The Motivic Cohomology of Stiefel Varieties

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Abstract

The main result of this paper is a computation of the motivic cohomology of varieties of $n \times m$-matrices of of rank m, including both the ring structure and the action of the reduced power operations. The argument proceeds by a comparison of the general linear group-scheme with a Tate suspension of a space which is $\mathbb{A}^1$-equivalent to projective $n-1$-space with a disjoint basepoint.

Key Words Motivic cohomology, higher Chow groups, reduced power operations, Stiefel varieties.

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1 Introduction

The results of this paper are theorem 19, which computes the motivic cohomology (or higher Chow groups) of varieties of $n \times m$-matrices of of rank m, the Stiefel varieties of the title, including the ring structure, and theorems 20, 21 which compute the action of the motivic reduced power operations, the first for the Steenrod squares, the second for odd primes. The ring structure on $H^{*,*}(\mathrm{GL}(n))$ has already been computed, in [Pus04], where the argument is via comparison with higher K-theory. We are able to offer a different computation of the same, which is more geometric in character, relying on an analysis of comparison maps of varieties rather than of cohomology theories.

The equivalent computation for étale cohomology appears in [Ray68]; the argument there is by comparison with singular cohomology, and that suffices to determine even the action of the reduced power operations on the étale cohomology of $\mathrm{GL}(n, \mathbb{Z})$, and from there on the étale cohomology of $\mathrm{GL}(n, k)$ where k is an arbitrary field. The universal rings of loc. cit. are not quite Stiefel varieties, but they are affine torsors over them. In principle, by slavish imitation of loc. cit. the methods of the present paper allow one to prove that with sporadic exceptions the universal stably-free module of rank at least 2 over a field is not free, and to do so without recourse to a non-algebraic category.

In this paper we deduce, by elementary means, the additive structure of the cohomology of Stiefel varieties, proposition 5, then we deduce the effect on cohomology of two comparison maps between the different Stiefel manifolds, these are propositions 7 and 8. By use of these, the ring structure and the reduced power operations in all cases may be deduced from the case of $\mathrm{GL}(n)$. We present a map in homotopy $\mathbb{G}_m \wedge \mathbb{P}^{n-1}_+ \to \mathrm{GL}(n)$ that is well-known in the classical cases of $\mathbb{R}^* \wedge \mathbb{R}P^{n-1}_+ \to O(n)$ and $\mathbb{C}^* \wedge \mathbb{C}P^{n-1}_+ \to U(n)$, [Jam76, Chapter 3], where it fits into a larger pattern of maps from suspensions of so-called "stunted quasiprojective spaces" into Stiefel manifolds. In the classical case, with some care, one may show that the cohomology of a Stiefel manifold is generated as a ring by classes detected by maps to stunted projective spaces. The comparison of Stiefel manifolds with the appropriate stunted projective-spaces underlies much of the classical homotopy-theory of Stiefel manifolds, the theory is presented thoroughly in [Jam76], we mention in addition only the spectacular resolution of the problem of vector fields on spheres in [Ada62]. Care is required even in cohomology calculations because the products of classes in the cohomology of Stiefel manifolds muddy the water. In $\mathbb{A}^1$-homotopy, there are also analogues of stunted projective spaces, but the bigrading on motivic cohomology means that the products of generating classes may be disregarded in articulating the range in which the cohomology $H^{*,*}(\mathbb{G}_m \wedge \mathbb{P}^{n-1}_+)$ and $H^{*,*}(\mathrm{GL}(n))$ coincide, and we arrive at a much simpler statement, theorem 18 without having to mention the stunted spaces at all.

In proving theorem 18, we employ a calculation in higher Chow groups. This calculation is on the one hand attractive in its geometric and explicit character but on the other hand it is the chief obstacle to extending the scope of the arguments presented here to other theories than motivic cohomology.

2 Preliminaries

We compute motivic cohomology as a represented cohomology theory in the motivic- or $\mathbb{A}^1$-homotopy category of Morel & Voevodsky, see [MV99] for the construction of this category. The best reference for the theory of motivic cohomology is [MVW06], and the proof that the theory presented there is representable in the category we claim can be found in [Del], subject to the restriction that the field k is perfect. Motivic cohomology, being a cohomology theory (again at least when k is perfect) equipped with suspension isomorphisms for both suspensions, Σ_s, Σ_t, is represented by a motivic spectrum, of course, but we never deal explicitly with such objects.

We therefore fix a perfect field k. We shall let R denote a fixed commutative ring of coefficients. We denote the terminal object, $\operatorname{Spec} k$, by pt. If X is a finite type smooth k-scheme or more generally an element in the category $s\mathbf{Shv}_{Nis}(\mathbf{Sm}/k))$, we write $H^{*,*}(X; R)$ for the bigraded motivic cohomology ring of X. This is graded-commutative in the first grading, and commutative in the second. If $R \to R'$ is a ring map, then there is a map of algebras

$H^{*,*}(X;R) \to H^{*,*}(X;R')$. It will be of some importance to us that most of our constructions are functorial in R, when the coefficient ring is not specified, therefore, it is to be understood that arbitrary coefficients R are meant and that the result is functorial in R.

We write $\mathbb{M}_R$ for the ring $H^{*,*}(\mathrm{pt}; R)$. Since pt is a terminal object, the ring $H^{*,*}(X;R)$ is in fact an $\mathbb{M}_R$ algebra. We assume the following vanishing results for a d-dimensional smooth scheme X: $H^{p,q}(X;R) = 0$ when $p > 2q$, $p > q + d$ or $q < 0$.

At one point we employ the comparison theorem relating motivic cohomology and the higher Chow groups. For all these, see [MVW06].

Let $\mathbf{Sm}/k$ denote the category of smooth k-schemes, and let $\mathbf{Aff}/k$ denote the category of affine regular finite-type k-schemes. We shall frequently make use of the following version of the Yoneda lemma

Lemma 1. *The functor* $\mathbf{Sm}/k \to \mathbf{Pre}(\mathbf{Aff}/k)$ *given by* $X \mapsto h_X$*, where* $h_X(\operatorname{Spec} R)$ *denotes the set of maps* $\operatorname{Spec} R \to X$ *over* $\operatorname{Spec} k$ *is a full, faithful embedding.*

Proof. The standard version of Yoneda's lemma is that there is a full, faithful embedding $\mathbf{Sm}/k \to \mathbf{Pre}(\mathbf{Sm}/k)$. The functor we are considering is the composition $\mathbf{Sm}/k \to \mathbf{Pre}(\mathbf{Sm}/k) \to \mathbf{Pre}(\mathbf{Aff}/k)$ obtained by restricting the domain of the functors in $\mathbf{Pre}(\mathbf{Sm}/k)$. One can write any Y in $\mathbf{Sm}/k$ as a colimit of spectra of finite-type k-algebras. We have

$$\mathbf{Sm}/k(Y,X) = \mathbf{Sm}/k(\operatorname{colim} \operatorname{Spec} A_i, X) = \lim \mathbf{Sm}/k(\operatorname{Spec} A_i, X) = \lim X(A_i)$$

from which it follows that the functor $\mathbf{Sm}/k \to \mathbf{Aff}/k$ inherits fullness and fidelity from the Yoneda embedding by abstract-nonsense arguments. □

We shall generally write $X(R)$ for $h_X(\operatorname{Spec} R)$.

In practice this result means that rather than specifying a map of schemes $X \to Y$ explicitly, we shall happily exhibit a set-map $X(R) \to Y(R)$, where R is an arbitrary finite-type k-algebra, and then observe that this set map is natural in R. The result is a map in $\mathbf{Pre}(\mathbf{Aff}/k)$, which is therefore (by the fullness and fidelity of Yoneda) also understood as a map of schemes $X \to Y$.

3 The Additive Structure

Proposition 2. *Let* X *be a smooth scheme and suppose* E *is an* $\mathbb{A}^n$*-bundle over* X*, and* F *is a sub-bundle with fiber* $\mathbb{A}^\ell$*, then there is an exact triangle of graded* $H^{*,*}(X)$*-modules*

$$\begin{tikzcd} H^{*,*}(X)\tau \arrow[rr, "j"] & & H^{*,*}(X) \arrow[dl] \\ & H^{*,*}(E \setminus F) \arrow[ul, "\partial"] & \end{tikzcd}$$

where $|\tau| = (2n - 2\ell, n - \ell)$

Proof. This is the localization exact triangle for the closed sub-bundle $F \subset E$, where

$$H^{*,*}(E) = H^{*,*}(F) = H^{*,*}(X)$$

arising from the cofiber sequence (see [MV99] for this and all other unreferenced assertions concerning $\mathbb{A}^1$-homotopy)

$$E \setminus F \longrightarrow E \longrightarrow \mathrm{Th}(N)$$

where N is the normal-bundle of F in E. There are in two possible choices for τ, since τ and $-\tau$ serve equally well. We make the convention that in any localization exact sequence associated with a closed immersion of smooth schemes $Z \to X$, viz.

$$\longrightarrow H^{*,*}(Z)\tau \longrightarrow H^{*,*}(X) \longrightarrow H^{*,*}(X \setminus Z) \longrightarrow$$

the class τ should be the class which, under the natural isomorphism of the above with the localization sequence in higher Chow groups, corresponds to the class represented by Z in $CH^*(Z,*)$.

In the case where $F = X$ is the zero-bundle, then j takes τ to $e(E)$, the Euler class, as proved in [Voe03]. In general, by identifying $H^{*,*}(X)\tau$ with $CH^{*-n+\ell}(F,*)$, the higher Chow groups of F as a closed subscheme of E, and employing covariant functoriality of higher Chow groups for the closed immersions $X \rightarrowtail F \rightarrowtail E$, we see that $j(\tau)e(F) = e(E)$. □

As shall be the case throughout, $H^{*,*}(X)$ denotes cohomology with unspecified coefficients, R, and the result is understood to be natural in R. For the naturality of the localization sequence in R, one simply follows through the argument in [MVW06], which reduces it to the computation of $H^{*,*}(\mathbb{P}^d) = \mathbb{M}_R[\theta]/(\theta^{d+1})$ which is natural in R by elementary means, c.f. [Ful84].

We note that $|j(\tau)| = (2c, c)$, so if, as often happens, $H^{2c,c}(X) = 0$, this triangle is a short exact sequence of $H^{*,*}(X)$-modules:

$$0 \longrightarrow H^{*,*}(X) \longrightarrow H^{*,*}(E \setminus F) \longrightarrow H^{*,*}(X)\rho \longrightarrow 0$$

Here $|\rho| = (2n - 2\ell - 1, n - \ell)$

Since $H^{*,*}(X)\rho$ is a free graded $H^{*,*}(X)$-module, this short exact sequence of graded modules splits, there is an isomorphism of $H^{*,*}(X)$-modules

$$H^{*,*}(E \setminus F) \cong H^{*,*}(X) \oplus H^{*,*}(X)\rho$$

We remark that $|\rho| = (2c-1, c)$, so that $2\rho^2 = 0$ by anti-commutativity, we now see that $(a + b\rho)(c + d\rho) = ac + (ad + (-1)^{\deg c} bc)\rho + (-1)^{\deg d} bd\rho^2$, so in many cases (e.g. when $1/2 \in R$) the multiplicative structure is fully determined, and $H^{*,*}(E \setminus F) = H^{*,*}(X)[\rho]/(\rho^2)$

Observe that if $H^{2n,n}(X) = 0$ for $n > 0$, as often happens, then the same applies to $H^{*,*}(E \setminus F)$.

We will have occasion later to refer to the following two results, which appear here for want of anywhere better to state them

Proposition 3. *Let X be a scheme and suppose E is a Zariski-trivializeable fiber bundle with fiber $F \simeq \mathrm{pt}$. Then $E \simeq X$.*

Proof. This is standard, see [DHI04]. $\square$

When we use the term 'bundle', we shall mean a Zariski-trivializable bundle over a scheme. The following two propositions allow us to identify affine bundles which are not necessarily vector-bundles.

Proposition 4. *Suppose X is a scheme, P is a projective bundle of rank n over X and Q is a projective subbundle of rank $n-1$, Then $P \setminus Q$ is a fiber bundle with fiber $\mathbb{A}^{n-1}$.*

Proof. This follows immediately by considering a Zariski open cover trivializing both bundles. $\square$

Let k be a field. Let $W(n,m)$ denote the variety of full-rank $n \times m$ matrices over k, that is to say it is the open subscheme of $\mathbb{A}^{nm}$ determined by the non-vanishing of at least one $m \times m$-minor. Without loss of generality, $m \leq n$. By a *Stiefel Variety* we mean such a variety $W(n,m)$.

Proposition 5. *The cohomology of $W(n,m)$ has the following presentation as an $\mathbb{M}_R$-algebra:*

$$H^{*,*}(W(n,m);R) = \frac{\mathbb{M}_R[\rho_n, \ldots, \rho_{n-m+1}]}{I} \qquad |\rho_i| = (2i-1, i)$$

The ideal I is generated by relations $\rho_i^2 - a_{n,i}\rho_{2i-1}$, where the elements $a_{n,i}$ lie in $\mathbb{M}_R^{1,1}$ and satisfy $2a_{n,i} = 0$.

We shall later identify $a_{n,i}$ as $\{-1\}$, the image of the class of -1 in $k^* = H^{1,1}(\operatorname{Spec} k; \mathbb{Z})$ under the map $H^{*,*}(\operatorname{Spec} k; \mathbb{Z}) \to H^{*,*}(\operatorname{Spec} k; R)$.

Proof. If $n = 1$, there is only one possibility to consider, that of $W(1,1) = \mathbb{A}^1 \setminus \{0\}$, the cohomology of which is already known from [Voe03], and is as asserted in the proposition. We therefore assume $n \geq 2$.

The proof proceeds by induction on m, starting with $m = 1$ (we could start with $W(n,0) = \mathrm{pt}$). In this case $W(n,1) = \mathbb{A}^n \setminus \{0\}$, and $H^{*,*}(W(n,1)) = \mathbb{M}[\rho_n]/(\rho_n^2)$.

$W(n,m-1)$ is a dense open set of $\mathbb{A}^{nm}$, and as such is a smooth scheme. If $m < n$, there is a trivial $\mathbb{A}^n$-bundle over $W(n,m-1)$, the fiber over a matrix A is the set of all $n \times m$-matrices whose first $m-1$ columns are the matrix A

$$\begin{pmatrix} & v_1 \\ A & \vdots \\ & v_n \end{pmatrix}$$

As a sub-bundle of this bundle, we find a trivial $\mathbb{A}^{m-1}$-bundle; the fiber of which over a $\bar{k}$-point (i.e. a matrix) A is the set of matrices where $(v_1, \ldots, v_n)$

is in the row-space of A. Proposition 2 applies in this setting, and we conclude that there exists an exact triangle

$$\begin{tikzcd} H^{*,*}(W(n,m-1))\tau \arrow[rr] & & H^{*,*}(W(n,m-1)) \arrow[dl] \\ & H^{*,*}(W(n,m)) \arrow[ul, "\partial"] & \end{tikzcd}$$

Since $H^{2i,i}(W(n,m-1)) = 0$ by induction, so this triangle splits to give

$$H^{*,*}(W(m,n)) \cong H^{*,*}(W(m-1,n)) \oplus H^{*,*}(W(m-1,n))\rho_{n-m+1}$$

where $|\rho_{n-m+1}| = (2(n-m+1)-1, n-m+1)$. By graded-commutativity we have, $2\rho_{n-m+1}^2 = 0$. By considering the bigrading on the motivic cohomology, and the vanishing results, we know that

$$\rho_r^2 \in H^{4r-2,2r}(W(n,m);R) = H^{4r-2,2r}(W(n,m-1);R) \text{ where } r = n-m+1$$

We can describe $H^{*,*}(W(n,m-1);R)$ as an $\mathbb{M}$-module as follows

$$H^{*,*}(W(n,m-1);R) \cong \mathbb{M} \oplus \bigoplus_i \mathbb{M}\rho_i \oplus \bigoplus_{i<j} \mathbb{M}\rho_i\rho_j \oplus J$$

where J is the submodule generated by multiples of at least three distinct classes of the form ρ_i. Since $\mathrm{wt}(\rho_i\rho_j) = i+j > 2(n-m+1)$ for all $i,j \geq n-m+2$, it follows the higher product terms are irrelevant to the determination of the cohomology group $H^{4n-4m+2,2n-2m+2}(W(n,m-1;R))$.

There are two possibilities to consider. First, that $n > 2m-1$, which by consideration of the grading forces $H^{4n-4m+2,2n-2m+2}(W(n,m-1);R) = 0$, and so $\rho_{n-m+1}^2 = 0$. The other is $n \leq 2m-1$, in which case

$$H^{4n-4m+2,2n-2m+2}(W(n,m-1);R) = \mathbb{M}_R^{1,1}\rho_{2n-2m+1}$$

so that $\rho_{n-m+1}^2 = a_{n,m}\rho_{2n-2m+1}$ as required. The bigrading alluded to above forces $2a_{n,m} = 0$. □

We denote the cohomology ring

$$H^{*,*}(W(n,m);R) = \mathbb{M}_R[\rho_n, \ldots, \rho_{n-m+1}]/I$$

where the ideal I is understood to depend on n, m. We shall need the following technical lemma

Lemma 6. *Let $Z \to X$ be a closed immersion of irreducible smooth schemes, and let $f : X' \to X$ be a map of smooth schemes such that $f^{-1}(Z)$ is again smooth and irreducible and so that is either f is flat or split by a flat map, in the sense that there* *exists a flat map $s : X \to X'$ such that $s \circ f = \mathrm{id}_{X'}$. Then there is a map of localization sequences in motivic cohomology*

$$\begin{tikzcd}
{} \arrow[r] & H^{*,*}(Z)\tau \arrow[r] \arrow[d] & H^{*,*}(X) \arrow[r] \arrow[d] & H^{*,*}(X \setminus Z) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & H^{*,*}(f^{-1}(Z))\tau' \arrow[r] & H^{*,*}(X') \arrow[r] & H^{*,*}(X' \setminus f^{-1}(Z)) \arrow[r] & {}
\end{tikzcd}$$

such that the last two vertical arrows are the functorial maps on cohomology and such that $\tau \mapsto \tau'$.

The giving of references for results concerning higher Chow groups is deferred to the beginning of section 4.

Proof. One begins by observing the existence in general of the following diagram

$$\begin{tikzcd}
X' \setminus f^{-1}(Z) \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d] & \mathrm{Th}\, N_{f^{-1}(Z) \to X'} \arrow[d, dashed] \\
X \setminus Z \arrow[r] & X \arrow[r] & \mathrm{Th}\, N_{Z \to X}
\end{tikzcd}$$

where the dotted arrow exists for reasons of general nonsense.

There is in general a map on cohomology arising from the given diagram of cofiber sequences, but we cannot at this stage predict the behavior of the map induced by the dotted arrow. When the map $X' \to X$ is flat, the pull-back $f^{-1}(Z) \to Z$ is too. We identify the motivic cohomology groups with the higher Chow groups, giving the localization sequence

$$\begin{tikzcd}
{} \arrow[r] & CH^*(Z,*) \arrow[r] \arrow[d, dashed] & CH^*(X,*) \arrow[r] \arrow[d] & CH^*(X \setminus Z,*) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & CH^*(f^{-1}(Z),*) \arrow[r] & CH^*(X',*) \arrow[r] & CH^*(X' \setminus f^{-1}(Z),*) \arrow[r] & {}
\end{tikzcd}$$

and in this case τ, τ' become the classes of the cycles $[Z]$, $[f^{-1}Z]$. Since the map

$$CH^*(Z,*) \to CH^*(f^{-1}(Z),*)$$

is the contravariant map associated with pull-back along a flat morphism, it follows immediately that $\tau \mapsto \tau'$. $\square$

The following results are analogues of classically known facts.

Proposition 7. *For $m' \leq m$, there is a projection $W(n,m) \to W(n,m')$ given by omission of the last $m - m'$-vectors. On cohomology, this yields an inclusion*

$$\mathbb{M}(\rho_n, \ldots, \rho_{n-m'+1})/I \to \mathbb{M}(\rho_n, \ldots, \rho_{n-m'+1}, \ldots, \rho_{n-m+1})/I$$

Proof. It suffices to prove the case $m' = m-1$. In this case, the map $W(n,m) \to W(n,m-1)$ is the fiber bundle from which we computed the cohomology of $W(n,m)$, and the result on cohomology holds by inspection of the proof. $\square$

Proposition 8. *Given a nonzero rational point, $v \in (\mathbb{A}^n \setminus 0)(k)$, and a complementary $n-1$-dimensional subspace U such that $\langle v \rangle \oplus U = \mathbb{A}^n(k)$, there is a closed immersion $\phi_{v,U} : W(n-1,m-1) \to W(n,m)$ given by identifying $W(n-1,m-1)$ with the space of independent $m-1$-frames in U, and then prepending v. On cohomology, this yields the surjection $\mathbb{M}[\rho_n,\ldots,\rho_{n-m+1}]/I \to \mathbb{M}[\rho_{n-1},\ldots,\rho_{n-m+1}]/I$ with kernel (ρ_n).*

Proof. We prove this by induction on the m, which is to say we deduce the case (n,m) from the case $(n,m-1)$. The base case of $m=1$ is straightforward.

Recall that we compute the cohomology of $W(n,m)$ by forming a trivial bundle $E_{n,m} \simeq W(n,m-1)$ over $W(n,m-1)$, which on the level of R-points consists of matrices

$$\begin{pmatrix} & & v_1 \\ & A & \vdots \\ & & v_n \end{pmatrix}$$

and removing the trivial closed sub-bundle $Z_{n,m}$ where the vector $(v_1,\ldots,v_n)$ is in the span of the columns of A. There is then an open inclusion

$$E_{n,m} \setminus Z_{n,m} \cong W(n,m) \to E_{n,m} \simeq W(n,m-1)$$

The inclusion $\phi_{v,U} : W(n-1,m-1) \to W(n,m)$, without loss of generality can be assumed to act on field-valued points as as

$$B \xmapsto{\phi_{v,U}} \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix}$$

We abbreviate this map of schemes to ϕ, and denote the analogous maps $W(n-1,m-2) \to W(n,m-1)$, $Z_{n-1,m-1} \to Z_{n,m}$, $E_{n-1,m-1} \to E_{n,m}$ etc. also by ϕ by abuse of notation. The following are pull-back diagrams

$$\begin{tikzcd} E_{n-1,m-1} \arrow[r] & E_{n,m} \\ Z_{n-1,m-1} \arrow[u] \arrow[r] & Z_{n,m} \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} E_{n-1,m-1} \arrow[r] & E_{n,m} \\ W(n-1,m-1) \arrow[u] \arrow[r, "\phi"] & W(n,m) \arrow[u] \end{tikzcd}$$

The second square above is homotopy equivalent to

$$\begin{tikzcd} W(n-1,m-2) \arrow[r, "\phi"] \arrow[d] & W(n,m-1) \arrow[d] \\ W(n-1,m-1) \arrow[r, "\phi"] & W(n,m) \end{tikzcd}$$

from which we deduce that the map

$$\phi^* : H^{*,*}(W(n,m)) \to H^{*,*}(W(n-1,m-1))$$

satisfies $\phi^*(\rho_j) = \rho_j$ for $n-m+2 \leq j \leq n-1$ and $\phi^*(\rho_n) = 0$, since this holds for $W(n-1,m-2) \to W(n,m-1)$ by induction.

The hard part is the behavior of the element ρ_{n-m+1}, which is in the kernel of

$$H^{*,*}(W(n,m)) \to H^{*,*}(W(n,m-1))$$

Recall that $\rho_{n-m+1} \in H^{*,*}(W(n,m))$ is the preimage of the Thom class τ under the map

$$\partial : H^{*,*}(W(n,m)) \to H^{*,*}(W(n,m-1))\tau = H^{*,*}(Z_{n,m})\tau$$

We should like to assert that the map of localization sequences

$$\begin{tikzcd}
\arrow[r] & H^{*,*}(W(n,m)) \arrow[r, "\partial"] \arrow[d, "\phi^*"] & H^{*,*}(Z_{n,m})\tau \arrow[r] \arrow[d] & H^{*,*}(E_{n,m}) \arrow[r] \arrow[d] & {} \\
\arrow[r] & H^{*,*}(W(n-1,m-1)) \arrow[r, "\partial"] & H^{*,*}(Z_{n-1,m-1})\tau' \arrow[r] & H^{*,*}(E_{n-1,m-1}) \arrow[r] & {}
\end{tikzcd} \tag{1}$$

one has $\tau \mapsto \tau'$, because then chasing the commutative square of isomorphisms

$$\begin{tikzcd}
\mathbb{Z}\rho_{n-m-1} = H^{2n-2m-1,n-m}(W(n,m)) \arrow[r] \arrow[d] & H^{0,0}(Z_{n,m}\tau = \mathbb{Z}\tau \arrow[d] \\
\mathbb{Z}\rho_{n-m-1} = H^{2n-2m-1,n-m}(W(n-1,m-1)) \arrow[r] & H^{0,0}(Z_{n-1,m-1}\tau = \mathbb{Z}\tau'
\end{tikzcd}$$

we have $\rho_{n-m-1} \mapsto \rho_{n-m-1}$ as required.

The difficulty is that the map $g : E_{n-1,m-1} \to E_{n,m}$ is a closed immersion, rather than a flat or split map, for which we have deduced this sort of naturality result in lemma 6. We can however factor g into such maps, which we denote only on the level of points, the obvious scheme-theoretic definitions are suppressed. Let $U_{n,m}$ denote the variety of n,m-matrices which (on the level of $\bar{k}$-points) have a decomposition as

$$\begin{pmatrix} u & * & * \\ * & A & * \end{pmatrix}$$

where $u \in \bar{k}^*$, and $A \in W(n-1,m-2)(\bar{k})$. It goes without saying that this is a variety, since the conditions amount to the nonvanishing of certain minors. It is also easily seen that $U_{n,m}$ is an open dense subset of $E_{n,m}$. We have a map $E_{n-1,m-1} \to E_{n,m}$, given by

$$B \mapsto \begin{pmatrix} 1 & * \\ * & B \end{pmatrix}$$

and this map is obviously split by the projection onto the bottom-right $n-1 \times m-1$-submatrix. The composition $E_{n-1,m-1} \to U_{n,m} \to E_{n,m}$ is a factorization of the map $E_{n-1,m-1} \to E_{n,m}$ into a split map followed by an open immersion. The splitting of $E_{n-1,m-1}$ is a projection onto a factor, and since all schemes are flat over pt, the splitting is flat as well. We may now use lemma 6 twice to conclude that in diagram (1) we have $\tau \mapsto \tau'$, so that $\rho_{n-m+1} \mapsto \rho_{n-m+1}$, as asserted. $\square$

4 Higher Intersection Theory

We shall, as is standard, denote the algebraic d-simplex $\operatorname{Spec} k[x_0, \dots, x_d]/(x_0 + \cdots + x_d - 1) \cong \mathbb{A}^d$ by Δ^d. The object $\Delta^\bullet$ is cosimplicial in an obvious way. It shall be convenient later to identify Δ^1 in particular with $\mathbb{A}^1 = \operatorname{Spec} k[t]$.

Let X be a scheme of finite type over a field. The higher Chow groups of X, denoted $CH^i(X,d)$ are defined in [Blo86] as the homology of a certain complex:

$$CH^i(X,d) = H_d(z^i(X,*))$$

where $z^i(X,d)$ denotes the free abelian group generated by cycles in $X \times \Delta^d$ meeting all faces of $X \times \Delta^d$ properly. We denote the differential in this complex by δ.

There is a comparison theorem, see [MVW06, lecture 19], [Voe02], which states that for any smooth scheme X over any field k, there is an isomorphism between the motivic cohomology groups and the higher Chow groups

$$CH^i(X,d) = H^{2i-d,i}(X,\mathbb{Z})$$

or the equivalent with $\mathbb{Z}$ replaced by a general coefficient ring R. The products on motivic cohomology and on higher Chow groups are known to coincide, see[Wei99]. In the difficult paper [Blo94], the following result is proven in an equivalent form (the strong moving lemma)

Theorem 9 (Bloch)**.** *Let X be an equidimensional scheme, Y a closed equidimensional subscheme of codimension c in X, $U \cong X \setminus Y$. Then for all i, there is an exact sequence of complexes*

$$0 \longrightarrow z^{i-c}(Y,*) \longrightarrow z^i(X,*) \longrightarrow z^i_a(U,*) \longrightarrow 0$$

where $z^i_a(U,*)$ is the subcomplex of $z^i(U,*)$ generated by subvarieties γ whose closure $\overline{\gamma} \rightarrowtail X \times \Delta^*$ meet all faces properly. The inclusion of complexes $z^i_a(U,*) \subset z^i(U,*)$ induces an isomorphism on homology groups.

For a cycle $\alpha \in z^i(U,d)$, we can write $\alpha = \sum_{i=1}^N n_i A_i$ for some subvarieties A_i of $U \times \Delta^d$, and $n_i \in \mathbb{Z} \setminus \{0\}$. We can form the scheme-theoretic closure of A_i in $X \times \Delta^d$, denoted $\overline{A_i}$. We remark that $\overline{A_i} \times_{X \times \Delta^d} (U \times \Delta^d) = A_i$ [Har77, II.3]. We define

$$\overline{\alpha} \sum_{i=1}^N n_i \overline{A_i}$$

We say that $\overline{\alpha}$ meets a subvariety $K \rightarrowtail X$ properly if every $\overline{A_i}$ meets K properly. Suppose α is such that $\overline{\alpha}$ meets the faces of $X \times \Delta^d$ properly, then $\alpha = (U \to X)^*(\overline{\alpha})$, so $\alpha \in z^i_a(U, d)$.

Proposition 10. *As before, let X be a quasiprojective variety, let Y be a closed subvariety of pure codimension c, let $U = X - Y$ and let $\iota : U \to X$ denote the open embedding. Suppose $\alpha \in z^i(U, d)$ is such that $\overline{\alpha}$ meets the faces of $X \times \Delta^d$ properly, then the connecting homomorphism $\partial : CH^i(U, d) \to CH^i(Y, d-1)$ takes the class of α to the class of $\delta(\overline{\alpha})$ which happens to lie in the subgroup $z^{i-c}(Y, d-1)$ of $z^{i-c}(X, d-1)$.*

Proof. First, since α is such that $\overline{\alpha}$ meets the faces of $X \times \Delta^d$ properly, it follows that $\alpha = \iota^*(\overline{\alpha})$, so $\alpha \in z^i_a(U, d)$.

The localization sequence arises from the short exact sequence of complexes

$$0 \longrightarrow z^{i-c}(Y, *) \longrightarrow z^i(X, *) \longrightarrow z^i_a(U, *) \longrightarrow 0$$

via the snake lemma. A diagram chase now completes the argument. □

Proposition 11. *Consider $\mathbb{A}^n \setminus \{0\}$ as an open subscheme of $\mathbb{A}^n$ in the obvious way, so there is a localization sequence in higher Chow groups for $\mathrm{pt}, \mathbb{A}^n$ and $\mathbb{A}^n \setminus \{0\}$. The higher Chow groups $CH(\mathrm{pt}) = \mathbb{M}$ are given an explicit generator, ν. Write $H^{2n-1,n}(\mathbb{A}^n \setminus \{0\}, \mathbb{Z}) = CH^n(\mathbb{A}^n \setminus \{0\}, 1) = \mathbb{Z}\gamma \oplus Q$, where $Q = 0$ for $n \geq 2$ and $Q = k^*$ for $n = 1$, and where γ is such that the boundary map*

$$\partial : CH^n(\mathbb{A}^n \setminus \{0\}, 1) \to CH^0(\mathrm{pt}, 0)$$

maps γ to ν. The element γ may be represented by any curve in

$$\mathbb{A}^n \times \Delta^1 = \operatorname{Spec} k[x_1, \ldots, x_n, t]$$

which fails to meet the hyperplane $t = 0$ and meets $t = 1$ with multiplicity one at $x_1 = x_2 = \cdots = x_n = 0$ only.

Proof. The low-degree part of the localization sequence is

$$\begin{array}{l} CH^0(\mathrm{pt}, 1) = 0 \longrightarrow CH^n(\mathbb{A}^n, 1) = Q \longrightarrow CH^n(\mathbb{A}^n \setminus 0, 1) \xrightarrow{\partial} CH^0(\mathrm{pt}, 0) \\ \qquad \longrightarrow CH^0(\mathbb{A}^n, 0) = \mathbb{Z} \longrightarrow CH^0(\mathbb{A}^n \setminus 0, 0) = \mathbb{Z} \longrightarrow 0 \end{array}$$

Suppose C is a curve which does not meet $t = 0$, and which meets $t = 1$ with multiplicity one at $x_1 = \cdots = x_n = 0$ only, then by proposition 10 the cycle $[C] \in CH^n(A^n - 0, 1)$ maps to the class of a point in $CH^0(\mathrm{pt}, 0) = \mathbb{Z}$, which is a generator, [Ful84]. The assertion now follows from straightforward homological algebra. □

Corollary 11.1. *Suppose $p \in \mathbb{A}^n \setminus \{0\}$ is a k-valued point. Write $p = (p_1, \ldots, p_n)$. The curve given by the equation*

$$\gamma_p : \quad (x_1 - p_1)t + p_1 = (x_2 - p_2)t + p_2 = \cdots = (x_n - p_n)t + p_n = 0$$

represents a canonical generator of $CH^n(\mathbb{A}^n \setminus \{0\}, 1)$.

Proof. One verifies easily that the proposition applies. □

Corollary 11.2. *Consider the map $\mathbb{A}^n \setminus \{0\} \to \mathbb{A}^n \setminus \{0\}$ given by multiplication by -1. This map induces the identity on cohomology.*

Proof. The preimage of the curve γ_p is the curve γ_{-p}, but both represent the same generator of $CH^n(\mathbb{A}^n \setminus \{0\}, 1)$, so the result follows. □

We can now prove two facts about the cohomology of $W(n, m)$ that should come as no surprise, the case of complex Stiefel manifolds being our guide.

Proposition 12. *Let $\gamma : W(n, m) \to W(n, m)$ denote multiplication of the first column by -1. Then γ^* is identity on cohomology*

Proof. By use of the comparison maps $\mathrm{GL}(n) \to W(n, m)$, see proposition 7, we see that it suffices to prove this for $\mathrm{GL}(n)$. By means of the standard inclusion $\mathrm{GL}(n-1) \to \mathrm{GL}(n)$ and induction we see that it suffices to prove $\gamma^*(\rho_n) = \rho_n$, where ρ_n is the highest-degree generator of $H^{*,*}(\mathrm{GL}(n); R)$. By the comparison map again, we see that it suffices to prove that

$$\tau : \mathbb{A}^n \setminus \{0\} \to \mathbb{A}^n \setminus \{0\}$$

has the required property, but this is corollary 11.2 □

Proposition 13. *Let $\sigma \in \Sigma_m$, the symmetric group on m letters. Let $f_\sigma : W(n, m) \to W(n, m)$ be the map that permutes the columns of $W(n, m)$ according to σ. Then f_σ^* is the identity on cohomology.*

Proof. We can reduce immediately to the case where σ is a transposition, and from there we can assume without loss of generality that σ interchanges the first two columns. Let R be a finite-type k-algebra. We view $W(n, m)$ as the space whose R-valued points are m-tuples of elements in R^n satisfying certain conditions which we do not particularly need to know. In the case $R = k$, the condition is that the matrix is of full-rank in the usual way.

We can act on $W(n, m)$ by the elementary matrix $e_{ij}(\lambda)$

$$e_{ij}(\lambda) : (v_1, \ldots, v_m, v_{m+1}) \mapsto (v_1, \ldots, v_i + \lambda v_j, \ldots, v_{m+1})$$

The two maps $e_{i,j}(\lambda)$ and $e_{i,j}(0) = \mathrm{id}$ are homotopic, so $e_{i,j}(\lambda)$ induces the identity on cohomology. There is now a standard method to interchange two columns and change the sign of one by means of elementary operation $e_{ij}(\lambda)$, to wit $e_{12}(1)e_{21}(-1)e_{12}(1)$. We therefore know that the map

$$(v_1, v_2, v_3, \ldots, v_m) \mapsto (-v_2, v_1, v_3, \ldots, v_m)$$

induces the identity on cohomology, but now proposition 12 allows us even to undo the multiplication by -1. □

5 The Comparison Map: $\mathbb{G}_m \wedge \mathbb{P}^{n-1}_+ \to \mathrm{GL}(n)$

It will be necessary in this section to pay attention to basepoints. The group schemes $\mathbb{G}_m$ and GL_n will be pointed by their identity elements. When we deal with pointed spaces, we compute reduced motivic cohomology for preference.

We establish a map in homotopy $\mathbb{G}_m \times \mathbb{P}^{n-1} \to \mathrm{GL}(n)$, in fact we have a map from the half-smash product

$$\mathbb{G}_m \wedge \mathbb{P}^{n-1}_+ \to \mathrm{GL}(n) \tag{2}$$

and we show this latter map induces isomorphism on a range of cohomology groups.

We view $\mathbb{P}^{n-1}$ as being the space of lines in $\mathbb{A}^n$, and $\check{\mathbb{P}}^{n-1}$ the space of hyperplanes in $\mathbb{A}^n$. Define a space F^{n-1} as being the subbundle of $\mathbb{P}^{n-1} \times \check{\mathbb{P}}^{n-1}$ consisting of pairs (L, U) where $L \cap U = 0$, or equivalently, where $L + U = \mathbb{A}^n$.

More precisely, we take $\mathbb{P}^{n-1}$ and construct

$$\mathbb{P}^{n-1} \times \check{\mathbb{P}}^{n-1} = \mathcal{P}\mathrm{roj}_S(\mathcal{O}_S[y_0, \ldots, y_{n-1}])$$

where $\mathcal{P}$roj denotes the global projective-spectrum functor. Let Z denote the closed subscheme of $\mathbb{P}^{n-1} \times \check{\mathbb{P}}^{n-1}$ determined by the bihomogeneous equation $x_0y_0 + x_1y_1 + \cdots + x_{n-1}y_{n-1} = 0$. Then we define F^{n-1} as the complement $(\mathbb{P}^{n-1} \times \check{\mathbb{P}}^{n-1}) \setminus Z$.

Proposition 14. *The composite $F^{n-1} \to \mathbb{P}^{n-1} \times \check{\mathbb{P}}^{n-1} \to \mathbb{P}^{n-1}$, the second map being projection, is a Zariski-trivializable bundle with fiber $\mathbb{A}^{n-1}$. In particular, $F^{n-1} \xrightarrow{\sim} \mathbb{P}^{n-1}$*

Proof. Taking $\mathbb{A}^{n-1}$ to be a coordinate open subscheme of $\mathbb{P}^{n-1}$ determined by e.g. $x_0 \neq 0$, we obtain the following pull-back diagram

$$\begin{tikzcd}
U = \mathcal{P}\mathrm{roj}_{\mathbb{A}^{n-1}}(\mathcal{O}_{\mathbb{A}^n}[y_0, \ldots, y_{n-1}]) \setminus Z|_{\mathbb{A}^{n-1}} \arrow[r] \arrow[d] & F^{n-1} \arrow[d] \\
\mathbb{A}^{n-1} \arrow[r] & \mathbb{P}^{n-1}
\end{tikzcd}$$

The scheme U is the complement of a hyperplane in $\mathbb{P}^{n-1} \times \check{\mathbb{P}}^{n-1}$, and so takes the form

$$U \cong \mathcal{S}\mathrm{pec}_{\mathbb{A}^{n-1}}(\mathcal{O}_{\mathbb{A}^n}[t_1, \ldots, t_{n-1}]) \cong \mathbb{A}^{n-1} \times \mathbb{A}^{n-1}$$

The projection $U \to \mathbb{A}^{n-1}$ is a projection onto a factor. Since the coordinate open subschemes $\mathbb{A}^{n-1}$ form an open cover of $\mathbb{P}^{n-1}$, it follows that $F^{n-1} \simeq \mathbb{P}^{n-1}$. $\square$

In order to prove results concerning F^{n-1}, it shall be useful to have the following definition to hand.

Definition: Let R be a commutative k-algebra. By an *n-generated split line-bundle* we mean the following data. First, an isomorphism class of projective R-modules of rank 1, denoted L by abuse of notation; second, a class of surjections $[f] : R^n \to L$, where two surjections are equivalent if they differ by a multiple of $R^\times$; third, a class of splitting maps $[g] : L \to R^n$, again the maps are considered up to action of $R^\times$, and where any $f' \in [f]$ is split by some $g' \in [g]$.

If $R \to S$ is a map of k-algebras, and if (L, f, g) is an n-generated split line-bundle over R, then application of $S \otimes_R \cdot$ yields the same over S. In this way, the assignment to R of the set of n-generated split line-bundles is a functor from the category of k-algebras to the category of sets.

Proposition 15. *If R is a finite-type k-algebra, then the set of R-points $F^{n-1}(R)$ is exactly the set of n-generated split line-bundles.*

Proof. It is generally known that $\operatorname{Spec} R \to \mathbb{P}^{n-1}$ classifies isomorphism classes of rank-1 vector bundles, L, over R, equipped with an equivalence class of surjections $R^n \to L$. This is a modification of a theorem of [Gro61, Chapter 4]. It follows that $\operatorname{Spec} R \to \mathbb{P}^{n-1} \times \check{\mathbb{P}}^{n-1}$ classifies pairs of equivalence classes of rank-1 projective modules, equipped with surjective maps $(f : R^n \to L_1, g : \check{R}^n \to L_2)$ considered up to scalar multiplication by $R^\times \times R^\times$.

For convenience, let $\{e_1, \ldots, e_n\}$ be a basis of R^n and let $\{\check{e}_1, \ldots, \check{e}_n\}$ be the dual basis. Let $h : R \to R^n \otimes_R \check{R}^n$ be the map given by the element $\sum_{i=1}^n e_i \otimes \check{e}_i$ of the latter module. Given f, g, one obtains a composite map

$$\phi = (f \otimes g) \circ h : R \to L_1 \otimes_R L_2 \tag{3}$$

Let Z be the closed subscheme of $\mathbb{P}^{n-1} \times \mathbb{P}^{\check{n}-1}$ determined by the equation $x_1 y_1 + \cdots + x_n y_n = 0$. Let $\mathfrak{m}$ be a maximal ideal of R. Suppose that there is a dashed arrow making the following diagram commute:

$$\begin{tikzcd} \operatorname{Spec} R/\mathfrak{m} \arrow[r, dashed] \arrow[d] & Z \arrow[d] \\ \operatorname{Spec} R \arrow[r] & \mathbb{P}^{n-1} \times \check{\mathbb{P}}^{n-1} \end{tikzcd}$$

If we take the composite, $\operatorname{Spec}(R/\mathfrak{m}) \to \operatorname{Spec} R \to \mathbb{P}^{n-1} \times \check{\mathbb{P}}^{n-1}$, then this has the effect of reducing our represented maps modulo $\mathfrak{m}$, and the result is two surjective maps over a field $\overline{f} : (R/\mathfrak{m})^n \to R/\mathfrak{m}$ and $\overline{g} : (\check{R}/\mathfrak{m})^n \to \check{R}/\mathfrak{m}$. These can be identified with two n-tuples $[a_1; \ldots; a_{n-1}]$ and $[b_1; \ldots; b_{n-1}]$ one in $(R/\mathfrak{m})^n$, the other in its dual, taken up to multiplication by $(R/\mathfrak{m})^\times$. The map $\operatorname{Spec} R/\mathfrak{m} \to \mathbb{P}^{n-1} \times \check{\mathbb{P}}^{n-1}$ factors through Z if and only if $a_1 b_1 + \cdots + a_n b_n = 0$, but this latter equation is precisely the statement that the reduction of the map ϕ of equation (3) above, $(\overline{f} \otimes \overline{g}) \cdot \overline{h}$, is nonzero.

Since R is a finite-type k-algebra, one has a factorization $\operatorname{Spec} R \to F^{n-1} = (\mathbb{P}^{n-1} \times \check{\mathbb{P}}^{n-1}) \setminus Z \to \mathbb{P}^{n-1} \times \check{\mathbb{P}}^{n-1}$ if and only if no closed point of $\operatorname{Spec} R$ lies in the closed subset Z of $\mathbb{P}^{n-1} \times \check{\mathbb{P}}^{n-1}$, but this is equivalent to the statement that no matter which maximal ideal $\mathfrak{m} \subset R$ is chosen, $\operatorname{Spec} R/\mathfrak{m} \to$

$\mathbb{P}^{n-1} \times \check{\mathbb{P}}^{n-1}$ does not factor through Z, and therefore to the statement that the reduction of ϕ at any maximal ideal of R is never 0. It follows that a map $\operatorname{Spec} R \to \mathbb{P}^{n-1} \times \check{\mathbb{P}}^{n-1}$ is the data of two equivalence classes of line bundles, L_1, L_2, each equipped with surjective maps $f : R^n \to L_1$, $g : R^n \to L_2$, considered only up to scalar multiple, and where there is a nowhere-vanishing map of modules $\phi : R^1 \to L_1 \otimes_R L_2$.

Such a nowhere-vanishing map must be an isomorphism, $R^1 \cong L_1 \otimes_R L_2$, and we therefore have an identifying isomorphism $L_2 = \check{L}_1$. Write $(a_1, \ldots, a_n)$ for the image of the map $f : R^n \to L_1$ and $(b_1, \ldots, b_n)$ for that of $g : \check{R}^n \to \check{L}_1$, then the identifying isomorphism has been constructed specifically so that the element $a_1 b_1 + \cdots + a_n b_n \in L_1 \otimes \check{L}_1$ corresponds exactly to a generator of R^1, to wit. a unit $u \in R$. Since we are working only with equivalence-classes of presentations of L_1, L_2, we may if need be replace L_1 by $u^{-1} L_1$, and so we find that the composite

$$L_1 \xrightarrow{\check{g}} R^n \xrightarrow{f} L_1$$

is the identity, as required. □

Proposition 16. *For all $n \in \mathbb{N}$, let X_n denote the motivic space*

$$X_n = \mathbb{G}_m \wedge (F_+^{n-1}) \tag{4}$$

Then there are maps $f_n : X_n \to \mathrm{GL}(n)$, and maps $\overline{h} : X_n \to X_{n+1}$, which make the following diagram commute

$$\begin{tikzcd}
\mathbb{G}_m \wedge \mathbb{P}_+^{n-1} \arrow[d, "\mathrm{id} \wedge h_+"] & X_n \arrow[l] \arrow[r] \arrow[d, dashed, "\overline{h}"] & \mathrm{GL}(n) \arrow[d, "\phi"] \\
\mathbb{G}_m \wedge \mathbb{P}_+^{n} & X_{n+1} \arrow[l] \arrow[r] & \mathrm{GL}(n+1)
\end{tikzcd} \tag{5}$$

Where the maps $h : \mathbb{P}^{n-1} \to \mathbb{P}^n$ are the standard inclusion of $\mathbb{P}^{n-1} \to \mathbb{P}^n$ as the first $n-1$ coordinates.

Proof. First we construct a map $\overline{f}_n : \mathbb{G}_m \times X_n \to \mathrm{GL}(n)$. The first scheme represents the functor taking a finite-type k-algebra, R, to the set of elements of the form $(\lambda, (L, \phi, \psi))$, that is to say, consisting of elements $\lambda \in R^\times$ and a surjection onto a rank-1 projective bundle $\phi : R^n \to L$ along with a splitting $\psi : L \to R^n$. The second scheme, $\mathrm{GL}(n)$, represents the functor taking R to $\mathrm{GL}_n(R)$. The map f_n can be constructed therefore as a natural transformation between functors. We set up such a transformation as follows: the data ϕ, ψ amount to an isomorphism

$$\Phi : \ker \phi \oplus L \xrightarrow{\cong} R^n$$

We can define a map

$$\Phi_\lambda : R^n \cong \ker \phi \oplus L \xrightarrow{(\mathrm{id}, \lambda)} \ker \phi \oplus L \cong R^n$$

it has inverse $(\mathrm{id}, \lambda^{-1})$, so it is an automorphism, and consequently an element of $\mathrm{GL}_n(R)$. The transformation taking $(\lambda, (L, \phi, \psi))$ to Φ_λ is natural, and so, by Yoneda's lemma, is a map of schemes.

The motivic space $\mathbb{G}_m \wedge F^{n-1}_+$ is the sheaf-theoretic quotient of the inclusion of sheaves of simplicial sets $1 \times F^{n-1} \to \mathbb{G}_m \times F^{n-1}$, in particular, $\overline{f}_n$ will descend to a map $f_n : \mathbb{G}_m \wedge F^{n-1}_+$ if and only if the composite $1 \times F^{n-1} \to \mathbb{G}_m \times F^{n-1} \to \mathrm{GL}(n)$ is contraction to a point. On the level of functors, however, the first scheme represents the functor taking R to pairs $(1, (L, \phi, \psi))$, and it is immediate that $\Phi_1 = \mathrm{id}$, so that the composite is indeed the constant map at the identity of $\mathrm{GL}(n)$. We have furnished therefore the requisite map $\mathbb{G}_m \wedge F^{n-1}_+$.

As for the commutativity of the diagram, we recall that the standard inclusion $\mathbb{P}^{n-1} \to \mathbb{P}^n$ represents the natural transformation taking a surjection such as $R^N \to L$ to the trivial extension $R^n \oplus R \to R^n \to L$. We lift this idea to F^{n-1}, given a triple $(L, \phi, \psi) \in F^{n-1}(R)$, one can extend the split maps ϕ, ψ to maps $\overline{\phi}$ and $\overline{\psi}$, where $\overline{\phi} : R^{n+1} \to L$ and $\overline{\phi}$ splits $\overline{\phi}$, simply by adding a trivial summand to R^n. This furnishes a natural transformation of functors, or a map of schemes, $F^{n-1} \to F^n$, that makes the diagram

$$\begin{tikzcd} \mathbb{P}^{n-1} \arrow[r] \arrow[d, "h"] & F^{n-1} \arrow[d] \\ \mathbb{P}^n \arrow[r] & F^n \end{tikzcd}$$

commute. One may define $\overline{h} : X_n \to X_{n+1}$ in diagram (5) as the map given by application of $\mathbb{G}_m \wedge (\cdot)_+$ to the map $F^{n-1} \to F^n$ immediately constructed.

The map $\phi : \mathrm{GL}(n) \to \mathrm{GL}(n+1)$ is obtained as a natural transformation by taking $A \in \mathrm{GL}_n(R)$ and constructing $A \oplus \mathrm{id} : R^n \oplus R \to R^n \oplus R$. It is now routine to verify that diagram (5) commutes.

□

The construction $\mathbb{G}_m \wedge X$ is denoted by $\Sigma^1_t X$ and is called the Tate suspension, [Voe03]. We have

$$H^{*,*}(\mathbb{G}_m; R) \cong \frac{\mathbb{M}[\sigma]}{\sigma^2 - \{-1\}\sigma}$$

the relation being derived in loc. cit. It is easily seen that as rings, we have

$$H^{*,*}(\mathbb{G}_m \times X; R) \cong H^{*,*}(X; R) \otimes_{\mathbb{M}} \frac{\mathbb{M}[\sigma]}{(\sigma^2 - \{-1\}\sigma)}$$

and that $\tilde{H}(\Sigma^1_t X; R)$ is the split submodule (ideal) generated by σ, leading to a peculiar feature of the Tate suspension

Proposition 17. *Suppose $x, y \in \tilde{H}^{*,*}(X; R)$, and that $\sigma x, \sigma y$ are their isomorphic images in $\tilde{H}^{*,*}(\Sigma^1_t X; R)$. Then $(\sigma x)(\sigma y) = \{-1\}\sigma(xy)$.*

We now come to the main theorem of this paper

Theorem 18. *The map f_n induces an isomorphism on cohomology*

$$H^{2j-1,j}(\mathrm{GL}(n);R) \to H^{2j-1,j}(\Sigma^1_t F^{n-1}_+) = H^{2j-1,j}(\Sigma^1_t \mathbb{P}^{n-1}_+)$$

in dimensions $(2j-1,j)$ where $j \geq 1$.

Proof. We first remark that when j is large, $j > n$,

$$H^{2j-1,j}(\mathrm{GL}(n);R) = H^{2j-1,j}(\Sigma^1_t \mathbb{P}^{n-1};R) = 0$$

so the result holds trivially in this range. We restrict to the case $1 \leq j \leq n$.

The following are known:

$$H^{*,*}(\mathrm{GL}_n) = \mathbb{M}[\rho_n,\dots,\rho_1]/I \quad |\rho_i| = (2i-1,i)$$
$$\tilde{H}^{*,*}(\Sigma^1_t F^{n-1}_+) = \tilde{H}^{*,*}(\Sigma^1_t \mathbb{P}^{n-1}_+) = \bigoplus_{i=0}^{n-1} \sigma\eta^i\mathbb{M} \quad |\sigma| = (1,1), \quad |\eta| = (2,1)$$

It suffices to show that $f_n^*(\rho_i) = \sigma\eta^{i-1}$. Since $f_n^*(\rho_i) = \tilde{f}_n^*(\rho_i)$, we may prove this for the map $\tilde{f}_n : \mathbb{G}_m \times F^{n-1}$, which has the benefit of being more explicitly geometric.

We prove this by induction on n. In the case $n = 1$, the space F^{n-1} is trivial, and $X_n = \mathbb{G}_m = \mathrm{GL}(n)$. The map f_1 is the identity map, so the result holds in this case.

There is a diagram of varieties

$$\begin{tikzcd}
\mathbb{G}_m \wedge \mathbb{P}^{n-1}_+ \arrow[d, "\mathrm{id}\wedge h_+"] & X_n \arrow[l] \arrow[r] \arrow[d, "\bar{h}"] & \mathrm{GL}(n) \arrow[d, "\phi"] \\
\mathbb{G}_m \wedge \mathbb{P}^{n}_+ & X_{n+1} \arrow[l] \arrow[r] & \mathrm{GL}(n+1)
\end{tikzcd} \tag{6}$$

as previously constructed. We understand the vertical map on the left since we can rely on the theory of ordinary Chow groups, [Ful84, chapter 1], we know that the induced map $H^{2j,j}(\mathbb{P}^{n-1}) \to H^{2j,j}(\mathbb{P}^{n-2})$ is an isomorphism for $j < n-1$, and so i^* is an isomorphism $H^{2j-1,j}(\mathbb{G}_m \times F^{n-1}) \cong H^{2j-1,j}(\mathbb{G}_m \times F^{n-2})$ for $j < n$.

The maps ϕ_i^* are also isomorphisms in this range, by proposition 8 and its corollary, so the diagram implies that the result holds except possibly for $f_n^*(\rho_n)$.

The argument we use to prove $f_n^*(\rho_n) = \sigma\eta^{n-1}$ is based on the composition

$$\begin{tikzcd}
\mathbb{G}_m \times F^{n-1} \arrow[r] & \mathrm{GL}(n) \arrow[r, "\pi"'] & \mathbb{A}^n \setminus \{0\}
\arrow[from=1-2, to=1-3, phantom]
\arrow[from=1-1, to=1-3, bend left, "g", start anchor={[xshift=3ex]}]
\end{tikzcd}$$

where the map π is projection on the first column. We write g for the composition of the two maps. Since $H^{*,*}(\mathbb{A}^n \setminus \{0\}) \cong \mathbb{M}[\iota]/(\iota^2)$, with $|\iota| = (2n-1, n)$, and $\pi^*(\iota) = \rho_n$, it suffices to prove that $g^*(\iota) = \sigma\eta^{n-1}$.

For the sake of carrying out computations, it is helpful to identify motivic cohomology and higher Chow groups, e.g. identify $H^{2n-1,n}(\mathbb{A}^n \setminus \{0\})$ and $CH^n(\mathbb{A}^n \setminus \{0\}, 1)$. We can write down an explicit generator for $CH^n(\mathbb{A}^n \setminus \{0\}, 1)$, see corollary 11.1, for instance the curve γ in $\mathbb{A}^n \setminus \{0\} \times \Delta^1$ given by $t(x_1 - 1) = -1$, $x_2 = x_3 = \cdots = x_n = 0$. We can also write down an explicit generator, μ, for a class $\sigma\eta_0^{n-1} \in \mathbb{G}_m \times \mathbb{P}^{n-1}$, writing x and $a_0, \ldots, a_{n-1}$ for the coordinates on each, and t for the coordinate function on Δ^1, $\sigma\eta^{n-1}$ is explicitly represented by the product of the subvariety of $\mathbb{P}^{n-1}$ given by $a_0 = 1, a_2 = 0 \ldots, a_{n-1} = 0$, which represents η^{n-1}, with the cycle given by $t(x-1) = -1$ on $\mathbb{G}_m \times \Delta^1$.

We have therefore a closed subvariety, γ, in $\mathbb{A}^n \setminus \{0\} \times \Delta^1$, and another closed subvariety $\mu \in \mathbb{G}_m \times \mathbb{P}^{n-1}$, representing the cohomology classes we wish to relate to one another. We shall show that the pull-backs of each to $\mathbb{G}_m \times F^{n-1} \times \Delta^1$ coincide. For this, it shall again be advantageous to take the functorial point of view.

The variety $\mathbb{A}^n \setminus \{0\} \times \Delta^1$ represents, when applied to a k-algebra R, unimodular columns $(x_1, \ldots, x_n)^t \in R^n$ of height n, along with a parameter $t \in R$. The subvariety γ represents the unimodular columns and parameters for which $t(x_1 - 1) = -1$ and $x_i = 0$ for $i > 1$. Note that for such pairs, we have $t \in R^\times$ and $t \neq 1$ (since otherwise $x_1 = 0$).

The pull-back of γ to $\mathrm{GL}(n) \times \Delta^1$ represents pairs (A, t), where $A \in \mathrm{GL}_n(R)$ and $t \in R$, satisfying $(A(1, 0, \ldots, 0)^t, t) \in \gamma(R)$. We observe that, writing $e_1 = (1, 0, \ldots, 0)^t$, this implies $Ae_1 = (1 - t^{-1})e_1$. Note further that $1 - t^{-1} \neq 1$.

Pulling γ back a second time, to $\mathbb{G}_m \times F^{n-1} \times \Delta^1$, we obtain the set of triples $(\lambda, (L, \phi, \psi), t)$, where $\lambda \in R^\times$, (L, ϕ, ψ) is a split n-generated line bundle, and t is a parameter, and where the invertible linear transformation $\Phi_\lambda : \ker \phi \oplus L \to \ker \phi \oplus L$ along with the parameter t lies in the pull-back of $\gamma(R)$ to $\mathrm{GL}_n(R)$. Decomposing $e_1 = v + w$, where $v \in \ker \phi$ and $w \in L$, we see that $\Phi_L(e_1) = v + \lambda w = (1 - t^{-1})e_1 = (1 - t^{-1})(v + w)$, which by uniqueness of the decomposition, forces $\lambda = (1 - t^{-1})$, and $v = 0$. Consequently, L is the rank-1 split subbundle of R^n generated by e_1, and we have $t(\lambda - 1) = -1$.

On the other hand, the variety $\mu \subset \mathbb{G}_m \times \mathbb{P}^{n-1} \times \Delta^1$ represents triples $(\lambda, L, t) \in R^\times \times \mathbb{P}^{n-1}(R) \times R$, where L is exactly the rank-1 free module generated by e_1, and where $t(\lambda - 1) = -1$. The pull-back of μ to $\mathbb{G}_m \times F^{n-1} \times \Delta^1$ coincides with that of γ, as claimed. $\square$

We are now in a position to pay off at last the debt we owe regarding the product structure of $H^{*,*}(W(n, m); R)$.

Theorem 19. *The cohomology of $W(n, m)$ has the following presentation as a graded-commutative $\mathbb{M}_R$-algebra:*

$$H^{*,*}(W(n, m); R) = \frac{\mathbb{M}_R[\rho_n, \ldots, \rho_{n-m+1}]}{I} \qquad |\rho_i| = (2i - 1, i)$$

The ideal I is generated by relations $\rho_i^2 - \{-1\}\rho_{2i-1}$, where $\{-1\} \in \mathbb{M}_R^{1,1}$ is the image of $-1 \in k^* = \mathbb{M}_{\mathbb{Z}}$ under the map $\mathbb{M}_{\mathbb{Z}} \to \mathbb{M}_R$.

Proof. It suffices to deal with the case $R = \mathbb{Z}$. It suffices also to consider only the case $m = n$, since we can use the inclusion $H^{*,*}(W(n,m)) \subset H^{*,*}(\mathrm{GL}(n))$ to deduce it for all n, m.

We have proved everything already in proposition 5, except that in the relation $\rho_i^2 - a\rho_{2i-1}$, we were unable to show a was nontrivial. We consider the map $\mathbb{G}_m \times F_+^{n-1} \to \mathrm{GL}(n)$, which induces a map of rings on cohomology. In the induced map, we have $\rho_i \mapsto \sigma\eta^{i-1}$, and so $\rho_i^2 \mapsto -\sigma^2\eta^{2i-2} = \{-1\}\sigma\eta^{2i-2}$. Since this is nontrivial if $2i - 2 \leq n - 1$, it follows that ρ_i^2 is similarly nontrivial. □

In the case $n = m$ this result, although computed by a different method, appears in [Pus04].

We can compute the action of the reduced power operations of [Voe03] on the cohomology $H^{*,*}(W(n,m); \mathbb{Z}/p)$ by means of the comparison theorem.

Theorem 20. *Suppose the ground-field k has characteristic different from 2. Represent $H^{*,*}(W(n,m); \mathbb{Z}/2)$ as $\mathbb{M}_2[\rho_n, \ldots, \rho_{n-k+1}]/I$. The even motivic Steenrod squares act as*

$$\mathrm{Sq}^{2i}(\rho_j) = \begin{cases} \binom{j-1}{i}\rho_{j+i} & \text{if } i + j \leq n \\ 0 & \text{otherwise} \end{cases}$$

The odd squares vanish for dimensional reasons.

Theorem 21. *Let p be an odd prime and suppose the ground-field has characteristic different from p. Represent $H^{*,*}(W(n,m); \mathbb{Z}/p)$ as $\mathbb{M}_p[\rho_n, \ldots, \rho_{n-k+1}]/I$. The reduced power operations act as*

$$P^i(\rho_j) = \begin{cases} \binom{j-1}{i}\rho_{ip+j-i} & \text{if } ip + j - i \leq n \\ 0 & \text{otherwise} \end{cases}$$

The Bockstein vanishes on these classes for dimensional reasons.

Observe that in both cases, since the cohomology ring is multiplicatively generated by the ρ_j, the given calculations suffice to deduce the reduced-power operations in full on the appropriate cohomology ring.

Proof. We prove only the case of $p = 2$, the other cases being much the same.

We observe that Sq^{2j} is honest squaring on $H^{2j,j}(\mathbb{P}^n; \mathbb{Z}/2)$, on the classes $\eta^i \in H^{2i,i}(\mathbb{P}^n; \mathbb{Z}/2)$ the Bockstein vanishes, and as a consequence the expected Cartan formula obtains for calculating $\eta^{i+i'}$, it is a simple matter of induction to show that $\mathrm{Sq}^{2i}(\theta^j) = \binom{j}{i}\theta^{j+i}$.

There is an inclusion of $H^{*,*}(W(n,m); \mathbb{Z}/2) \subset H^{*,*}(\mathrm{GL}(n); \mathbb{Z}/2)$ arising from the projection map, see proposition 7. It suffices therefore to compute the action of the squares on $H^{*,*}(\mathrm{GL}(n);\mathbb{Z}/2)$. Using the previous proposition and the decomposition in equation (4), we have isomorphisms

$$H^{2n-1,n}(\mathrm{GL}(n);\mathbb{Z}/2) \cong H^{2n-1,n}(\Sigma_t^1\mathbb{P}^{n-1};\mathbb{Z}/2)$$

The reduced power operations are stable not only with respect to the simplicial suspension, but are also stable with respect to the Tate suspension. This allows a transfer of the calculation on $\mathbb{P}^n$ to the calculation on GL_n via the comparison of theorem 18

To be precise, we have

$$\begin{aligned} &f_n^*(\mathrm{Sq}^{2i}\,\rho_j) = \mathrm{Sq}^{2i}\,f_n^*(\rho_j) = \mathrm{Sq}^{2i}\,\sigma\eta^{j-1} \\ = \sigma\,\mathrm{Sq}^{2i}\,\eta^{j-1} = &\begin{cases} \binom{j-1}{i}\sigma\eta^{j+i-1} = \binom{j-1}{i}f_n^*(\rho_{j+i}) & \text{if } i+j \leq n \\ 0 \quad \text{otherwise} \end{cases} \end{aligned}$$

Since f_n^* is an isomorphism on these groups, the result follows. $\square$

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