8 (Cancellation). *Assume* (3.50) *is true for* $0 \le t \le T\varepsilon^{-1}$. *Let* $\phi \in M \cap X_3$. *For* $1 \ll t \le T\varepsilon^{-1}$, *we have that*

$$\left\| \int_0^t \mathsf{P}(s,t)\phi ds \right\|_{H^1} \le C_{12} t^{1/2}\,\|\phi\|_{X_3}$$

for a constant $C_{12} = C_{12}(W,T)$ *independent of* ε. *Furthermore, for* $\phi(t) : [0, T\varepsilon^{-1}] \to M \cap X_3$,

$$\left\| \int_0^t \mathsf{P}(s,t)\phi(t) ds \right\|_{H^1} \le C_{12} t^{1/2} \sup_s \|\phi(s)\|_{X_3} + C_{12} t \sup_{|s-\sigma| \le t^{1/2}} \|\phi(s) - \phi(\sigma)\|_{H^1}. \tag{3.57}$$

The space X_3 *has been defined in* (3.34).

We also claim the following bounds on the main term $\phi(s) = P_M(-i\Omega_0(s)Q)$,

$$\begin{aligned} \|\phi(s)\|_{X_3} &\le CC_w\varepsilon^2, \\ \|\phi(s) - \phi(\sigma)\|_{H^1} &\le CC_w\varepsilon^3|s-\sigma|. \end{aligned} \tag{3.58}$$

We will prove the lemma and the claim in next subsection.

Assuming Lemma 3.8 and the claim, we get

$$\left\| \int_0^t \mathsf{P}(s,t)P_M(-i\Omega_0(s)Q)ds \right\|_{H^1} \le C_{12}t^{1/2}CC_w\varepsilon^2 + C_{12}tCC_w\varepsilon^3t^{1/2} \le C_{13}\varepsilon^{3/2},$$

where $C_{13} = CC_{12}C_w(T+1)^{3/2}$. Hence, by (3.56), $h_M(t)$ is bounded by

$$\|h_M(t)\|_{H^1} \le C_{13}\varepsilon^{3/2} + CC_{10}\int_0^t \varepsilon^{-1}\zeta(s)^2ds + Cc_0\varepsilon^{3/2}. \tag{3.59}$$

Recall that $\zeta(t) = |a(t)| + |\omega(t)| + \varepsilon^{1/2}\,\|h(t)\|_{H^1}$. Then we can combine all these bounds, (3.51), (3.52), and (3.59), to obtain the following estimate:

$$\begin{aligned} \zeta(t) &\le C(C_w + c_0(1+\varepsilon^{1/2}T) + C_{13})\varepsilon^2 + C\varepsilon^{-1}\zeta^2(t) + CC_{10}\int_0^t \varepsilon^{-1}\zeta(s)^2ds \\ &\le C\varepsilon^2(C_w + c_0(1+\varepsilon^{1/2}T) + C_{13} + C_{10}C_*^2\varepsilon(1+T)) \\ &\le C\varepsilon^2C_{14}, \qquad \text{where } C_{14} = C_w + 2c_0 + C_{13} + 1, \end{aligned}$$

if we require $\varepsilon^{1/2}T \le 1$ and $C_{10}C_*^2\varepsilon(1+T) < 1$, in addition to assumption (3.50). We now choose

$$C_* = 2CC_{14}$$

and then ε_0 such that

$$\varepsilon_0 \le (C_* + 100)^{-2}, \quad \varepsilon_0^{1/2}T \le 1, \quad C_{10}C_*^2\varepsilon_0(1+T) < 1.$$

With these choices, we have proven that

$$\zeta(t) \le \frac{1}{2}C_*\varepsilon^2 \tag{3.60}$$

under assumption (3.50) that $\zeta(t) \le C_*\varepsilon^2$. Suppose that $[0, t_1]$ is the maximal time interval such that (3.50) holds and $t_1 < T\varepsilon^{-1}$. Then the equality must hold at $t = t_1$ by local existence and continuity, and $\zeta(t)$ must be slightly less than $C_*\varepsilon^2$ for some $t < t_1$. This is a contradiction to (3.60) and hence (3.50) holds true for all $t \le T\varepsilon^{-1}$.

3.7. Proofs of lemmas.

3.7.1. Proof of Lemma 3.7. Here we prove that the flow given by $\mathsf{P}(s,t)$ is bounded in M:

Proof. Let $u(t)=\mathsf{P}(s,t)\phi\in M$, and

$$f(t)=\mathrm{Im}(\mathcal{L}u(t),u(t))\ge 0.$$

Recall the second assertion of Lemma 3.4: It implies that $\hat f(t)=\mathrm{Im}(\mathcal{L}g(t),g(t))$, with $g(t):=e^{t\mathcal{L}}\phi$, is constant. We propose that $f(t)$ does not grow in t very fast, for $s,t\in(0,T\varepsilon^{-1})$. More precisely, we will prove that

$$\frac{d}{dt}f(t)\le C\varepsilon f(t),$$

which implies $f(t)\le Cf(s)$, and hence Lemma 4.7 follows.

We recall the third assertion of Lemma 3.4. In our case, $\partial_t u=\mathcal{L}u+P_M\frac{1}{i}\Omega u$, hence

$$\begin{aligned}\frac{d}{dt}f(t)/2&=\mathrm{Im}(\mathcal{L}u,P_M\tfrac{1}{i}\Omega u)=\mathrm{Im}(\mathcal{L}u,-i\Omega u)-\mathrm{Im}(\mathcal{L}u,P_S(-i\Omega u))\\&=\mathrm{Im}\int\frac{1}{2}\nabla\bar u(\nabla\Omega)u+O(\varepsilon^2\|u\|_2^2)-\mathrm{Im}(\mathcal{L}u,P_S(-i\Omega u))\,.\end{aligned}$$

If $\{e^j\}$ and $\{e_j\}$ denote dual bases of S, then $P_S(-i\Omega u)=\sum(e^j,-i\Omega u)e_j=\sum(i\Omega e^j,u)e_j$. Hence $\|P_S(-i\Omega u)\|_{H^1}\le C\varepsilon^2\|u\|_{L^2}$, and

$$\mathrm{Im}(\mathcal{L}u,P_S(-i\Omega u))\le C\|u\|_{H^1}\cdot\|P_S(-i\Omega u)\|_{H^1}\le C\varepsilon^2\|u\|_{H^1}^2\,.$$

Since $\|\nabla\Omega\|_\infty=\|\varepsilon\nabla W(\varepsilon x)-\varepsilon\nabla W(\varepsilon r)+a\|_\infty\le 2C_w\varepsilon+C_*\varepsilon^2$, (with no y dependence), the term $\mathrm{Im}\int\frac{1}{2}\nabla\bar u(\nabla\Omega)u$ dominates, and

$$\frac{d}{dt}f(t)\le\mathrm{Im}\int\nabla\bar u(\nabla\Omega)u+C\varepsilon^2\|u\|_{H^1}^2\le(2C_w\varepsilon+C\zeta(t))\|u\|_{H^1}^2\le CC_w\varepsilon f.$$

The last inequality follows from (3.50) and Lemma 3.3. Hence

$$f(t)\le e^{CC_w\varepsilon t}f(0)\le e^{CC_wT}f(0)$$

for $t\le T\varepsilon^{-1}$. $\square$

3.7.2. Proof of Lemma 3.8. Next we prove the key cancellation lemma. The cancellation is due to oscillatory behavior in time. We first prove a variant of Lemma 3.8 for the original flow $e^{t\mathcal{L}}$, which will help us to visualize the oscillation. Then we will prove a weaker result for the modified flow in Lemma 3.8.

Suppose $\rho(t)\in M$ satisfies $\rho(t)=O(1)$ and $\frac{d}{dt}\rho(t)=O(\varepsilon)$ in H^1. (One such example is $\varepsilon^{-2}\Omega_0(t)Q$.) Then there is a $C>0$ such that

$$\left\|\int_0^t e^{(t-s)\mathcal{L}}\rho(s)ds\right\|_{H^1}\le C(1+\varepsilon t).\tag{3.61}$$

By Lemma 3.5, $\mathcal{L}^{-1}$ is defined on M and commutes with $e^{(t-s)\mathcal{L}}$. Thus

$$\begin{aligned}\int_0^t e^{(t-s)\mathcal{L}}\rho(s)ds &= \int_0^t \frac{d}{ds}\left(-e^{(t-s)\mathcal{L}}\right)\mathcal{L}^{-1}\rho(s)ds \\ &= \left[-e^{(t-s)\mathcal{L}}\mathcal{L}^{-1}\rho(s)\right]_0^t + \int_0^t e^{(t-s)\mathcal{L}}\mathcal{L}^{-1}\frac{d}{ds}\rho(s)ds \\ &= O(1) + \int_0^t e^{(t-s)\mathcal{L}}O(\varepsilon)ds \\ &= O(1) + O(\varepsilon t), \qquad \text{in } H^1.\end{aligned}$$

Here we have used Lemma 3.3. (Notice the analogy with the integration of e^{it}, which does not increase the order of e^{it} because of its oscillation.)

Now we prove the lemma.

Proof. Choose $\tau \sim t^{1/2} \gg 1$. We have

$$\begin{aligned}\int_0^t \mathsf{P}(s,t)\phi ds &= \sum_j \int_{j\tau}^{(j+1)\tau} \mathsf{P}(s,t)\phi ds \\ &= \sum_j \mathsf{P}((j+1)\tau, t)\int_{j\tau}^{(j+1)\tau} \mathsf{P}(s,(j+1)\tau)\phi ds.\end{aligned}$$

We write each summand as

$$\begin{aligned}&\int_{j\tau}^{(j+1)\tau} \mathsf{P}(s,(j+1)\tau)\phi ds \equiv \text{(I)} \\ &\quad = \int_{j\tau}^{(j+1)\tau} e^{((j+1)\tau - s)\mathcal{L}}\phi ds \\ &\qquad + \int_{j\tau}^{(j+1)\tau}\int_s^{(j+1)\tau} \mathsf{P}(\sigma,(j+1)\tau)P_M\tfrac{1}{i}\Omega(\sigma)e^{(\sigma-s)\mathcal{L}}\phi d\sigma ds \\ &\qquad \equiv \text{(II)} + \text{(III)}.\end{aligned}$$

We have

$$\|\text{(II)}\|_{H^1} \le C\,\|\phi\|_{H^1}(1+\varepsilon\tau) \le C\,\|\phi\|_{H^1}$$

by (3.61) and (3.50). For (III), since ϕ is localized, we expect it is not affected much by the large potential in $P_M\frac{1}{i}\Omega(\sigma)$ for large y. To prove this, we use the finite propagation speed estimate (3.35): For $s \in (0,\tau)$,

$$\begin{aligned}\left\|P_M\tfrac{1}{i}\Omega(\cdot)e^{s\mathcal{L}}\phi\right\|_{H^1} &\le C\left\|\left[C_w\varepsilon^2 y^2 + C_*\varepsilon^2(1+|y|)\right]e^{s\mathcal{L}}\phi\right\|_{H^1} \\ &\le CC_w\varepsilon^2\left\{\left\|(1+y^2)\phi\right\|_{H^1} + (1+s^2)\|\phi\|_{H^3}\right\} \\ &\quad + CC_*\varepsilon^2\left\{\|(1+|y|)\phi\|_{H^1} + (1+s)\|\phi\|_{H^2}\right\}.\end{aligned}$$

Hence

$$\|\mathrm{III}\|_{H^1} \le CC_{10}\varepsilon^2\tau^2 \left\{(C_w + C_*) \left\|(1+y^2)\phi\right\|_{H^1} + (C_w\tau^2 + C_*\tau) \|\phi\|_{H^3}\right\}$$
$$\le CC_{10}C_w \|\phi\|_{X_3},$$

since $\tau^2\varepsilon < 2$ and $\varepsilon^{1/2}C_* \le 1$, see (3.50).

Therefore $\|(I)\|_{H^1} \le C_{11} \|\phi\|_{X_3}$ with $C_{11} = C + CC_{10}C_w$ and

$$\left\|\int_0^t \mathsf{P}(s,t)\phi ds\right\|_{H^1} \le \sum_j C_{10}C_{11} \|\phi\|_{X_3} \le CC_{10}C_{11}t^{1/2} \|\phi\|_{X_3}.$$

Next, for a suitably localized function $\phi(t) \in M \cap X_3$,

$$\left\|\int_0^t \mathsf{P}(s,t)\phi(t)ds\right\|_{H^1}$$
$$= \left\|\sum_j \mathsf{P}((j+1)\tau, t) \int_{j\tau}^{(j+1)\tau} \mathsf{P}(s,(j+1)\tau) \left\{\Big[\phi(j\tau)\Big] + \Big[\phi(s) - \phi(j\tau)\Big]\right\} ds\right\|_{H^1}$$
$$\le \sum_j C_{10}C_{11} \|\phi(j\tau)\|_{X_3} + \sum_j C_{10}^2\tau \sup_{|s-\sigma|\le\tau} \|\phi(s) - \phi(\sigma)\|_{H^1}$$
$$\le C_{12}t^{1/2} \sup_s \|\phi(s)\|_{X_3} + C_{12}t \sup_{|s-\sigma|\le t^{1/2}} \|\phi(s) - \phi(\sigma)\|_{H^1}$$

with $C_{12} = CC_{11}^2 = C(1 + C_w e^{CC_wT})^2$. $\square$

This estimate is mainly used for $\phi(s) = P_M(-i\Omega_0(s)Q)$.

3.7.3. Proof of claim (3.58). We rewrite Ω_0 in the form

$$\Omega_0(x,t) = W(\varepsilon r + \varepsilon y) - W(\varepsilon r) - \nabla W(\varepsilon r)\cdot\varepsilon y$$
$$= \int_0^1 \{\nabla W(\varepsilon r + u\varepsilon y)\cdot\varepsilon y\} du - \nabla W(\varepsilon r)\cdot\varepsilon y$$
$$= \int_0^1 \int_0^1 \left\{\nabla^2 W(\varepsilon r + vu\varepsilon y) : \varepsilon y \otimes u\varepsilon y\right\} dvdu.$$

From the first line we have that $\left\|\nabla^3\Omega_0\right\|_\infty \le \varepsilon^3 \left\|\nabla^3 W\right\|_\infty$. From the third line we obtain $\left\|\Omega_0 e^{-\nu|y|}\right\|_\infty \le \varepsilon^2 \left\|\nabla^2 W\right\|_\infty$. Hence, for $\phi(s) = P_M(-i\Omega_0(s)Q)$, we have that

$$\|\phi(s)\|_{X^3} \le C \left\|\nabla^3\Omega_0\right\|_\infty + C \left\|\Omega_0 e^{-\nu|y|}\right\|_\infty \le C \|W\|_{W^{3,\infty}} \varepsilon^2,$$

where the factor $e^{-\nu|y|}$ is due to the exponential decay of Q. Furthermore, since

$$\left|\nabla^2 W(\varepsilon r(s) + vu\varepsilon y) - \nabla^2 W(\varepsilon r(\sigma) + vu\varepsilon y)\right| \le \sup|\nabla^3 W| \cdot \varepsilon |r(s) - r(\sigma)|$$

and $|r(s) - r(\sigma)| \le CC_*|s-\sigma|$, (note $|v(t)| \le CC_*$), we conclude that

$$\|\phi(s) - \phi(\sigma)\|_{L^2} \le C \left\|\nabla^3 W\right\|_\infty C_*\varepsilon^3|s-\sigma|.$$

By rewriting $\nabla\Omega_0(x,t) = \int_0^1 \left\{\nabla^2 W(\varepsilon r(t) + u\varepsilon y)\cdot\varepsilon^2 y\right\} du$, we get the same bound for $\|\nabla[\phi(s) - \phi(\sigma)]\|_{L^2}$.

4. Møller Wave Operator

In this section we prove Theorem 1.2. We assume for simplicity that the space dimension $n = 3$. All arguments can be modified easily to $n > 3$.

In the main argument of this section, we assume $v_0 = 0$ and work with the profile $\xi_\infty = h_{as,0}$, with $\hat{\xi}_\infty(0) = 0$. At the end of this section we deal with general v_0 by applying a Galilei transform. In either case, we have $h_{as,0}(x) = \xi_\infty(x)e^{iv_0\cdot x}$, and

$$\hat{h}_{as,0}(v_0) = \hat{\xi}_\infty(0) = 0.$$

4.1. Dynamical linearization. We recall the Hartree equation

$$i\partial_t\psi = -\frac{1}{2}\Delta\psi - (\Phi * |\psi|^2)\psi,$$

and the equation for the ground state Q,

$$-\frac{1}{2}\Delta Q - (\Phi * Q^2)Q = EQ.$$

We consider solutions of the Hartree equation of the form

$$\psi = (Q(y) + h(y,t))e^{i\theta(y,t)},$$

where

$$y = x - r(t), \quad \dot{r}(t) = v(t), \quad \dot{v}(t) = a(t),$$
$$\theta(y,t) = v(t)y - Et + \theta_1(t), \quad \theta_1(t) = -\int_t^\infty \left(\tfrac{1}{2}v^2 + \omega\right) ds\ .$$

The argument here is the same as that in Subsect. 4.2, with $W \equiv 0$. We obtain the equation for h:

$$\partial_t h = \mathcal{L}h + G(h), \tag{4.1}$$

where the linear part

$$\mathcal{L} = \frac{1}{i}\left\{-\tfrac{1}{2}\Delta - E + A\right\},$$
$$A(h) = -(\Phi * Q^2)h - Q(\Phi * (Q(h+\bar{h}))),$$

and the nonlinear part

$$G = \frac{1}{i}\left\{\Omega(Q+h) + F(h)\right\}, \qquad \Omega = \omega + ay,$$
$$F(h) = -(\Phi * |h|^2)(Q+h) - (\Phi * [Q(h+\bar{h})])h.$$

We take projections of Eq. (4.1) onto S and M. The equations on S are

$$\begin{aligned}
\begin{pmatrix} 0 \\ Q \end{pmatrix}: &\quad \dot{\alpha} = -\delta + \kappa_1(\mathrm{Im}G, \Gamma),\\
\begin{pmatrix} \nabla Q \\ 0 \end{pmatrix}: &\quad \dot{\beta} = -\gamma + \kappa_2(\mathrm{Re}G, yQ),\\
\begin{pmatrix} 0 \\ yQ \end{pmatrix}: &\quad \dot{\gamma} = \qquad \kappa_2(\mathrm{Im}G, \nabla Q),\\
\begin{pmatrix} \Gamma \\ 0 \end{pmatrix}: &\quad \dot{\delta} = \qquad \kappa_1(\mathrm{Re}G, Q).
\end{aligned}$$

(See Proposition 4.2.) The equation on M is

$$\partial_t h_M = \mathcal{L} h_M + P_M G(h).$$

Next we consider the wave operator. Given a profile ξ_∞ at $t = \infty$, we hope to find a function $h(y, t)$ such that

$$h(y, t) - e^{t\mathcal{L}_0}\xi_\infty \to 0,$$

as $t \to \infty$, in a sense to be made more precise. Here

$$\mathcal{L}_0 = -i\left\{-\tfrac{1}{2}\Delta - E\right\}$$

so that $\mathcal{L} = \mathcal{L}_0 - iA$. Our strategy is to write

$$h(\cdot, t) = \xi(\cdot, t) + g(\cdot, t),$$

where $\xi(t)$ is the main term, which satisfies a **linear** equation and has the desired profile explicitly; $g(t)$ is an error and converges to zero, as $t \to \infty$, in a suitable sense.

In view of the equation for h, we would like ξ to satisfy the linear equation

$$\xi(t) \in M, \quad \partial_t \xi = \mathcal{L}\xi + P_M J\xi \tag{4.2}$$

with $\xi(t) \to e^{t\mathcal{L}_0}\xi_\infty$, as $t \to \infty$. The operator J is a modification of the multiplication operator $-i\Omega$ and is to be defined later in (4.9).

Define the propagator $\widetilde{\mathsf{P}}(s, t)$ such that $u(t) := \widetilde{\mathsf{P}}(s, t)\phi$ solves the equation

$$\partial_t u = \mathcal{L}u + P_M J u, \quad u(s) = \phi \in M.$$

Clearly, $\widetilde{\mathsf{P}}(s, t)$ leaves the space M invariant so that $u \in M$. Note that $t < s$ in this section, cf. Sect. 3. We define ξ to be given by

$$\xi(t) = P_M e^{t\mathcal{L}_0}\xi_\infty - \int_t^\infty P_M \widetilde{\mathsf{P}}(s, t) P_M [\frac{1}{i}A + P_M J(s)] e^{s\mathcal{L}_0}\xi_\infty ds. \tag{4.3}$$

We have that $\xi \in M$, by definition, and that ξ satisfies (4.2) (differentiate (4.3) and use that $[\mathcal{L}, P_M] = 0$!). We shall prove later on that

$$\xi(t) \to e^{t\mathcal{L}_0}\xi_\infty \quad \text{in } L^2, \quad \text{as } t \to \infty, \tag{4.4}$$

under the assumption

$$\boxed{\hat{\xi}_\infty(0) = 0.} \tag{4.5}$$

The potential $\Omega = \omega + ay$ is unbounded and complicates the analysis. One may prove certain finite propagation speed estimates, so that y is effectively cut off, as in Sect. 3. Alternatively, we can modify the form of ψ so that the unbounded potential is cut off. We shall follow the second option in this section. Specifically, we would like h not to "see" the fast phase change vy in θ when y is large. Let $\chi(\cdot)$ be a smooth cutoff function with $\chi(x) = 1$, for $|x| \le 1$, and $\chi(x) = 0$, for $|x| \ge 2$. We consider ψ of the form:

$$\psi = Q(y)e^{i\theta} + h(y, t)e^{i(\chi v y - Et + \theta_1)} = \left(Q + \mu^{-1}h\right)e^{i\theta},$$

where $\theta = vy - Et + \theta_1$, $\mu = \exp(i(1-\chi)vy)$, $y = x - r(t)$ and $\chi = \chi(C_* y/t)$, ($C_* > 0$ is a constant to be chosen later). Then $\mu^{-1}h$ satisfies (4.1)

$$\partial_t(\mu^{-1}h) = \mathcal{L}(\mu^{-1}h) - i\Omega(Q + \mu^{-1}h) - iF(\mu^{-1}h). \tag{4.6}$$

Now $\mu\partial_t(\mu^{-1}h) = \partial_t h + h\partial_t(-i(1-\chi)vy)$ and $\partial_t(-i(1-\chi)vy) =: -i(ay + J^{(1)})$, where

$$J^{(1)} = \Big[-\chi ay - (1-\chi)v^2 + (\nabla\chi)(vt + y)t^{-2}vy\Big].$$

Also $\mu\mathcal{L}(\mu^{-1}h) = \mathcal{L}h + \mu[\mathcal{L}, \mu^{-1}]h$. Explicit computation gives

$$\mu\nabla\mu^{-1} = i\Big[-(1-\chi)v + (\nabla\chi)t^{-1}vy\Big] = iJ^{(2)},$$
$$\mu\Delta\mu^{-1} = -(J^{(2)})^2 + i\nabla\cdot J^{(2)}, \qquad \nabla\cdot J^{(2)} = 2(\nabla\chi)\cdot t^{-1}v + (\Delta\chi)t^{-2}vy.$$

Recall that $\mathcal{L} = -i(-\Delta/2 - E + A)$. Thus,

$$\begin{aligned}\mu[\mathcal{L}, \mu^{-1}]h &= \frac{i\mu}{2}[\Delta, \mu^{-1}]h - i\mu[A, \mu^{-1}]h \\ &= \left(-\frac{i}{2}(J^{(2)})^2 - \frac{\nabla\cdot J^{(2)}}{2}\right)h - J^{(2)}\cdot\nabla h + J^{(3)}h,\end{aligned}$$

where

$$J^{(3)}h := -i\mu[A, \mu^{-1}]h = i\mu Q\Phi * [Q(\mu^{-1}h + \mu\bar{h})]. - iQ\Phi * [Q(h + \bar{h})]$$

This yields the following equation for h:

$$\partial_t h = \mathcal{L}h + Jh - i\Omega\mu Q - i\mu F(\mu^{-1}h), \tag{4.7}$$

where

$$Jh = \left(-i\omega + iJ^{(1)} - \frac{i}{2}(J^{(2)})^2 - \frac{\nabla\cdot J^{(2)}}{2}\right)h - J^{(2)}\cdot\nabla h + J^{(3)}h.$$

Notice that J depends on ω, a and v with $\dot{v}(t) = a(t)$. Throughout the rest of this section we assume that there is a constant C_* such that

$$t^3|a(t)| + t^2|\omega(t)| \le C_*. \tag{4.8}$$

We shall prove later on that this assumption holds. Under this assumption, one finds that

$$\|J^{(1)}\|_\infty \le O(t^{-2}), \quad \|J^{(2)}\|_\infty \le O(t^{-2}),$$
$$\|\nabla\cdot J^{(2)}\|_\infty \le O(t^{-3}), \quad \|J^{(3)}\|_\infty \le O(e^{-t}).'$$

We write

$$J = J_a + J_b\cdot\nabla + J_c, \tag{4.9}$$

with

$$
\begin{aligned}
J_a &= i[-\omega - \chi a y + (\nabla\chi)t^{-2}(vy)y],\\
J_b &= -J^{(2)} = -\Big[-(1-\chi)v + (\nabla\chi)t^{-1}vy\Big],\\
J_c &= -i(1-\chi)v^2 + i(\nabla\chi)vt^{-1}(vy) - \frac{i}{2}(J^{(2)})^2 - \frac{\nabla\cdot J^{(2)}}{2} + J^{(3)}.
\end{aligned}
$$

Note that J_b is real. Furthermore, the only appearance of μ in J is in $J^{(3)}$, which is exponentially small. Assuming the bound (4.8) on a and ω, we can check the following bounds on J:

$$
\|J_a\|_\infty + \|J_b\|_\infty \le O(t^{-2}), \quad \|J_c\|_\infty \le O(t^{-3}). \tag{4.10}
$$

Once $J(t)$ is defined, so is $\xi(t)$ by (4.2). We can now use (4.7) and (4.2) to obtain an equation for $g := h - \xi$:

$$
\partial_t g = (\mathcal{L} + J)g + P_S J\xi - i\Omega\mu Q - i\mu F(\mu^{-1}(\xi + g)). \tag{4.11}
$$

Let

$$
G_\mu^{(1)} := Jg_S - i\Omega(\mu - 1)Q - i\mu F(\mu^{-1}(\xi + g)). \tag{4.12}
$$

Since $-i\Omega Q \in S$, we have that $P_M G = P_M J g_M + P_M G_\mu^{(1)}$, and the equation for g on M is

$$
g_M(t) = -\int_t^\infty \widetilde{\mathsf{P}}(s,t) P_M G_\mu^{(1)} ds, \tag{4.13}
$$

Let

$$
G_\mu^{(2)} := Jg + P_S J\xi - i\Omega(\mu - 1)Q - i\mu F(\mu^{-1}(\xi + g)). \tag{4.14}
$$

Then $P_S G = -i\Omega Q + P_S G_\mu^{(2)}$, and the equations on S are

$$
\begin{aligned}
\tbinom{0}{Q}:&\quad \dot\alpha = -\delta - \omega + \kappa_1(\mathrm{Im} G_\mu^{(2)}, \Gamma),\\
\tbinom{\nabla Q}{0}:&\quad \dot\beta = -\gamma \qquad + \kappa_2(\mathrm{Re} G_\mu^{(2)}, yQ),\\
\tbinom{0}{yQ}:&\quad \dot\gamma = \qquad -a + \kappa_2(\mathrm{Im} G_\mu^{(2)}, \nabla Q),\\
\tbinom{\Gamma}{0}:&\quad \dot\delta = \qquad\qquad \kappa_1(\mathrm{Re} G_\mu^{(2)}, Q).
\end{aligned} \tag{4.15}
$$

Here we have used that $\kappa_1(-\Omega Q, \Gamma) = -\omega$, $\kappa_2(-\Omega Q, \nabla Q) = -a$.

4.2. Bounds on the Propagator $\widetilde{\mathsf{P}}(s,t)$. The following lemma shows that $\widetilde{\mathsf{P}}(s,t)$ conserves the H^1-norm in M.

Lemma 4.1. *Assume the bound* (4.8). *Then* $\widetilde{\mathsf{P}}(s,t)$ *is bounded in* $M \cap H^k$, $k = 1, 2, 3$. *More precisely, there is a constant* C *such that for any* C_* *(the bound in* (4.8)*), any* $T \geq 1$, *and any* $\phi \in M \cap H^k$, *we have*

$$\left\| \widetilde{\mathsf{P}}(s,t)\phi \right\|_{H^k} \leq e^{CC_*/T} \|\phi\|_{H^k}, \, k = 1, 2, 3,$$

provided that $s, t \geq T$. *(The larger* T *is, the better the estimate.)*

Proof. We first consider the case $k = 1$. Assume $u(t) \in M$, $\partial_t u(t) = \mathcal{L}u + P_M J(t)u$, $u(s) = \phi$. Let $f(t) = \operatorname{Im}(\mathcal{L}u, u) \geq 0$. Then

$$\frac{d}{2dt} f(t) = \operatorname{Im}(\mathcal{L}u, P_M Ju) = \operatorname{Im}(\mathcal{L}u, Ju) - \operatorname{Im}(\mathcal{L}u, P_S Ju).$$

Here we have used Lemma 4.3. Note $|(\mathcal{L}u, P_S Ju)| \leq CC_* t^{-2} \|u\|_{L^2}^2$ and $\operatorname{Im}(\mathcal{L}u, Ju) = C\operatorname{Re}(\Delta u, J_b \cdot \nabla u) + O(t^{-2}) \|u\|_{H^1}^2$, also, (recall J_b is real)

$$\begin{aligned} 2\operatorname{Re}(\Delta u, J_b \cdot \nabla u) &= -\int J_b \cdot \nabla |\nabla u|^2 - 2\operatorname{Re} \int (\nabla \bar{u} \cdot \nabla) J_b \cdot \nabla u \\ &= \int (\nabla \cdot J_b) |\nabla u|^2 - 2\operatorname{Re} \int (\nabla \bar{u} \cdot \nabla) J_b \cdot \nabla u \leq CC_* t^{-3} \|u\|_{H^1}^2 . \end{aligned} \tag{4.16}$$

Hence we have

$$\left| \frac{d}{dt} f(t) \right| \leq CC_* t^{-2} \|u\|_{H^1}^2 \leq CC_* t^{-2} f(t). \tag{4.17}$$

Hence we get

$$\left| [\ln f]_s^t \right| \leq -CC_* \left[t^{-1} \right]_s^t \leq CC_* T^{-1}.$$

In particular,

$$\frac{f(t)}{f(s)}, \frac{f(s)}{f(t)} \leq e^{CC_* T^{-1}}.$$

Now we consider the case $k = 3$. The case $k = 2$ follows by interpolation. Let $u(t)$ be as above and $w = \mathcal{L}u \in M$. We have

$$\partial_t w = \mathcal{L}w + \mathcal{L}P_M Ju = \mathcal{L}w + Jw + [\mathcal{L}, J]u - \mathcal{L}P_S Ju.$$

This time we let $f_3(t) = \operatorname{Im}(\mathcal{L}w, w)$ and have

$$\frac{d}{2dt} f_3(t) = \operatorname{Im}(\mathcal{L}w, Jw + [\mathcal{L}, J]u - \mathcal{L}P_S Ju).$$

We have $|(\mathcal{L}w, \mathcal{L}P_S Ju)| \leq CC_* t^{-2} \|w\|_2 \|u\|_2$, and we already showed $|\operatorname{Im}(\mathcal{L}w, Jw)| \leq CC_* t^{-2} \|w\|_{H^1}^2$ when we considered $f(t)$, see especially (4.16). Finally

$$\begin{aligned} |\operatorname{Im}(\mathcal{L}w, [\mathcal{L}, J]u)| &= |\operatorname{Im}(\mathcal{L}w, -i(\nabla J_b) \cdot \nabla u + O(t^{-2})u)| \\ &\leq CC_* t^{-3} \|w\|_{H^1} \|u\|_{H^2} + CC_* t^{-2} \|w\|_{H^1} \|u\|_{H^1} , \end{aligned}$$

by integration by parts. Since $\|w\|_{H^1}^2$ is comparable with f_3, we conclude

$$\left|\frac{d}{dt} f_3(t)\right| \leq CC_* t^{-2}\left[f_3(t) + \sqrt{f_3(t)}\,\|u(t)\|_{H^1}\right] \leq CC_* t^{-2}\left[f_3(t) + f(t)\right].$$

Together with (4.17), we see $(f + f_3)$ satisfies the same inequality in (4.17), and hence the same bound. Since $(f + f_3) \sim \|u(t)\|_{H^3}^2$, the lemma is proved. □

Remark. Due to the spatial cut-off in our Eq. (4.7), we do not need to prove a finite speed estimate for $\widetilde{\mathsf{P}}$, (as we did in Lemma 4.6 for P), in order to prove the above lemma.

4.3. Estimates of ξ. We now estimate ξ precisely. Recall (4.2) and (4.3), the equations of ξ. Our goal is to estimate the term $-\int_t^\infty \widetilde{\mathsf{P}}(s,t)P_M(\frac{1}{i}A + P_M J(s))e^{s\mathcal{L}_0}\xi_\infty ds$.

We need the following standard results on the free evolution.

Lemma 4.2 (Decay of $e^{it\Delta/2}$). *Let $k > 0$ be a positive integer and assume $\nabla_p^m \hat{\xi}_\infty(0) = 0$ for all non-negative integers $m \leq 2k-2$, then*

$$\left|\left(\nabla_x^n e^{it\Delta/2}\xi_\infty\right)(x)\right|_{x=O(1)} \leq \frac{C}{t^{d/2+k}} \int (1+|y|^{2k})|\nabla_y^n \xi_\infty(y)|dy, \tag{4.18}$$

for any integer $n \geq 0$.

Proof. We first consider the case $n = 0$. Write $r = \frac{i|x-y|^2}{2t}$. We have

$$\begin{aligned}(e^{it\Delta/2}\xi_\infty)(x) &= \frac{1}{(2\pi i t)^{d/2}} \int e^{\frac{i|x-y|^2}{2t}} \xi_\infty(y)dy \\ &= \frac{1}{(2\pi i t)^{d/2}} \int \left\{1 + r + \frac{1}{2}r^2 + \cdots + \frac{1}{(k-1)!}r^{k-1} + O(r^k)\right\} \xi_\infty(y)dy.\end{aligned}$$

Therefore, the conclusion of the lemma holds if

$$\int |x-y|^{2l}\xi_\infty(y)dy = 0 \quad \text{for all } x, \text{ for all } l < k,$$

which is true under the assumption of the lemma. For general n, we take the derivative first and then proceed as above. Note $\nabla_p^m(\widehat{\nabla_x^n \xi_\infty})(0) = \nabla_p^m(p^n\hat{\xi}_\infty)(0) = 0$ for all $m \leq 2k-2$. □

We now use that $\widetilde{\mathsf{P}}(s,t)$ is bounded in H_1 (Lemma 4.1) to have

$$\left\|\int_t^\infty \widetilde{\mathsf{P}}(s,t)P_M \frac{1}{i} A e^{s\mathcal{L}_0}\xi_\infty ds\right\|_{H^1} \leq \int_t^\infty \left\|P_M \frac{1}{i} A e^{s\mathcal{L}_0}\xi_\infty\right\|_{H^1} ds\ .$$

From Lemma 4.2 with $k = 1$, the last term is bounded by

$$\int_t^\infty \left\|e^{s\mathcal{L}_0}\xi_\infty\right\|_{W^{1,\infty}(y\sim 1)} ds \leq \int_t^\infty s^{-5/2} ds \leq Ct^{-3/2}.$$

Notice that this is the only place we use assumption (4.5). Now we recall $J = J_a + J_b \cdot \nabla + J_c$. Since $\|J_c(s)\|_\infty \le s^{-3}$, we have

$$\left\| -\int_t^\infty \widetilde{\mathsf{P}}(s,t) P_M J_c(s) e^{s\mathcal{L}_0} \xi_\infty ds \right\|_{H^1} \le \int_t^\infty C s^{-3} ds \|\xi_\infty\|_{H^1} \le C t^{-2} \|\xi_\infty\|_{H^1}.$$

We now expand $\widetilde{\mathsf{P}}(s,t)$ once more to get

$$-\int_t^\infty \widetilde{\mathsf{P}}(s,t) P_M (J_a + J_b \cdot \nabla)(s) e^{s\mathcal{L}_0} \xi_\infty ds = \xi_J + \xi_{AJ} + \xi_{JJ},$$

where

$$\begin{aligned}
\xi_J &= -P_M \int_t^\infty e^{(t-s)\mathcal{L}_0} P_M (J_a + J_b \cdot \nabla)(s) e^{s\mathcal{L}_0} \xi_\infty ds,\\
\xi_{AJ} &= \int_t^\infty \left\{ \int_t^s \widetilde{\mathsf{P}}(\sigma,t) P_M \frac{1}{i} A e^{(\sigma - s)\mathcal{L}_0} d\sigma \right\} P_M (J_a + J_b \cdot \nabla)(s) e^{s\mathcal{L}_0} \xi_\infty ds,\\
\xi_{JJ} &= \int_t^\infty \left\{ \int_t^s \widetilde{\mathsf{P}}(\sigma,t) P_M J(s) e^{(\sigma - s)\mathcal{L}_0} d\sigma \right\} P_M (J_a + J_b \cdot \nabla)(s) e^{s\mathcal{L}_0} \xi_\infty ds.
\end{aligned}$$

Recall from J.-L. Journe, A. Soffer and C. D. Sogge [15],

$$\left\| e^{is_0 H_0} V e^{is_1 H_0} \right\|_{(L^1, L^\infty)} \le \frac{C \|\hat{V}\|_1}{(s_0 + s_1)^{d/2}}. \tag{4.19}$$

Suppose that we can neglect the second projection P_M in the definition of ξ_J. Since $J_b \nabla e^{s\mathcal{L}_0} = J_b e^{s\mathcal{L}_0} \nabla$, and we can write $J_a = -i\omega + J_{a2}$, $J_b = v + J_{b2}$, where J_{a2} and J_{b2} have compact supports and $\|\widehat{J_{a2}}(s)\|_{L^1(p)} + \|\widehat{J_{b2}}(s)\|_{L^1(p)} = O(s^{-2})$, the L^∞-norm of the integrand of ξ_J is bounded by $C t^{-3/2} s^{-2}$. Integrating in s we get

$$\|\xi_J(t)\|_{L^\infty} \le C t^{-3/2} \|\xi_\infty\|_{W^{1,1}}.$$

To handle the P_M, we simply use that $P_M = 1 - P_S$. Since P_S is a projection onto local smooth functions, the same proof applies. We shall not repeat the argument to handle the projection P_M later on.

We can also bound $\xi_J(t)$ in the H^1 norm by brutal force as we deal with J_c:

$$\|\xi_J(t)\|_{H^1} \le C t^{-1} \|\xi_\infty\|_{H^2},$$

since ξ_J involves only free evolution.

We now use that $\widetilde{\mathsf{P}}(s,t)$ is bounded in H_1 to have

$$\|\xi_{AJ}(t)\|_{H^1} \le \int_t^\infty \int_t^s \left\| A e^{(\sigma - s)\mathcal{L}_0} P_M (J_a + J_b \cdot \nabla)(s) e^{s\mathcal{L}_0} \xi_\infty \right\|_{H^1} d\sigma ds.$$

From the definition of A, we have

$$\begin{aligned}
&\left\| A e^{(\sigma - s)\mathcal{L}_0} P_M (J_a + J_b \cdot \nabla)(s) e^{s\mathcal{L}_0} \xi_\infty \right\|_{H^1}\\
&\quad \le \left\| e^{(\sigma - s)\mathcal{L}_0} P_M (J_a + J_b \cdot \nabla)(s) e^{s\mathcal{L}_0} \xi_\infty \right\|_{L^\infty}\\
&\qquad + \left\| \nabla e^{(\sigma - s)\mathcal{L}_0} P_M (J_a + J_b \cdot \nabla)(s) e^{s\mathcal{L}_0} \xi_\infty \right\|_{L^\infty}.
\end{aligned}$$

Again we use (4.19) to have

$$\left\| e^{(\sigma-s)\mathcal{L}_0} P_M (J_a + J_b \cdot \nabla)(s) e^{s\mathcal{L}_0} \xi_\infty \right\|_{L^\infty} \leq \sigma^{-3/2} s^{-2} \left\| \xi_\infty \right\|_{W^{1,1}} \ .$$

Since ∇ and $e^{(\sigma-s)\mathcal{L}_0}$ commute, we can bound the term with $\nabla e^{(\sigma-s)\mathcal{L}_0}$ in the same way by also using $\|\widehat{\nabla_y J_{a2}}\|_{L^1(p)} + \|\widehat{\nabla_y J_{b2}}\|_{L^1(p)} \leq O(s^{-2})$. We conclude that

$$\left\| \xi_{AJ}(t) \right\|_{H^1} \leq \int_t^\infty \int_t^s \sigma^{-3/2} s^{-2} d\sigma ds \left\| \xi_\infty \right\|_{W^{2,1}} \leq t^{-3/2} \left\| \xi_\infty \right\|_{W^{2,1}} . \tag{4.20}$$

Finally, we can bound $\xi_{JJ}(t)$ by

$$\left\| \xi_{JJ}(t) \right\|_{H^1} \leq \int_t^\infty \int_t^s \sigma^{-2} s^{-2} d\sigma ds \left\| \xi_\infty \right\|_{H^3} \leq t^{-2} \left\| \xi_\infty \right\|_{H^3} . \tag{4.21}$$

Let $\xi(t) = \xi^{(0)}(t) + \xi^{(1)}(t) + \xi^{(2)}(t)$, where $\xi^{(0)}(t) = P_M e^{t\mathcal{L}_0}\xi_\infty$, $\xi^{(1)} = \xi_J$ and $\xi^{(2)}(t)$ denotes the rest. Then we have proved that

$$\begin{aligned} &\left\| \xi^{(0)}(t) \right\|_{L^\infty} + \left\| \xi^{(1)}(t) \right\|_{L^\infty} \leq C t^{-3/2}, \\ &\left\| \xi^{(1)}(t) \right\|_{H^1} \leq C t^{-1}, \quad \left\| \xi^{(2)}(t) \right\|_{H^1} \leq C t^{-3/2}, \end{aligned} \tag{4.22}$$

with the constants depending on ξ_∞. In fact, tracking the proof we see that, since ∇ commutes with $e^{s\mathcal{L}_0}$, we actually have

$$\begin{aligned} &\left\| \xi^{(0)}(t) \right\|_{W^{2,\infty}} + \left\| \xi^{(1)}(t) \right\|_{W^{2,\infty}} \leq C t^{-3/2}, \\ &\left\| \xi^{(1)}(t) \right\|_{H^2} \leq C t^{-1}, \quad \left\| \xi^{(2)}(t) \right\|_{H^2} \leq C t^{-3/2}. \end{aligned} \tag{4.23}$$

Of course we need to use a stronger norm for ξ_∞. The following norm is sufficient:

$$\left\| \xi_\infty \right\|_{H^4} + \left\| \xi_\infty \right\|_{W^{3,1}} + \left\| \xi_\infty \right\|_{W^{2,1}((1+x^2)dx)} \leq C^{-1} C_* \ , \tag{4.24}$$

where C_* is a small constant to be chosen in the next subsection.

4.4. Existence of g. In this section we construct the solution via a contraction mapping argument. After defining the map in Step 1, we show the following bounds in Step 2:

$$t^2|\omega(t)| + t^3|a(t)| + t^2 \left\| g(t) \right\|_{H^2} < C_*, \quad (t > T) \tag{4.25}$$

provided that $\|\xi_\infty\| \leq C^{-1}C_*$ with $C_* > 0$ sufficiently small (see (4.24)) and T sufficiently large. Finally in Step 3 we show that the contraction mapping converges in the norm

$$t^2|\omega(t)| + t^3|a(t)| + t^2 \left\| g(t) \right\|_{H^1}$$

in the ball $t^2|\omega(t)| + t^3|a(t)| + t^2 \|g(t)\|_{H^2} < C_*$. Notice that we use the H^1 norm for $g(t)$ in the contraction, which is weaker than the H^2 norm appearing in (4.25). Our approach is certainly not the shortest. Once a certain apriori bound is established, we can follow standard existence construction by taking weak limits. This will avoid the proof of the contraction completely. Our approach however provides more information to the scattering operator.

$\boxed{\text{STEP 1}}$ We first define the map

$$(\omega, a, g) \longrightarrow (\omega^\triangle, a^\triangle, g^\triangle) \tag{4.26}$$

with the convention $g_S^\triangle = P_S g^\triangle$ and $g_M^\triangle = P_M g^\triangle$, and so on. Recall that $J(t)$ and $\xi(t)$, defined by (4.9) and (4.3) respectively, depend on ω and a. To solve the equation on the S (4.15), we first solve β and δ from (4.15). Since we plan to solve the equation by iteration, we define (we think $\gamma = 0$)

$$\delta^\triangle(t) = -\int_t^\infty \kappa_1(\mathrm{Re}G_\mu^{(2)}(s), Q)ds \ , \tag{4.27}$$
$$\beta^\triangle(t) = -\int_t^\infty \kappa_2(\mathrm{Re}G_\mu^{(2)}(s), yQ)ds \ .$$

Instead of solving the equation for α and γ, we choose ω and a such that $\dot\alpha = \dot\gamma = 0$. Therefore, we define $\omega^\triangle$, $a^\triangle$ to be

$$\omega^\triangle = -\delta^\triangle + \kappa_1(\mathrm{Im}G_\mu^{(2)}, \Gamma), \tag{4.28}$$
$$a^\triangle = \kappa_2(\mathrm{Im}G_\mu^{(2)}, \nabla Q).$$

With this choice, the component of $g^\triangle$ in the S direction is simply

$$g_S^\triangle(t) = \beta^\triangle(t)\begin{pmatrix}\nabla Q\\ 0\end{pmatrix} + \delta^\triangle(t)\begin{pmatrix}\Gamma\\ 0\end{pmatrix}.$$

Finally, the component on the M direction is given by

$$g_M^\triangle(t) = -\int_t^\infty \widetilde{\mathsf{P}}(s,t) P_M G_\mu^{(1)} ds, \tag{4.29}$$

where $G_\mu^{(1)}$ is defined in (4.12). Note the definition of $\widetilde{\mathsf{P}}(s,t)$ depends on a and ω, so is μ. Our next step is to prove this map is bounded in a certain norm.

$\boxed{\text{STEP 2}}$ Suppose that $\|\xi_\infty\| \le C^{-1}C_*$ (see (4.24)) and

$$t^2|\omega(t)| + t^3|a(t)| + t^2\,\|g(t)\|_{H^2} < C_*. \tag{4.30}$$

We will prove the following bound:

$$t^2|\omega^\triangle(t)| + t^3|a^\triangle(t)| + t^2\left\|g^\triangle(t)\right\|_{H^2} < C_*/2 \tag{4.31}$$

provided that C_* is sufficiently small. The last statement seems to be contradictory as the norm is getting smaller after each iteration and we can drive the constant to zero. But this is impossible as the constant on the estimate of ξ_∞ remains unchanged. Indeed, the right hand side of the last bound depends mainly on the constant appearing in the estimate of ξ_∞, i.e., in the inequality $\|\xi_\infty\| \le C^{-1}C_*$.

Since $a(t)$ satisfies (4.30) and $a = \dot v$, $v = \dot r$, we have $|v(t)| \le CC_*t^{-2}$ and $|r(t)| \le CC_*t^{-1}$. We now estimate $\left\|\mu F(\mu^{-1}(\xi + g))\right\|_{H^2}$. By definition,

$$\mu F(\mu^{-1}h) = -\left(\Phi * |h|^2\right)(\mu Q + h) - 2\left(\Phi * [Q\mathrm{Re}(\mu^{-1}h)]\right)h.$$

Recall the decomposition and the estimate for ξ (4.23) from Subsect. 5.3. Write $h = \xi + g = \xi^{(0)} + \xi^{(1)} + (\xi^{(2)} + g)$. Because of the bound (1.27) on Φ,

$$\begin{aligned}
&\left\| \left(\Phi * |h|^2\right) (\mu Q + h) \right\|_{H^2} \\
&\qquad \le \left\| \Phi * |h|^2 \right\|_{W^{2,\infty}} \cdot \|\mu Q + h\|_{H^2} \\
&\qquad \le C \left\| \Phi * |\xi^{(0)} + \xi^{(1)}|^2 \right\|_{W^{2,\infty}} + C \left\| \Phi * |\xi^{(2)} + g|^2 \right\|_{W^{2,\infty}} \\
&\qquad \le C \|\Phi\|_{W^{2,1}} \cdot \left\| \xi^{(0)} + \xi^{(1)} \right\|^2_{W^{2,\infty}} + C \|\Phi\|_{W^{2,\infty}} \cdot \left\| \xi^{(2)} + g \right\|^2_{H^2} \\
&\qquad \le C C_*^2 t^{-3}.
\end{aligned}$$

Since $\left(\Phi * [Q\mathrm{Re}(\mu^{-1}h)]\right) h$ is a local term by the presence of Q, using the bound (1.27) on Φ we have

$$\left\| \left(\Phi * [Q\mathrm{Re}(\mu^{-1}h)]\right) h \right\|_{H^2} \le C \|h\|^2_{H^2(y\sim 1)} \le C C_*^2 t^{-3}.$$

We conclude that

$$\left\| \mu F(\mu^{-1}(\xi + g)) \right\|_{H^2} \le C C_*^2 t^{-3}.$$

From the bound of J (4.10) and the assumption on the norm of g (4.25), we have $\|J g_S(t)\|_{H^2} \le C C_*^2 t^{-2-2}$. For any $f \in S$, we also have

$$|(f, J g_M)| \le C C_* t^{-2} \|f\|_{H_1} \|g_M\|_{L^2} \le C C_*^2 t^{-4} \|f\|_{H_1}.$$

Also, $|(f, P_S J \xi)| \le C C_*^2 t^{-2-3/2}$. Finally $-i\Omega(\mu - 1)Q$ is exponentially small in t. Hence we conclude that $|(f, G_\mu^{(2)})| \le C C_*^2 t^{-3} \|f\|$. Thus

$$\begin{aligned}
|\beta^\triangle(t)| + |\delta^\triangle(t)| &\le \frac{1}{8} C_* t^{-2}, \qquad & |\omega^\triangle(t)| &\le \frac{1}{8} C_* t^{-2}, \\
|a^\triangle(t)| &\le \frac{1}{8} C_* t^{-3}, \qquad & \left\| g_S^\triangle(t) \right\|_{H^2} &\le \frac{1}{8} C_* t^{-2},
\end{aligned}$$

provided that C_* is sufficiently small. One can also easily check that

$$\left\| g_M^\triangle(t) \right\|_{H^2} \le \int_t^\infty \left\| G_\mu^{(1)}(s) \right\|_{H^2} ds \le \int_t^\infty C C_*^2 s^{-3} ds \le \frac{1}{8} C_* t^{-2}.$$

The claim (4.31) is proved.

$\boxed{\text{STEP 3}}$ Given two data (ω_1, a_1, g_1) and (ω_2, a_2, g_2) we denote by $\boldsymbol{\delta}$ their differences: $\boldsymbol{\delta}\omega = \omega_1 - \omega_2$, $\boldsymbol{\delta}a = a_1 - a_2$, $\boldsymbol{\delta}g = g_1 - g_2$, $\boldsymbol{\delta}g^\triangle = g_1^\triangle - g_2^\triangle$, and so on. We also let

$$\boldsymbol{\delta}_0 = \sup_t \left\{ t^2 |\boldsymbol{\delta}\omega(t)| + t^3 |\boldsymbol{\delta}a(t)| + t^2 \|\boldsymbol{\delta}g(t)\|_{H^1} \right\}. \tag{4.32}$$

Note: different $a(t)$ gives different μ, ($\mu = e^{i(1-\chi)vy}$), but χ is the same. Also, from the definition of J, we have $\|\boldsymbol{\delta}J_a(t)\|_\infty + \|\boldsymbol{\delta}J_b(t)\|_\infty + \|\boldsymbol{\delta}J_c(t)\|_\infty \le C\boldsymbol{\delta}_0 t^{-2}$.

Our goal is to estimate $t^2|\boldsymbol{\delta}\omega^\triangle(t)| + t^3|\boldsymbol{\delta}a^\triangle(t)| + t^2 \left\|\boldsymbol{\delta}g^\triangle(t)\right\|_{H^1}$. Recall the definition of $\omega^\triangle$, $a^\triangle$ and $g^\triangle$ from (4.27), (4.28) and (4.29). In order to estimate the difference of

$\omega^\triangle$, $a^\triangle$ from two initial data, we need to control the difference of $\boldsymbol{\delta} G_\mu^{(2)} := G_{\mu,1}^{(2)} - G_{\mu,2}^{(2)}$, where $G_{\mu,k}^{(2)} := J_k g_k - i\mu_k F(\mu_k^{-1}(\xi_k + g_k)), k = 1, 2$. Here μ_k, ξ_k and J_k denote the corresponding μ, ξ and J, $k = 1, 2$ and thus $\partial_t \xi_k = (\mathcal{L} + P_M J_k)\xi_k$. We shall first estimate $\boldsymbol{\delta}\xi$, then $\boldsymbol{\delta} F = \mu_1 F(\mu_1^{-1}(\xi_1 + g_1)) - \mu_2 F(\mu_2^{-1}(\xi_2 + g_2))$ and finally $\boldsymbol{\delta}(Jg) := J_1 g_1 - J_2 g_2$ and $\boldsymbol{\delta}(J\xi) := J_1\xi_1 - J_2\xi_2$.

From the equation of ξ, $\boldsymbol{\delta}\xi$ satisfies

$$\partial_t \boldsymbol{\delta}\xi = (\mathcal{L} + P_M J_1)\boldsymbol{\delta}\xi + P_M(\boldsymbol{\delta} J)\xi_2.$$

Since $\boldsymbol{\delta}\xi(t) \to 0$ in H^1 as $t \to \infty$, (see (4.23)), we have

$$\boldsymbol{\delta}\xi = -\int_t^\infty \widetilde{\mathsf{P}}_1(s,t) P_M(\boldsymbol{\delta} J(s))\xi_2(s)ds$$

in H^1. We now derive a bound on $\boldsymbol{\delta}\xi$. The last term can be decomposed into two parts $A + B$ with

$$A := -\int_t^\infty \widetilde{\mathsf{P}}_1(s,t) P_M(\boldsymbol{\delta} J(s)) P_M e^{s\mathcal{L}_0}\xi_\infty ds,$$
$$B := -\int_t^\infty \widetilde{\mathsf{P}}_1(s,t) P_M(\boldsymbol{\delta} J(s)) \left\{\xi_2(s) - P_M e^{s\mathcal{L}_0}\xi_\infty\right\} ds.$$

Since $\|\boldsymbol{\delta} J(s)\|_\infty \le C\boldsymbol{\delta}_0 s^{-2}$ and $\|\xi_2(s) - e^{s\mathcal{L}_0}\xi_2\|_{H^2} \le Cs^{-1}$ from (4.23), we can bound B by

$$\|B\|_{H^1} \le \int C\boldsymbol{\delta}_0 s^{-2} C_* s^{-1} ds = CC_*\boldsymbol{\delta}_0 t^{-2}.$$

We can bound A exactly as in Subsect. 4.5. In other words, it can be written as a sum of three terms satisfying (4.23). More precisely, $A = A^{(0)} + A^{(1)} + A^{(2)}$ and

$$\left\|A^{(0)}(t)\right\|_{W^{2,\infty}} + \left\|A^{(1)}(t)\right\|_{W^{2,\infty}} \le C\boldsymbol{\delta}_0 t^{-3/2},$$

$$\left\|A^{(1)}(t)\right\|_{H^2} \le C\boldsymbol{\delta}_0 t^{-1}, \qquad \left\|A^{(2)}(t)\right\|_{H^2} \le C\boldsymbol{\delta}_0 t^{-3/2}.$$

(In fact, $A^{(0)} = 0$.) Notice that the constants on the right hand side now have a $\boldsymbol{\delta}_0$ factor. In particular, we can write $\boldsymbol{\delta}\xi = (\boldsymbol{\delta}\xi)_a + (\boldsymbol{\delta}\xi)_b$ with $(\boldsymbol{\delta}\xi)_a = A^{(0)} + A^{(1)}$ and $(\boldsymbol{\delta}\xi)_b = A^{(2)} + B$ such that

$$\|(\boldsymbol{\delta}\xi)_a(t)\|_{W^{1,\infty}} \le C\boldsymbol{\delta}_0 t^{-3/2}, \quad \|(\boldsymbol{\delta}\xi)_b(t)\|_{H^1} \le C\boldsymbol{\delta}_0 t^{-3/2}. \tag{4.33}$$

From the definition of $\boldsymbol{\delta} F$, we can bound $\boldsymbol{\delta} F$ in terms of $\boldsymbol{\delta}\xi$ and $\boldsymbol{\delta} g$. (Note that $(\mu_1 - \mu_2)Q$ is exponentially small in t.) The previous bound on $\boldsymbol{\delta}\xi$ and the bound (4.32) on $\boldsymbol{\delta} g$ thus yields that

$$\|\boldsymbol{\delta} F\|_{H^1} \le CC_*\boldsymbol{\delta}_0 t^{-3}.$$

Also, $\boldsymbol{\delta}(Jg) = (\boldsymbol{\delta} J)g_1 + J_2(\boldsymbol{\delta} g)$. Thus, for any $f \in S$ with $\|f\|_{H^1} \le 1$ we have

$$|(f, \boldsymbol{\delta}[Jg])| \le CC_*\boldsymbol{\delta}_0 t^{-4}.$$

Similarly,

$$|(f, \delta[P_S J\xi])| \le CC_* \delta_0 t^{-7/2}.$$

Finally, $\delta(-i\Omega(\mu-1)Q) \le CC_* t^{-2} e^{-Ct}$. We conclude for any $f \in S$ with $\|f\|_{H^1} \le 1$ that

$$|(f, \delta G_\mu^{(2)})| \le CC_* \delta_0 t^{-3}.$$

Simple calculations then show that

$$\begin{aligned} |\delta a^\triangle(t)| &\le \frac{1}{8}\delta_0 t^{-3}, \\ |\delta \omega^\triangle(t)| &\le \frac{1}{8}\delta_0 t^{-2}, \\ \left\| \delta g_S^\triangle(t) \right\|_{H^1} &\le \frac{1}{8}\delta_0 t^{-2}, \end{aligned}$$

provided that C_* is sufficiently small.

Finally, the equation of $g_M^\triangle$ (4.29) can be written explicitly as

$$\partial_t g_M^\triangle = \mathcal{L} g_M^\triangle + P_M \left\{ J(g_M^\triangle + g_S) - i\Omega(\mu-1)Q - i\mu F(\mu^{-1}(g+\xi)) \right\}.$$

Hence for $\delta g_M^\triangle = g_{1,M}^\triangle - g_{2,M}^\triangle$ we have

$$\partial_t \delta g_M^\triangle = (\mathcal{L} + P_M J_1)\delta g_M^\triangle + P_M \left\{ (-\delta J) g_{2,M}^\triangle + \delta(J g_S) - i\delta(\Omega(\mu-1))Q - i\delta F \right\}.$$

Since $(\delta g_M^\triangle)(t) \to 0$ as $t \to \infty$ in H^1, we can put it in integral form:

$$\begin{aligned} &(\delta g_M^\triangle)(t) = \\ &\quad -\int_t^\infty \widetilde{\mathsf{P}}_1(s,t) P_M \left\{ (-\delta J) g_{2,M}^\triangle + \delta(J g_S) - i\delta(\Omega(\mu-1))Q - i\delta F \right\} ds. \end{aligned} \tag{4.34}$$

Therefore, we can bound the H_1 norm of $\delta g_M^\triangle$ by

$$\left\| \delta g_M^\triangle \right\|_{H^1} \le C \int_t^\infty \left\| (-\delta J) g_{2,M}^\triangle + \delta(J g_S) - i\delta(\Omega(\mu-1))Q - i\delta F \right\|_{H^1} ds\ .$$

Since

$$\left\| (\delta J) g_{2,M}^\triangle \right\|_{H^1} \le C\delta_0 s^{-2} \left\| g_{2,M}^\triangle \right\|_{H^2},$$

(that is why we needed to prove a stronger bound for $g^\triangle$ in Step 2), together with previous bounds on $\delta(J g_S)$, $-i\delta(\Omega(\mu-1))Q$ and $i\delta F$, we can bound the integrand by $C_* \delta_0 s^{-3}$. Thus we have

$$\left\| \delta g_M^\triangle \right\|_{H^1} \le \frac{1}{8}\delta_0 t^{-2}$$

provided that C_* is sufficiently small.

$\boxed{\text{Conclusion:}}$ For the case $v_0 = 0$, we have proved that

$$\left\{ t^2 |\delta\omega^{\triangle}(t)| + t^3 |\delta a^{\triangle}(t)| + t^2 \left\| \delta g^{\triangle}(t) \right\|_{H^1} \right\} \leq \delta_0 / 2$$

under the assumptions (4.24), (4.30) and (4.32). Thus the map (4.26) is a contraction. Since (4.30) holds for a nonempty set of functions (including zero), we obtain a solution (ω, a, g), together with ξ. Furthermore, we have proved that

$$\left\{ t^2 |\omega(t)| + t^3 |a(t)| + t^2 \left\| g(t) \right\|_{H^2} \right\} \leq C_*$$

for t greater than an aboulute constant T. Hence $v(t) = -\int_t^\infty a(s)ds = O(t^{-2})$. Similarly $r(t) = O(t^{-1})$ and $\theta_0(t) = O(t^{-1})$. Also recall $y = x - r(t)$ and $h_{as,0} = \xi_\infty$. Therefore, by Taylor expansion,

$$\begin{aligned} &\psi(x,t) - \psi_{as}(x,t) \\ &= \left(Q(y) e^{i(vy - Et + \theta_0)} + h(y,t) e^{i(\chi vy - Et + \theta_0)} \right) - \left(Q(x) e^{-iEt} + \left(e^{it\Delta/2} \xi_\infty \right)(x) \right) \\ &= O(t^{-1}) \qquad \text{in } H^2. \end{aligned}$$

Note that our result is true for $t > T$. However, if we replace all previous estimates of the form t^{-m} by $(t+T)^{-m}$, our contraction argument still holds. Hence Theorem 1.2 is proved for the case $v_0 = 0$. To conclude Theorem 1.2 for general v_0, we apply the following Galilei transform (boost):

$$\psi(x,t) \longrightarrow \psi(x - v_0 t, t) e^{i(v_0 \cdot x - \frac{1}{2} v_0^2 t)}.$$

(Recall $h_{as,0}(x) = \xi_\infty(x) e^{iv_0 \cdot x}$ and $\hat{h}_{as,0}(v_0) = \hat{\xi}_\infty(0) = 0$.) Also, for general r_0 we apply a translation, which does not require a change of assumption. The proof is complete.

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