Notes on Homotopy and $\mathbb{A}^1$ homotopy—v1.24

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Part I

Abstract Homotopy theory

Chapter 1

Model Categories

1.1 Model Categories

The following definition [Hov99] is a modification of one due to Quillen [Qui67].

Definition 1.1.1. Let $\mathbf{C}$ be a category having all (small) limits and colimits. A *model structure* on $\mathbf{C}$ consists of three subcategories of $\mathbf{C}$ called *weak equivalences*, *cofibrations* and *fibrations* and two functorial ways of factoring maps f in $\mathbf{C}$, either as $f = \alpha \circ \beta$ where α is a weak equivalence and a fibration and β is a cofibration, or as $f = \gamma \circ \delta$ where γ is a fibration and δ is a weak equivalence and a cofibration.

These data have to satisfy the following axioms:

1. If f, g are morphisms in $\mathbf{C}$ such that gf is defined, then if any two of f, g, gf are weak equivalences, so is the third. (2-out-of-3 property)

2. If f is a retract of g and g is a weak equivalence, cofibration or fibration, so is f.

3. If, in the solid commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

   the map i is a cofibration and the map p is a fibration, and at least one of these two maps is also a weak equivalence, then there exists a dotted arrow making the diagram commute.

Notation 1.1.2. A map that is a weak equivalence and a (co)fibration will be called a *trivial (co)fibration*. Weak equivalences will be written $A \xrightarrow{\sim} B$, cofibrations will be written $A \rightarrowtail B$ and fibrations will be written $A \twoheadrightarrow B$.

A category equipped with a model structure will be called a *model category*.

Notation 1.1.3. In the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

if the dotted arrow exists, we say that $A \to B$ has the *left lifting property* with respect to $X \to Y$ and that $X \to Y$ has the *right lifting property* with respect to $A \to B$.

Remark 1.1.4. The model structure axioms are dual with respect to fibrations and cofibrations, in that if $\mathbf{C}$ is a model category, then $\mathbf{C}^{\mathrm{op}}$ is a model category as well, with the opposite of the category of cofibrations functioning as fibrations and vice versa. This duality manifests frequently in the theory, in that it is often sufficient to prove something for cofibrations and then say the case of fibrations is dual.

Remark 1.1.5. The fibrations are exactly the maps having the right lifting property with respect to trivial cofibrations, and the trivial cofibrations are exactly the maps having the right lifting property with respect to cofibrations. Therefore, in a given model category, the weak equivalences and cofibrations determine the fibrations. Dually, the weak equivalences and fibrations determine the cofibrations.

Remark 1.1.6. A pushout of a (trivial) cofibration is a (trivial) cofibration, since we can detect (trivial) cofibrations by means of a left lifting property. The dual statement for fibrations and pullbacks also holds.

Example 1.1.7. Let $\mathbf{Top}$ denote the category of topological spaces and continuous maps. We say a map $f : X \to Y$ in this category is a *weak equivalence* if

$$f_* : \pi_n(X, x) \to \pi_n(Y, f(y))$$

is an isomorphism (of pointed sets, groups or abelian groups) for all $x \in X$ and all $n \in \{0, 1, \dots\}$.

Let J denote the set of all inclusions $D^n \to D^n \times I$ sending x to $(x, 0)$ for all n. We say $f : X \to Y$ is a *Serre fibration* if it has the left lifting property with respect to all maps in J.

There are two ways of defining *Serre cofibrations*. In the first place, we can define a Serre cofibration as a map $A \to B$ having the left lifting property with respect to all maps having the right lifting property with respect to the inclusions $\partial D^n \to D^n$ (yes, really).

Alternatively, we can define a *relative cell complex* to be a map $f : B \to A$ such that there is a sequence $B = B_0 \subseteq B_1 \subseteq B_2 \subseteq \dots$ where B_{i+1} is obtained from B_i by attaching cells and such that $A = \operatorname{colim} B_i$. A map is a Serre cofibration if it is a retract of a relative cell complex. It is not proved here that these two definitions are equivalent, you can look at [Hov99, Chapter 2]

This gives us a model structure on $\mathbf{Top}$, called the *Quillen* or *classical* model structure.

The main purpose of a model structure is to allow one to form and work with an associated homotopy category.

Definition 1.1.8. Let $\mathbf{C}$ be a model category. Then the *homotopy category*, $\mathrm{Ho}\,\mathbf{C}$, of $\mathbf{C}$ is the category with the same objects as $\mathbf{C}$ and with morphisms obtained by adjoining (formal) inverses to weak equivalences. The definition is made more fully and precisely as [Hov99, Definition 1.2.1].

Remark 1.1.9. The first problem with $\mathrm{Ho}\,\mathbf{C}$ is that we do not know that $\mathrm{Ho}\,\mathbf{C}(X, Y)$ is a set. The second, and related problem, is that it is hard to calculate $\mathrm{Ho}\,\mathbf{C}(X, Y)$. The role of the fibrations and cofibrations in the model structure is to solve these two problems.

Definition 1.1.10. Write $\emptyset$ for the colimit of the empty diagram and pt for the limit.

An object X of a model category $\mathbf{C}$ is *cofibrant* if the unique map $\emptyset \to X$ is a cofibration. It is *fibrant* if $X \to \mathrm{pt}$ is a fibration.

Example 1.1.11. In the ordinary model structure on **Top**, the retracts of CW complexes are precisely the cofibrant objects. All objects are fibrant.

Example 1.1.12. We can factor any map $\emptyset \to X$ as $\emptyset \rightarrowtail QX \overset{\sim}{\twoheadrightarrow} X$. Then QX is cofibrant and weakly equivalent to X. Any cofibrant object weakly equivalent to X is called a *cofibrant replacement*. The dual concept is of *fibrant replacement* $X \overset{\sim}{\rightarrowtail} RX \twoheadrightarrow \mathrm{pt}$. Observe that the fibrant replacement of a cofibrant object is cofibrant and vice versa. This means we can write down cofibrant–fibrant replacements.

In the standard model structure, we have no need for fibrant replacements, but we know cofibrant replacements as "CW approximations".

Proposition 1.1.13 (Ken Brown's Lemma)**.** *Suppose* $\mathbf{C}$ *is a model category and* $\mathbf{D}$ *is a category with weak equivalences (satisfying 2-out-of-3). Suppose* $F : \mathbf{C} \to \mathbf{D}$ *is a functor taking trivial cofibrations between cofibrant objects to weak equivalences, then* F *takes all weak equivalences between cofibrant objects to weak equivalences.*

The dual statement for fibrations and fibrant objects also holds.

In fact, as [Hov99] uses this lemma, it is necessary only for $\mathbf{C}$ to satisfy the axioms of a model category and to contain finite coproducts (or finite products in the fibrant case).

Proof. This proof is slightly tricky.

Suppose $f : X \to Y$ is a weak equivalence of cofibrant objects. Factor $X \coprod Y \to Y$ into a cofibration $q : X \coprod Y \rightarrowtail Z$ followed by a trivial fibration $p : Z \overset{\sim}{\twoheadrightarrow} Y$. Each of the two inclusion maps $i_1 : X \to X \coprod Y$ and $i_2 : Y \to X \coprod Y$. is a cofibration. Each of the two composite maps $X, Y \to Z$ is a weak equivalence (2-out-of-3) and a cofibration. Both $F(q \circ i_1)$ and $F(q \circ i_2)$ are weak equivalences, as is $F(p \circ q \circ i_2) = F(\mathrm{id}_Y)$, so that $F(p)$ is a weak equivalence, and so too is $F(f) = F(p \circ q \circ i_1)$. □

Notation 1.1.14. Let $\mathbf{C}_{cf}$ denote the full subcategory of $\mathbf{C}$ consisting of cofibrant–fibrant objects. This may not be a model category, since it may not be complete or cocomplete. Nonetheless, we can talk about $\mathrm{Ho}\,\mathbf{C}_{cf}$, the category obtained from $\mathbf{C}_{cf}$ by formally inverting weak equivalences, and Ken Brown's lemma applies to $\mathbf{C}_{cf}$. It is not difficult to see ([Hov99, Prop 1.2.3]) that the obvious functor $\mathrm{Ho}\,\mathbf{C}_{cf} \to \mathrm{Ho}\,\mathbf{C}$ is an equivalence of categories.

1.1.1 Homotopy

Definition 1.1.15. Let X be an object in a model category. A *cylinder object* on X is a factorization of the fold map $X \coprod X \to X$ as

$$X \coprod X \rightarrowtail \mathrm{Cyl}\, X \overset{\sim}{\to} X.$$

Note that we do not require this to be the specific functorial factorization of the model structure, although it certainly provides a cylinder object. A *left homotopy* between two morphism $f, g : X \to Y$ is a map $H : \mathrm{Cyl}\, X \to Y$ that yields f, g when composed with the two inclusions $X \to X \coprod X \to \mathrm{Cyl}\, X$.

Example 1.1.16. In the standard model structure on spaces, the inclusion $X \coprod X \to X \times I$, including the copies of X at either end, is a cofibration (it's a relative cell complex). Therefore the definition of $\operatorname{Cyl} X$ is a cylinder object in this model structure, and a left homotopy is just what we would ordinarily call a "homotopy".

Remark 1.1.17. Let X be an object in a model category. A *path object*, PX, on X is a factorization of the diagonal $X \to X \times X$ as

$$X \simeq PX \twoheadrightarrow X \times X.$$

A *right homotopy* between $f : Y \to X$ and $g : Y \to X$ is a map $H : Y \to PX$ that composes to give f or g in the analogous way. For instance, if X is a topological space, we can form PX as X^I in the compact-open topology, with evaluation at 0 and 1 being the maps to X. Using the adjunction between maps $Y \to X^I$ and $Y \times I \to X$, we see that here again we have recovered the usual notion of homotopy, but in a slightly harder-to-visualize way.

Proposition 1.1.18. *If X is cofibrant and Y is fibrant then the relations of left- and right-homotopy between maps $X \to Y$ agree. Moreover, this relation is an equivalence relation.*

Proof of a selected part of this statement. Among the more technical parts of this statement is the assertion that left-homotopy is a transitive relation on maps $X \to Y$ when X is cofibrant. Let's do this, as an example of the sort of argument that model categories require. Suppose we have three maps f_1, f_2 and $f_3 : X \to Y$, and left homotopies $H_1 : X' \to Y$ and $H_2 : X'' \to Y$ where X' and X'' are cylinders for X.

Form Z as the pushout

$$\begin{tikzcd} X \arrow[r, "i_1"] \arrow[d, "i_0'"] & X' \arrow[d] \\ X'' \arrow[r] & Z \end{tikzcd}$$

and we get a factorization of the fold map $X \coprod X \overset{j_0+j_1}{\to} Z \overset{t}{\to} X$. We remark that if X is cofibrant, then $X \to X'$ (or $X \to X''$) is a cofibration, being the pushout of a cofibration. It is also a weak equivalence, due to the 2-out-of-3 property. Therefore, since X is cofibrant, the map $X'' \to Z$ is a trivial cofibration.

Unfortunately, the map $j_0 + j_1$ may not be a cofibration, although t is necessarily a weak equivalence, since $X'' \to Z$ is a trivial cofibration and $Z \to X$ fits in a factorization of the identity $X \overset{\sim}{\to} X'' \overset{\sim}{\rightarrowtail} Z \to X$ so that 2-out-of-3 implies $Z \to X$ is a weak equivalence. Now take $X \coprod X \to Z$ and factor it as a cofibration followed by a trivial fibration: $X \coprod X \rightarrowtail Z' \overset{\sim}{\twoheadrightarrow} Z$. The object Z' is a cylinder object for X and it supports a left homotopy from f_1 to f_3. □

Proposition 1.1.19. *There is a well-defined functor $\mathbf{C}_{cf} \to \mathbf{C}_{cf}/\sim$ sending objects to objects and sending a map $f : X \to Y$ to the homotopy class of f.*

The proof is not given here. See [Hov99, Section 1.2].

Proposition 1.1.20. *Let Y be a cofibrant–fibrant object and let $\mathbf{C}/\sim(Y,\cdot) : \mathbf{C}_{cf} \to \mathbf{Set}$ denote the functor taking an object X to the set of homotopy classes of maps $\mathbf{C}(X,Y)/\sim$. Then $\mathbf{C}/\sim(Y,\cdot)$ sends weak equivalences to bijections.*

Proof. By Ken Brown's lemma (using isomorphisms of sets as weak equivalences in that category), it's sufficient to prove that it sends trivial fibrations to isomorphisms. That is, if $X \overset{\sim}{\twoheadrightarrow} Z$ is a trivial fibration, we want to show that $\mathbf{C}(Y,X)/\sim \to \mathbf{C}(Y,Z)/\sim$ is a bijection. Since $X \overset{\sim}{\twoheadrightarrow} Z$ is a trivial fibration, we can lift any map $Y \to Z$ to a map $Y \to Z$. This establishes surjectivity.

To show injectivity, suppose we have two maps $f, g : Y \to X$ that become (left) homotopic when we compose with $X \to Z$. Choose a left homotopy $H : Y' \to Z$ between them and consider

$$\begin{tikzcd}
Y \coprod Y \arrow[r, "f+g"] \arrow[d] & X \arrow[d] \\
Y' \arrow[r] & Z
\end{tikzcd}$$

the obvious lift gives us a homotopy between f and g. □

Proposition 1.1.21. *A map $f : X \to Y$ between cofibrant–fibrant objects is a homotopy equivalence if and only if it is a weak equivalence.*

Proof. The previous result says that if $f : X \to Y$ is a weak equivalence, then the induced map $\mathbf{C}(Y,Y)/\sim \to \mathbf{C}(Y,X)/\sim$ is a bijection. The class of the identity map then maps to a homotopy class of maps $g : Y \to X$, any one of which is a homotopy inverse for f—admittedly we are skipping many details here.

The converse statement that a homotopy equivalence is a weak equivalence is surprisingly intricate. First observe that a map that is left homotopy equivalent to a weak equivalence is a weak equivalence. This follows from the diagram

$$\begin{tikzcd}
A \arrow[dr, "\sim"] \arrow[drr, bend left] & & \\
& \mathrm{Cyl}\, A \arrow[r] & B \\
A \arrow[ur, "\sim"] \arrow[urr, bend right, "\sim"'] & &
\end{tikzcd}$$

and the 2-out-of-3 property.

Now suppose $f : X \to Y$ is a homotopy equivalence between cofibrant–fibrant objects. We can factor this as $g : X \overset{\sim}{\rightarrowtail} Z$ followed by $p : Z \twoheadrightarrow Y$. It suffices to show that p is a weak equivalence. Note that Z is cofibrant–fibrant and that g is therefore known to be a homotopy equivalence. Let f' and g' be homotopy inverses for f and g, and let $H : Y' \to Y$ be a left homotopy from ff' to id_Y. Define H' as a lift

$$\begin{tikzcd}
Y \arrow[r, "gf'"] \arrow[d, "i_0"'] & Z \arrow[d, "p"] \\
Y' \arrow[r] \arrow[ur, dotted, "H'"] & Y
\end{tikzcd}$$

and let $q = H'i_1 : Y \to Z$. The map q has been produced specifically so that $pq = \mathrm{id}_Y$, and so that H' is a left homotopy from gf' to q. Then $qp \sim gf'p \sim gf'pgg' = gf'fg' \sim id_Z$.

In particular, we have reduced the problem to showing that p is a weak equivalence when $p : Z \to Y$ is the retraction map of a deformation retract. This is relatively easy:

$$\begin{tikzcd} Z \arrow[r, equal] \arrow[d, "p"] & Z \arrow[r, equal] \arrow[d, "qp"] & Z \arrow[d, "p"] \\ Y \arrow[r, "q"] & Z \arrow[r, "p"] & Y \end{tikzcd}$$

and the middle map is a weak equivalence, so the outer map is as well (weak equivalences being closed under retracts). ☐

Corollary 1.1.22 (Whitehead's theorem)**.** *If $X \to Y$ is a weak equivalence of CW complexes, then it is a homotopy equivalence.*

Now we come to the point of the whole discussion:

Corollary 1.1.23. *The equivalent categories $\mathrm{Ho}\,\mathbf{C}_{cf} \equiv \mathrm{Ho}\,\mathbf{C}$ are equivalent to the category $\mathbf{C}_{cf}/\sim$.*

This means we have a method of calculating $\mathrm{Ho}\,\mathbf{C}(X, Y)$: namely, replace X and Y by weakly equivalent cofibrant-fibrant objects X' and Y', then calculate $\mathbf{C}(X', Y')/\sim$.

In fact, you can do a bit better

Corollary 1.1.24. *Let $X' \to X$ be a cofibrant replacement of X in $\mathbf{C}$ and $Y \to Y'$ a fibrant replacement. Then $\mathrm{Ho}\,\mathbf{C}(X, Y) = \mathbf{C}(X', Y')/\sim$.*

This follows from [Hov99, Proposition 1.2.5] and the previous corollary.

Example 1.1.25. There is another model structure on **Top**, the *Strøm model structure*, established in [Str72], in which the role of the weak equivalences is played by the homotopy equivalences of spaces, the fibrations are the maps satisfying the right lifting property with respect to the inclusions $X \to X \times I$ for all spaces X and the closed topological cofibrations.

Example 1.1.26. This example is [Hov99, Section 2.3]. Let R be a ring, and work in the category of left R-modules. We assume you know what a complex of R-modules is (with homological grading), and what a map of complexes looks like. A map $f : A_\bullet \to B_\bullet$ is a *quasi-isomorphism* if $f_* : H_i(A_\bullet) \to H_i(B_\bullet)$ is an isomorphism for all i.

For any integer n, let S^n denote the complex that has R in the nth position and 0 elsewhere. Let D^{n+1} denote the complex that has R in the n-th and the $n+1$st position, and where the nontrivial differential is an identity. There is an inclusion map of complexes $i_n : S^n \to D^{n+1}$.

Call a map of complexes a *fibration* if it has the right lifting property with respect to all maps $0 \to D^n$, and a *cofibration* if it has the left lifting property with respect to all maps having the right lifting property with respect to the inclusions i_n.

This produces a model structure on the category of chain complexes of R-modules. Some claims, all of which are proved in [Hov99]:

1. The fibrations are precisely the levelwise surjective maps.

2. A map is a trivial fibration if and only if it has the right lifting property w.r.t. the i_n.

3. A bounded-below chain complex of projective modules is cofibrant. Any cofibrant object is levelwise projective, but there exist unbounded, levelwise projective and non-cofibrant objects.

1.2 Quillen adjunctions

According to the experts, the "right" notion of a morphism between model categories is the following.

Definition 1.2.1. Suppose $\mathbf{C}$ and $\mathbf{D}$ are model categories (i.e., categories and model structures). Then a functor $F : \mathbf{C} \to \mathbf{D}$ is a *left Quillen functor* if it has a right adjoint U and F preserves cofibrations and trivial cofibrations. A functor $U : \mathbf{D} \to \mathbf{C}$ is a *right Quillen functor* if it is a right adjoint and preserves fibrations and trivial fibrations.

Proposition 1.2.2. *Let $F \dashv U$ be an adjoint pair in which F is left Quillen or U is right Quillen. Then F is left Quillen and U is right Quillen.*

Remark 1.2.3. Ken Brown's lemma implies that a left Quillen functor preserves weak equivalences between cofibrant objects, and a right Quillen functor preserves weak equivalences between fibrant objects.

This means that we can form derived functors of Quillen functors.

Construction 1.2.4. Suppose $F : \mathbf{C} \to \mathbf{D}$ is a left Quillen functor. The *total left derived functor LF* of F is the functor

$$\operatorname{Ho}\mathbf{C} \to \operatorname{Ho}\mathbf{D}$$

that on objects $X \in \mathbf{C}$ is $F(QX)$—Q being the cofibrant replacement functor. The *total right derived functor RU* of U is defined dually.

Remark 1.2.5. We haven't quite justified the assertion that this functor is defined, and we're not going to do so explicitly. You can do it yourself.

Proposition 1.2.6. *If $F \dashv U$ is a Quillen adjunction, with $F : \mathbf{C} \to \mathbf{D}$, then the pair of functors $QF : \operatorname{Ho}\mathbf{C} \rightleftarrows \operatorname{Ho}\mathbf{D} : RU$ is an adjoint pair of functors.*

Proof. We wish to establish a natural isomorphism of sets

$$\operatorname{Ho}\mathbf{D}(FQX, Y) \xrightarrow{\cong} \operatorname{Ho}\mathbf{C}(X, URY).$$

It is enough to establish a natural isomorphism

$$\mathbf{D}(FQX, RY)/\sim \xrightarrow{\cong} \mathbf{C}(QXURY)/\sim$$

which looks a lot like the adjunction we already had between F and U, applied to the objects QX and RY. The only question is whether this adjunction isomorphism (for $\mathbf{C}$, $\mathbf{D}$) preserves the relation "is homotopic to". Call the adjunction isomorphism ϕ.

Suppose $f, g : FX \to Y$ are two maps in $\mathbf{D}$, where X is cofibrant and Y fibrant. Let's show that if ϕf is homotopic to ϕg, then f is homotopic to g. The other argument is dual (using right instead of left homotopy). So suppose X' is a cylinder object for X and H a left homotopy $H : X' \to UY$ between ϕf and ϕg. Then FX' is a cylinder object for FX, since F preserves coproducts, cofibrations and trivial cofibrations, and $\phi^{-1}H$ is a left homotopy from f to g. $\square$

Definition 1.2.7. A Quillen adjunction $F \dashv U$ is a *Quillen equivalence* if the following holds: For all cofibrant $X \in \mathbf{C}$ and all fibrant $Y \in \mathbf{D}$, the map $f : FX \to Y$ is a weak equivalence if and only if the adjoint map $X \to UY$ is a weak equivalence.

Remark 1.2.8. The derived functors of a Quillen equivalence are equivalences of categories.

Chapter 2

Simplicial Sets

2.1 Definition

Let Δ denote the *simplicial category* where the objects are

$$[n] = \{0, 1, \dots, n\}$$

for $n \geq 0$ and the maps $\Delta([n], [k])$ are the set of weakly order-preserving maps of sets.

Lemma 2.1.1. *Every map f in Δ can be factored uniquely as a surjection followed by an injection.*

Definition 2.1.2. Let $d^i : [n-1] \to [n]$ denote the injective map $[n-1] \to [n]$ skipping $i \in \{0, 1, \dots, n\}$. This is called the i-th *(standard) coface map*. Let $s^i : [n] \to [n-1]$ be the surjective map identifying i and $i+1$, the i-th *(standard) codegeneracy map*.

Lemma 2.1.3. *Every injective map in Δ can be factored as a composite of coface maps, and every surjective map can be factored as a composite of codegeneracy maps. Consequently, every map in Δ is a composite of coface and codegeneracy maps.*

Lemma 2.1.4. *The following relations all hold*

$$\begin{aligned}
d^j d^i &= d^i d^{j-1} \quad i < j \\
s^j d^i &= d^i s^{j-1} \quad i < j \\
s^j d^i &= \mathrm{id} \quad i = j,\ i = j+1 \\
s^j d^i &= d^{i-1} s^j \quad i > j+1 \\
s^j s^i &= s^{i-1} s^j \quad i > j
\end{aligned}$$

and suffice to generate all relations in the simplicial category.

Definition 2.1.5. Let $\mathbf{C}$ be a category. A *simplicial object* in $\mathbf{C}$ is a functor $X_\bullet : \Delta^{\mathrm{op}} \to \mathbf{C}$. The notation X_n is used for $X_\bullet([n])$. The images $X_\bullet(d^i) : X_{n+1} \to X_n$ are written d_i and are called *face maps*. Similarly, $X_\bullet(s^i) = s_i$ are called *degeneracy maps*. These are subject to the *simplicial identities* which are the duals of the relations of Lemma 2.1.4.

A map of simplicial objects is a natural transformation of functors, or equivalently, a sequence of levelwise maps $X_n \to Y_n$ compatible with the simplicial maps. The category of simplicial objects in $\mathbf{C}$ will be written $\mathbf{sC}$.

Lemma 2.1.6. *If $\mathbf{C}$ is a category with limits and colimits, then so is $\mathbf{s}C$; limits and colimits being formed objectwise.*

Example 2.1.7. The category of *simplicial sets*, $\mathbf{sSet}$ is particularly important. If $X_\bullet$ is a simplicial set, then we can distinguish two kinds of element in X_n: those that are in the image of some degeneracy map the *degenerate elements*, and those that are not.

Let us define some standard objects in this category. In the first place, there is $\Delta[n]$ which sends $[k]$ to $\Delta([k],[n])$. This is a the (simplicial) *n-simplex*. There is also $\partial\Delta[n]$, which is the subobject of $\Delta[n]$ consisting of all those maps $[k] \to [n]$ that are not surjective.

For each r, there is $\Lambda^r[n]$ that can be constructed as follows: Let $\mathcal{D}$ be the category where the objects are non-identity injective maps $[k] \to [n]$ where the image contains r. Each map yields a map $\Delta[k] \to \Delta[n]$ of simplicial sets, then $\Lambda^r[n]$ is the colimit over $\mathcal{D}$ of the $\Delta[k]$.

When we draw a picture of a simplicial set, we usually draw the non-degenerate simplices only. There are several reasons why degenerate simplices are included in the structure. For instance, $\Delta[1] \times \Delta[1]$ has two nondegenerate 2-simplices that arise from the degenerate simplices of $\Delta[1]$.

Definition 2.1.8. A *cosimplicial object* in $\mathbf{C}$ is a functor $X^\bullet : \Delta \to \mathbf{C}$. A cosimplicial object is a sequence of objects in $\mathbf{C}$ equipped with coface and codegeneracy maps satisfying the relations of Lemma 2.1.4.

Example 2.1.9. There is a standard *cosimplicial topological space*. This is given on objects by

$$\Delta^n_t = \left\{ \mathbf{x} \in \mathbb{R}^{n+1} \;:\; \sum_{i=0}^{n} x_i = 1, \quad x_i \geq 0, \quad \forall i \right\}.$$

The i-th coface map $\Delta^{n-1}_t \to \Delta^n_t$ is given by including $(x_1,\ldots,x_n) \to (x_1,\ldots,0,\ldots,x_n)$, inserting a 0 in the i-th position, and the i-th codegeneracy map is given by the map $\Delta^n_t \to \Delta^{n-1}_t$ sending $\mathbf{x} \mapsto (x_0, x_1, \ldots, x_i + x_{i+1}, \ldots, x_n)$.

Construction 2.1.10. Let $\mathbf{C}$ be a category with all colimits and let $A^\bullet$ be a cosimplicial object in $\mathbf{C}$. We can construct a functor

$$|\cdot|_A : \mathbf{sSet} \to \mathbf{C} \quad \text{(realization)}$$

in the following way:

Suppose $X_\bullet$ is a simplicial set. Let ΔX denote the category where the objects are maps $\Delta[n] \to X$ (i.e., elements of X_n) and a map from $a \in X_n$ to $b \in X_m$ is a commuting triangle

$$\begin{tikzcd} \Delta[n] \arrow[rr] \arrow[dr] & & \Delta[m] \arrow[dl] \\ & X & \end{tikzcd}$$

This is called the *category of simplices* in X. (Added later: watch out for the variance in this diagram. You could easily reverse the arrows here, i.e., consider the opposite category of the category of simplices).

Tautologically, $\operatorname{colim}_{(\Delta[n]\to X)\in\Delta X}\Delta[n] = X$. The A-realization functor is then

$$|X| = \operatorname*{colim}_{(\Delta[n]\to X)\in\Delta X} A^n.$$

In the case where $\mathbf{C} = \mathbf{Top}$ (or similar) and $A = \Delta_t^\bullet$, this construction is called the *geometric realization* of the simplicial set $X_\bullet$, and is written $|X_\bullet|$.

Remark 2.1.11. In order to produce $|X_\bullet|$, it is sufficient to consider the nondegenerate simplices of $X_\bullet$.

Construction 2.1.12. Let $\mathbf{C}$ be a category with all colimits and let $A^\bullet$ be a cosimplicial object in $\mathbf{sC}$. We can construct a functor $\operatorname{Sing}^A : \mathbf{C} \to \mathbf{sSet}$ by setting $\operatorname{Sing}^A(Y)_n = \mathbf{C}(A^n, X)$. The cosimplicial structure maps in $A^\bullet$ then immediately yield the simplicial structure maps in $\operatorname{Sing}^A(Y)_\bullet$.

When $A = \Delta_t^\bullet$, this is a familiar construction: $\operatorname{Sing}^A(Y)_n = \mathbf{Top}(\Delta_t^n, Y)$ is the collection of singular n-simplices in Y. This is the basis for the free abelian group $C_n^{\text{sing}}(Y)$ used to calculate singular homology.

Proposition 2.1.13. *Let $\mathbf{C}$ be a category having all colimits. Then the two constructions above form an adjoint pair of functors $|\cdot|_A \dashv \operatorname{Sing}^A$.*

Proof.

$$\mathbf{C}(\operatorname*{colim}_{\Delta X} A^n, Y) = \lim_{\Delta X}\mathbf{C}(A^n, Y) = \lim_{\Delta X}\operatorname{Sing}^A(Y)_n = \lim_{\Delta X}\mathbf{sSet}(\Delta[n], \operatorname{Sing}^A(Y)_\bullet) = \mathbf{sSet}(X_\bullet, \operatorname{Sing}^A(Y)_\bullet)$$

$\square$

Definition 2.1.14. Let $\mathbf{K}$ denote the category of compactly generated spaces, also known as "Kelly spaces". The category $\mathbf{K}$ is a subcategory of $\mathbf{Top}$ and the inclusion $\mathbf{K} \to \mathbf{Top}$ is left-adjoint to a "Kellification", functor so that $\mathbf{K}$ is closed under all colimits in $\mathbf{Top}$. It is notably not closed under ordinary products of spaces, instead there is a product in $\mathbf{K}$ given by replacing $X \times Y$ by its Kellification—this has the same underlying set, but possibly different closed subsets.

Lemma 2.1.15. *The geometric realization functor $|\cdot| : \mathbf{sSet} \to \mathbf{K}$ preserves finite products.*

Outline of proof. That is, we assert $|X_1 \times X_2| \approx |X_1| \times |X_2|$. Observe that there is a map in the forward direction here, and it suffices to prove it is a homeomorphism.

It is a feature of both the category of sets and of $\mathbf{K}$ that there is a functor $\cdot \times A$ that is a left adjoint—see the Appendix to Gaunce Lewis' thesis for a proof of this for $\mathbf{K}$. In particular, finite products commute with all colimits in both categories.

Moreover, $|\cdot|$ is a left adjoint, and commutes with all colimits. We consequently have a reduction

$$|X_1 \times X_2| = |\operatorname*{colim}_{\Delta X_1}\Delta[n] \times X_2| = \operatorname*{colim}_{\Delta X_1}|\Delta[n] \times X_2| = \operatorname*{colim}_{\Delta X_1, \Delta X_2}|\Delta[n] \times \Delta[m]|$$

and

$$|X_1| \times |X_2| = \operatorname*{colim}_{\Delta X_1, \Delta X_2}|\Delta[n]| \times |\Delta[m]|$$

so it suffices to prove that the natural map

$$\nu : |\Delta[n] \times \Delta[m]| \to |\Delta[n]| \times |\Delta[m]|$$

is a homeomorphism. The target here is clearly compact.

One sees that the nondegenerate simplices of $\Delta[n]\times\Delta[m]$ correspond to totally ordered subsets of the partially-ordered set $[n] \times [m]$. There are only finitely many of these, so $|\Delta[n] \times \Delta[m]|$ is compact. Therefore it suffices to show ν is bijective.

It then suffices to show that every point in $|\Delta[n]| \times |\Delta[m]|$ arises from realizing one of the maximal (i.e., $n+m$-dimensional) nondegenerate simplices in $\Delta[n]\times\Delta[m]$. This is done in the last part of [Hov99, Lemma 3.1.8]. □

2.2 The model structure on simplicial sets

2.2.1 Summary

Definition 2.2.1. A map $f : X \to Y$ is a *weak equivalence of simplicial sets* if $|f|$ is a weak equivalence of topological spaces. Since $|X|$ and $|Y|$ are CW complexes, this is the same as being a homotopy equivalence of spaces.

Example 2.2.2. Note that the three spaces $|\Lambda^2_i|$ are all homeomorphic as spaces, but the Λ^2_i are pairwise non-isomorphic as simplicial sets.

Definition 2.2.3. A map $f : X \to Y$ is a *cofibration of simplicial sets* if it is levelwise injective.

Definition 2.2.4. A map $f : X \to Y$ is a *fibration of simplicial sets* or a *Kan fibration* if it has the following right lifting property

$$\begin{tikzcd} \Lambda^i[n] \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \Delta[n] \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd} \tag{2.1}$$

for all n and all $i \in \{0, \dots, n\}$.

Notation 2.2.5. A simplicial set K such that $K \to \Delta[0]$ is a fibration (i.e., a fibrant object in this structure) is called a *Kan complex*.

Remark 2.2.6. A simplicial set $X_\bullet$ such that $X_\bullet \to \mathrm{pt}$ satisfies the lifting property in (2.1) for all n and all $i \in \{1, \dots, n-1\}$ is called a *quasicategory*. We may return to this definition later.

Theorem 2.2.7. *The weak equivalences, cofibrations and fibrations defined above form a model structure.*

2.2.2 Cofibrant generation

We won't prove this in full detail, since we've allotted no more than two lectures to it. Here are the main ideas. The full proof can be assembled from [Hov99, Section 2.1] and [Hov99, Section 3.2-3.6].

Notation 2.2.8. Let I be a collection of maps in a category $\mathbf{C}$ having all colimits. The notation $I-\mathrm{inj}$ denotes the collection of maps that have the r.l.p. with respect to the maps in I. Let $I-\mathrm{cof}$ denote the set of maps having the l.l.p. w.r.t. $I-\mathrm{inj}$. Let $I-\mathrm{cell}$ (the relative I cell complexes) denote the smallest collection of maps that is

1. closed under direct colimits and
2. closed under coproducts (this actually follows from the other two axioms)
3. contains all pushouts of maps in I.

Lemma 2.2.9. *Any retract of a map in* $I - \mathrm{cof}$ *is in* $I - \mathrm{cof}$.

Lemma 2.2.10. *Any map in* $I - \mathrm{cell}$ *is in* $I - \mathrm{cof}$.

Definition 2.2.11. Let I be a collection of morphisms in a category $\mathbf{C}$. An object $X \in \mathbf{C}$ is *small* (relative to I) if, for all direct systems $Y_1 \to Y_2 \to \dots$ of maps in I, the map

$$\operatorname*{colim}_i \mathbf{C}(X, Y_i) \to \mathbf{C}(X, \operatorname*{colim}_i Y_i)$$

is a bijection.

Remark 2.2.12. The small objects in the category of sets are the finite sets. The small objects in **sSet** are the simplicial sets having finitely many nondegenerate simplices.

Remark 2.2.13. There is a generalization of smallness to κ-smallness, where the direct systems are indexed over other ordinals than ω.

Theorem 2.2.14 (The small object argument, finite version)**.** *Let* $\mathbf{C}$ *be a category having all colimits, and that* I *is a set of maps. Suppose the domains of the maps in* I *are small relative to* $I - \mathrm{cell}$. *There is a functorial factorization of all maps* f *in* $\mathbf{C}$ *into* $\delta(f) \circ \gamma(f)$ *where* $\delta(f)$ *is in* $I - \mathrm{inj}$ *and* $\gamma(f)$ *is in* $I - \mathrm{cell}$ *(and in particular, in* $I - \mathrm{cof}$*).*

There is also a version of this for more general notions of smallness.

General idea of proof. Let $f : X \to Y$ be a map in $\mathbf{C}$. We want to produce a factorization $X \to X' \to Y$ where $X \to X'$ is in $I - \mathrm{cell}$ and $X' \to Y$ has the r.l.p. w.r.t. I. To what extent does $X \to Y$ fail to have that lifting property already? Suppose there is a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] & Y \end{tikzcd}$$

where $A \to B$ is in I. There may not be a lift along $X \to Y$, but we can replace X by X', the pushout of $B \leftarrow A \to X$. Then there is a factorization $X \to X' \to Y$ where the first map is I-cellular and the second map is closer to being in $I - \mathrm{inj}$ because at least in the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X' \arrow[d] \\ B \arrow[r] & Y \end{tikzcd}$$

where A is the composite $A \to X \to X'$, there is a lift.

The "small object argument" is an argument that says some (infinitely repeated) application of this idea does actually lead to a functorial factorization. $\square$

Lemma 2.2.15 (The Retract Argument). *Suppose $f = pi$ is a factorization of a map in a category where f has the l.l.p. w.r.t. p. Then f is a retract of i.*

Proof. Write $i : A \to B$ and $p : B \to C$ and consider the lift in

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[d, "p"] \\ C \arrow[r, equal] \arrow[ur, dotted, "r"] & C \end{tikzcd}$$

Then the diagram

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "f"] & A \arrow[r, equal] \arrow[d, "i"] & A \arrow[d, "f"] \\ C \arrow[r, "r"'] & B \arrow[r, "p"'] & C \end{tikzcd}$$

does the job. □

Lemma 2.2.16. *Assume I is small relative to $I-\mathrm{cell}$. Any map in $I-\mathrm{cof}$ is a retract of a map in $I-\mathrm{cell}$.*

Proof of lemma. Let $f : X \to Y$ be a map in $I-\mathrm{cof}$. Using the small object argument, we can factor it as a composite $X \to X' \to Y$ where the first map is in $I-\mathrm{cell}$ (and so in $I-\mathrm{cof}$) and the second is in $I-\mathrm{inj}$. Then use the retract argument. □

Theorem 2.2.17 (Cofibrantly generated model structures). *Let $\mathbf{C}$ be a category containing all limits and colimits. Let $\mathbf{W}$ be a subcategory of weak equivalences closed under retracts and satisfying the two-out-of-three property. Let I and J be two sets of map in $\mathbf{C}$. Suppose the domains of the maps in I and J are each small relative to $I-\mathrm{cell}$, $J-\mathrm{cell}$ respectively, and further that*

1. $J-\mathrm{cell} \subseteq \mathbf{W} \cap I-\mathrm{cof}$
2. $I-\mathrm{inj} = \mathbf{W} \cap J-\mathrm{inj}$

Then there is a model structure on $\mathbf{C}$ where $I-\mathrm{cof}$ are the cofibrations, $J-\mathrm{inj}$ are the fibrations and $I-\mathrm{inj}$ are the trivial fibrations.

Remark 2.2.18. A model category admitting this sort of description is called *cofibrantly generated*.

Partial proof. For notational convenience, set $I-\mathrm{cof}$ to be the cofibrations and $J-\mathrm{inj}$ to be the fibrations. It is also easy to show that these are closed under composition. The hypotheses ensure that $I-\mathrm{inj}$ is exactly the trivial fibrations.

It is easy to verify that cofibrations and fibrations are closed under retracts—this is just a diagram chase in each case.

The small-object argument can be applied to give functorial factorizations of any map f into a map in $J-\mathrm{cell}$ followed by one in $J-\mathrm{inj}$, and similarly for I. The first of these is the functorial $\overset{\sim}{\rightarrowtail} \cdot \twoheadrightarrow$ factorization. The second is the $\rightarrowtail \cdot \overset{\sim}{\twoheadrightarrow}$ factorization.

Finally we turn to the lifting axioms.

A cofibration has the l.l.p. w.r.t. $I-\mathrm{inj}$, the trivial fibrations, by hypothesis. The case of a trivial cofibration is a little worse. A trivial cofibration $f : A \overset{\sim}{\rightarrowtail} B$ can be factored into a map $h : A \overset{\sim}{\rightarrowtail} A'$

in $J-\mathrm{cell}$, followed by a trivial fibration $g : A' \overset{\sim}{\twoheadrightarrow} B$. Since f has the lifting property w.r.t. the trivial fibration g, the retract lemma tells us that f is a retract of h, a map in $J-\mathrm{cell}$. It follows that f is in $J-\mathrm{cof}$, so has the lifting property against fibrations. It even follows that $J-\mathrm{cof}$ consists of precisely the trivial cofibrations. □

2.2.3 Cofibrant generation of the structure on simplicial sets

Now we have to outline why the structure on simplicial sets fits into this structure.

Notation 2.2.19. Until further notice, I will denote the set of all canonical inclusions $\partial\Delta[n] \hookrightarrow \Delta[n]$, and J the set of all the canonical inclusions $\Lambda^r[n] \to \Delta[n]$.

Proposition 2.2.20. *The following are equivalent:*

1. $I-\mathrm{cof}$.
2. *Relative I-cell complexes*
3. *Injective maps of simplicial sets*

Proof. It is easy to verify that relative I-cell complexes are injective, and it is easy also to verify that retracts of injective maps are injective. Since every element of $I-\mathrm{cof}$ is a retract of a relative cell complex, this implies that all the maps we are considering here are injective.

To finish the argument suffices to show that any injective map is actually a relative cell complex. This is routine enough, and is done in detail in [Hov99, 3.2.2] □

We now try to apply Theorem 2.2.17.

Remark 2.2.21. For convenience, let us recapitulate the hypotheses of that theorem:

1. **C** contains all limits and colimits—done.
2. **W** is closed under retracts and satisfies 2-out-of-3—obvious.
3. Smallness—satisfied (the domains have only finitely many nondegenerate simplices in each case).
4. $J-\mathrm{cell} \subseteq \mathbf{W} \cap I-\mathrm{cof}$.
5. $I-\mathrm{inj} = \mathbf{W} \cap J-\mathrm{inj}$.

Proposition 2.2.22. *Axiom 4 is satisfied.*

In fact, more is true. The maps in $J-\mathrm{cof}$ are called *anodyne extensions*, and they are all trivial cofibrations.

Proof. Since $J \subseteq I-\mathrm{cof}$, it is formal that $J-\mathrm{cof} \subseteq I-\mathrm{cof}-\mathrm{cof} = I-\mathrm{cof}$.

It remains to show that anodyne extensions are weak equivalences. It is formal from the $|\cdot| \dashv \mathrm{Sing}$ adjunction that the realization of an anodyne extension is in $|J|-\mathrm{cof}$, where $|J|$ denotes the geometric realization of the maps in J. But the maps in $|J|-\mathrm{inj}$ are exactly the Serre fibrations, and so $|J|-\mathrm{cof}$ are the trivial Serre cofibrations, in particular, they are weak equivalences. □

So only Axiom 5 remains. This would take a long time to prove, so we will not do it.

Proposition 2.2.23 (Hovey 3.2.6)**.** *If $f \in I - \text{inj}$, then f is a trivial fibration.*

You can look in [Hov99] for a proof of this. Proving f is a fibration does not require anything that we haven't done already, since we can produce the maps J as I-cell complexes, against which f has the r.l.p. Proving that f is a homotopy equivalence is not difficult either, but it does involve the long exact sequence of a fibration (in classical homotopy theory), which we will discuss later.

Proposition 2.2.24. *If f is a trivial fibration, then f is in $I - \text{inj}$.*

Heavy combinatorics (i.e., the theory of *minimal fibrations* [Hov99, Definition 3.5.5] or [GJ99, Section I.10]) is used to reduce this to a statement about homotopy groups of simplicial sets. We will therefore devote some time to homotopy groups later, and you will have to trust me that the argument can be made in a not-circular way.

In the process of proving this, one proves most of the following theorem

Theorem 2.2.25. *The realization and singular functors*

$$|\cdot| : \mathbf{sSet} \rightleftarrows \mathbf{K} : \mathrm{Sing}$$

form a Quillen equivalence.

Proof. To see this is a Quillen adjunction, we concentrate on the right adjoint, Sing. Suppose $f : X \to Y$ is a Serre fibration, and there is a diagram

$$\begin{tikzcd} |\Lambda^i[n]| \arrow[r] \arrow[d] & X \arrow[d] \\ |\Delta[n]| \arrow[r] & Y \end{tikzcd} .$$

Since the left hand vertical map is a cellular inclusion and a weak equivalence, there is a lift in this diagram, and so, after applying the adjunction, we get a lift in

$$\begin{tikzcd} \Lambda^i[n] \arrow[r] \arrow[d] & \mathrm{Sing}\, X \arrow[d] \\ \Delta[n] \arrow[r] \arrow[ur, dotted] & \mathrm{Sing}\, Y \end{tikzcd} .$$

The same argument also applies to $\partial\Delta[n] \to \Delta[n]$ when f is a trivial Serre fibration. This handles the "Quillen adjunction" part of the theorem.

For the equivalence, let K be a simplicial set and X be a k-space, we have to show that $|K| \to X$ is a weak equivalence if and only if $K \to \mathrm{Sing}\, X$ is a weak equivalence. This is means showing that $|K| \to X$ is a weak equivalence if and only if $|K| \to |\mathrm{Sing}\, X|$ is a weak equivalence, which in turn is equivalent to showing that the natural map $|\mathrm{Sing}\, X| \to X$ is a weak equivalence. This is true, but we will postpone explaining why until after we describe homotopy groups. □

Remark 2.2.26*.* As part of this Quillen equivalence, there is the statement that if f is a cofibration, then $|f|$ is a cofibration. In fact, if f is a cofibration, then by 2.2.20, it is a relative cell complex (in what we called $I - \text{cell}$) and we can see directly that $|f|$ is a relative cell complex of topological spaces.

In particular, $\mathrm{Ho}\,\mathbf{sSet}$ is an equivalent category to $\mathrm{Ho}\,\mathbf{K}$.

Two further facts about realizations of simplicial sets that we will not prove are given.

Theorem 2.2.27. *The realization functor $|\cdot|$ preserves all finite limits, i.e., limits of finite diagrams.*

Sketch of proof. This appears as [Hov99, Lemma 3.2.4]. The idea is that all finite limits can be constructed by iterating finite products (for which we already know this result) and equalizers.

Definition 2.2.28. Let $f, g : A \to B$ be two maps in a category. The *equalizer* of f, g is the limit of the diagram $A \rightrightarrows B$.

To show that $|\cdot|$ preserves equalizers, argue as follows. Let $f, g : A \to B$ be two maps of simplicial sets, let K be the simplicial set equalizer. This is a subobject of A. Let Z be the topological equalizer of $|f|, |g|$. That is, this is the subspace of $|A|$ on which the two cellular maps $|f|$ and $|g|$ agree. The space Z is a closed CW subspace of $|A|$, and the functorial inclusion $|K| \to |A|$ factors through Z. It suffices to verify that the inclusion $|K| \to Z$ is surjective, and this can be done on a simplex-by-simplex basis. $\square$

Theorem 2.2.29 (Quillen)**.** *If $f : X \to Y$ is a Kan fibration of simplicial sets, then $|f|$ is a Serre fibration.*

This appears as [GJ99, Theorem 10.10].

Chapter 3

Homotopy theory of simplicial sets and spaces

3.1 Pointed model categories

Notation 3.1.1. Every model category has an initial object $\emptyset$ and a terminal object pt. A model category is said to be *pointed* if $\emptyset \to \mathrm{pt}$ is an isomorphism. In particular, this implies that every object X is equipped with a unique map $x_0 : \mathrm{pt} \to X$ called the *basepoint* of X.

Construction 3.1.2. If $\mathbf{C}$ is a model category, then we can form the *associated pointed model category* $\mathbf{C}_+$ where the objects are pairs (X, x_0), with $X \in \mathbf{C}$ and $x_0 : \mathrm{pt} \to X$ being a morphism (i.e., a choice of basepoint). Morphisms in $\mathbf{C}_+$ are required to send basepoints to basepoints. Weak equivalences (resp. cofibrations, fibrations) are the maps that are weak equivalences (resp. cofibrations, fibrations) after forgetting the basepoint.

There is a functor $\mathbf{C} \to \mathbf{C}_+$ given by sending the object X to $X \coprod \mathrm{pt}$, pointed at the disjoint basepoint, and a forgetful functor $\mathbf{C}_+ \to \mathbf{C}$, forgetting basepoints. These functors form a Quillen adjunction.

Remark 3.1.3. Frequently, when working in a pointed category, we will write X but mean (X, x_0).

Remark 3.1.4. The Quillen equivalence of $\mathbf{sSet}$ and $\mathbf{K}$ extends to an equivalence of pointed categories.

3.2 Cartesian structure

Definition 3.2.1. Let X and Y be simplicial sets. Let $\mathrm{Map}(X, Y)_\bullet$ denote the simplicial set having $\mathbf{sSet}(X \times \Delta^n, Y)$ as its n-th level. Since $\Delta^\bullet$ form a cosimplicial set, this makes sense. The construction $\mathrm{Map}(X, Y)$ is functorial in both variables (contravariantly in the first).

Proposition 3.2.2. *For a fixed simplicial set Y, the functors $\cdot \times Y \dashv \mathrm{Map}(Y, \cdot)$ form an adjoint pair.*

Proof. Exercise. $\square$

The following theorem sets a pattern for many similar theorems in homotopy theory, and is very useful.

Theorem 3.2.3 (Mapping theorem)**.** *Suppose $i : K \to L$ is a cofibration (injective map) and $p : X \to Y$ is a fibration of simplicial sets, then the induced map*

$$\mathrm{Map}(L, X) \to \mathrm{Map}(K, X) \times_{\mathrm{Map}(K,Y)} \mathrm{Map}(L, Y)$$

is a fibration. It is a trivial fibration if either i or p is also a weak equivalence.

Construction 3.2.4. Note that $\mathrm{Map}(X, \mathrm{pt}) = \mathrm{pt}$. There is a pointed version of Map, denoted Map_+ given by a pullback

$$\begin{tikzcd} \mathrm{Map}_+(X, Y) \arrow[r] \arrow[d] & \mathrm{Map}(X, Y) \arrow[d] \\ \mathrm{Map}(\mathrm{pt}, \mathrm{pt}) \arrow[r] & \mathrm{Map}(\mathrm{pt}, Y) \end{tikzcd} .$$

The morphism $\mathrm{pt} = \mathrm{Map}_+(X, \mathrm{pt}) \to \mathrm{Map}_+(X, Y)$ gives $\mathrm{Map}_+(X, Y)$ the structure of a pointed simplicial set. It is contravariantly functorial in the first variable and covariantly in the second.

The functor $\mathrm{Map}_+(X, \cdot)$ from $\mathbf{sSet}_+$ to itself is right adjoint to a functor $\cdot \wedge X$, the *smash product*. You can construct the pointed space $X \wedge Y$ as the pushout

$$\begin{tikzcd} X \times y_0 \coprod x_0 \times Y \arrow[r] \arrow[d] & \mathrm{pt} \arrow[d] \\ X \times Y \arrow[r] & X \wedge Y \end{tikzcd} .$$

Remark 3.2.5. The analogue of Theorem 3.2.3 holds in the pointed case.

Remark 3.2.6. An entirely analogous story can be told about the usual model structure on $\mathbf{K}$ (i.e., the restriction of the structure on $\mathbf{Top}$). Here we have an internal mapping object $\mathrm{Map}(X, Y) = \mathcal{C}(X, Y)$, a pointed analogue, $\mathrm{Map}_+(X, Y)$ and the *smash product* $X \wedge Y$. The mapping theorem, 3.2.3, also applies in this case.

Corollary 3.2.7 (Corollary of Theorem 3.2.3)**.** *Work in either the category of simplicial sets or the usual model structure on* $\mathbf{K}$*. Let L be a cofibrant object. Then the adjoint functors $\cdot \times L$ and $\mathrm{Map}(L, \cdot)$ form a Quillen adjunction.*

Similarly, in the pointed cases, $\cdot \wedge L$ and $\mathrm{Map}_+(L, \cdot)$ form a Quillen adjunction.

Proof. Apply the theorem with $i : \emptyset \to L$ and $p : X \to Y$ a (trivial) fibration. This suffices to show that $\mathrm{Map}(L, \cdot)$ preserves (trivial) fibrations, and so is a right Quillen functor. The pointed case is an exercise. $\square$

In particular, the adjunction descends to an adjunction on homotopy categories.

Proposition 3.2.8. *Suppose $i : L \to K$ is a weak equivalence of simplicial sets, then the natural transformation $\mathrm{Map}(K, \cdot) \to \mathrm{Map}(L, \cdot)$ is a natural weak equivalence.*

This relies on Ken Brown's lemma and the fact that all simplicial sets are cofibrant. There is a dual statement for the other variable in the mapping space, but it applies only when the objects are Kan complexes.

Example 3.2.9. Let S^n denote the usual n-sphere in $\mathbf{K}$ with a basepoint x_0. For a pointed topological space X, define $\Sigma^n X = X \wedge S^n$ and $\Omega^n X$ as $\mathrm{Map}_+(S^n, X)$. Similarly, for a pointed simplicial sets X, define $S^n = \Delta[n]/\partial\Delta[n]$ and then $\Sigma^n X = X \wedge S^n$ and $\Omega^n X = \mathrm{Map}_+(S^n, X)$. These are called the *$n$-fold (reduced) suspension* and *n-fold loop space* of X, respectively.

There are adjunctions in the pointed homotopy category $[\Sigma^n X, Y]_+ = [X, \Omega^n Y]_+$ in each case. In both cases, there is a weak equivalence $S^n \wedge S^1 \to S^{n+1}$ and similarly in the simplicial set case (in the topological case, this is actually a homeomorphism). This implies that $\Sigma^i \Sigma^j X \simeq \Sigma^{i+j} X$ and $\Omega^i \Omega^j X \simeq \Omega^{i+j} X$.

Remark 3.2.10. If A and B are simplicial sets, we know that $|A \times B| \approx |A| \times |B|$ and similarly $|A \wedge B| \approx |A| \wedge |B|$. Clearly $|\partial\Delta[n+1]| \equiv S^n$. This is sufficient to prove that $|\Sigma^n A| \approx \Sigma^n |A|$.

A homework assignment asks you to show that $|\mathrm{Map}(A, B)|$ is related by a chain of weak equivalences (in either direction) to $\mathrm{Map}(|A|, |B|)$ provided B is a Kan complex. A variation on this argument shows that $|\Omega^n Z| \simeq \Omega^n |Z|$ provided Z is Kan.

3.3 Simplicial model categories

This section is just general building up of vocabulary. It contains no real theorem.

Definition 3.3.1. A *simplicial category* is a category $\mathbf{C}$ equipped with three extra pieces of structure:

1. A *simplicial enrichment*: between any two objects X, Y there is a simplicial set $\mathcal{S}(X, Y)$ of maps, such that $\mathcal{S}(X, Y)_0 = \mathbf{C}(X, Y)$ and satisfying the usual associative composition axioms.
2. A *simplicial tensor* structure: for any object X and any simplicial set K there is an object $X \otimes K$ in $\mathbf{C}$, functorial in both variables, and so that there is a natural isomorphism: $X \otimes (K \times K') \cong (X \otimes K) \otimes K'$.
3. A *simplicial cotensor* structure: for any object X and any simplicial set K, there is a mapping object X^K in $\mathbf{C}$.

These structures must further be related by natural isomorphisms

$$\mathbf{sSet}(K, \mathcal{S}(X, Y)) \cong \mathbf{C}(X \otimes K, Y) \cong \mathbf{C}(X, Y^K).$$

Remark 3.3.2. In the above circumstances, $\mathcal{S}(X, Y)_n = \mathbf{C}(X \otimes \Delta[n], Y)$, so that $\mathcal{S}(X, Y)$ is determined by the rest of the structure.

Note also that $\cdot \otimes K$ and $X \otimes \cdot$ are both left adjoints, so both preserve colimits and in particular preserve initial objects.

Definition 3.3.3. A *simplicial model category* $\mathbf{C}$ is a category equipped with a simplicial structure and a model structure and so that given a cofibration $f : U \to V$ and cofibration $g : W \to X$ in $\mathbf{sSet}$, the induced map

$$f \Box g : (V \otimes W) \coprod_{U \otimes W} (U \otimes X) \to V \otimes X$$

is a cofibration which is trivial if either f or g is.

Remark 3.3.4. This definition has consequences for the adjoints of $\otimes$. For instance, it follows that for a fixed cofibrant X, the functors $X \otimes \cdot$ and $\mathcal{S}(X, \cdot)$ form a Quillen adjunction.

Example 3.3.5. The model structure on $\mathbf{sSet}$ is simplicial. Both the cotensor and the simplicial mapping object are given by Map.

A more interesting example is given by $\mathbf{K}$. This is made into a simplicial model structure by defining

$$\begin{aligned} X \otimes A &:= X \times |A| \\ \mathcal{S}(X, Y)_n &:= \mathbf{K}(X[n], Y) \\ X^A &:= \mathrm{Map}(|A|, X). \end{aligned}$$

The pointed versions of these simplical model structures also have simplicial structures. For instance, when X is a pointed space, $X \otimes A$ is $X \wedge A_+$. The mapping space $\mathcal{S}(X, Y)$ is a simplicial set already (forgetting the basepoint) and X^A is given by $\mathrm{Map}_+(A_+, X)$ (which is just $\mathrm{Map}(A, X)$, forgetting the basepoint.

Remark 3.3.6. There are model categories that are not simplicial (or at least, carry no obvious simplicial structure), but we will not encounter these in this course.

3.4 Homotopy groups

Notation 3.4.1. We work throughout in the category of pointed simplicial sets, but we will generally not mention the basepoints. It will be enough to know they are there. The basepoint of a simplicial set K will be denoted k. We will say a map $f : X \to Y$ is "constant" if it factors through $y \to Y$.

The notation $\{0\}$ and $\{1\}$ will be used for the two vertices of $\Delta[1]$.

Definition 3.4.2. Let K be a simplicial set, and let $k_a, k_b \in K_0$. We say that $k_a \sim k_b$ if there is a 1-simplex $j \in K_1$ such that $d_0 j = k_a$ and $d_1 j = k_b$. If K is a Kan complex, then $\sim$ is an equivalence relation on K_0, and we define $\pi_0(K) = K_0/\sim$. This is a set. If $k_a \in K_0$ is a particular element of K_0, then $\pi_0(K_0, k_a)$ is the set $\pi_0(K_0)$ where the class of k_a is a distinguished element.

Remark 3.4.3. π_0 is a functor.

Remark 3.4.4. There is a map of spaces $K_0 \to |K|$, and this map induces a bijection $\pi_0(K) \to \pi_0(|K|)$.

Definition 3.4.5. If $\mathbf{C}$ is a pointed category with limits and colimits, and $f : X \to Y$ is a map, then the *fibre* of f is the pullback

$$\begin{tikzcd} F \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{pt} \arrow[r] & Y \end{tikzcd}$$

and the *cofibre* is the pushout

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ \mathrm{pt} \arrow[r] & C \end{tikzcd}$$

Remark 3.4.6. The cofibre doesn't require a pointed category for its definition, but it is a pointed object. For a given map f, changing the basepoint in Y yields different fibres.

Definition 3.4.7. Suppose K is a pointed Kan complex, with basepoint k. Let $n \geq 1$ be an integer. Give $\Delta[n]$ a basepoint at 0, and apply this to subobjects of $\Delta[n]$. There is a fibration of fibrant objects $\mathrm{Map}_+(\Delta[n], K) \to \mathrm{Map}_+(\partial\Delta[n], K)$. The basepoint of $\mathrm{Map}_+(\partial\Delta[n], K)$ is the map sending all of $\partial\Delta[n]$ to k. Let F denote the fibre of $\mathrm{Map}_+(\Delta[n], K) \to \mathrm{Map}_+(\partial\Delta[n], K)$, and define $\pi_n(K, k) = \pi_0(F)$.

Remark 3.4.8. I think there is a gap in the argument proving [Hov99, Proposition 3.6.3], in that he has not established the group structure on the higher homotopy groups, but does seem to use it. So as not to spend our lives worrying about group structures, let us refer to [GJ99, Theorem 7.2] for a proof that $\pi_n(K, k)$ is a group when $n \geq 1$.

Proposition 3.4.9. *This defines a functor from pointed Kan complexes to pointed sets, and to groups if $n \geq 1$.*

Remark 3.4.10. There are many different ways of defining $\pi_n(K, k)$, all of them ultimately equivalent. What we are really defining is $[\Delta[n]/\partial\Delta[n], K]$ in the pointed homotopy category. The definition just given is the easiest to work with before you know you've set up a model structure on $\mathbf{sSet}$.

We defined the homotopy groups using a specific homotopy relation on maps, essentially a left homotopy in the unpointed category. To get a different construction, we can produce a different cylinder object for $\Delta[n]/\partial\Delta[n]$ in $\mathbf{sSet}_+$. This is what is behind the following technical lemma [Hov99, Lemma 3.4.5], which is proved in the source without assuming there is a model structure.

Lemma 3.4.11. *Let K be a Kan complex, with basepoint k. Let $s : \Delta[n] \to K$ be a map that is constant on the boundary. Then the class of s is trivial if and only if there is a map $x : \Delta[n+1] \to K$ such that $x \circ d^{n+1} = s$ and $x \circ d^i$ is constant for all other i.*

Proof. We do the "only if" direction here. You can look up the "if" direction yourself. Suppose there is a homotopy $\tilde{s} : \Delta[n] \times \Delta[1] \to K$ that is constant on $\partial\Delta[n] \times \Delta[1]$ and on $\Delta[n] \times \{1\}$.

We define $G : \partial\Delta[n+1] \times \Delta[1] \to K$ to be constant when restricted to $\Lambda^n[n] \times \Delta[1]$, and to be $\tilde{s}$ on the other face of the prism. Then we have a commutative diagram

$$\begin{tikzcd}
\partial\Delta[n+1]\times\Delta[1] \coprod_{\partial\Delta[n+1]\times\{1\}} \Delta[n+1]\times\{1\} \arrow[r, "G \coprod k"] \arrow[d] & K \arrow[d] \\
\Delta[n+1]\times\Delta[1] \arrow[r] & \mathrm{pt}
\end{tikzcd}$$

Since K is Kan, this admits a lift $F : \Delta[n+1] \times \Delta[1]$. Then the $n+1$-simplex $F|_{\Delta[n+1]\times\{0\}} : \Delta[n+1] \times \{0\} \to K$ is the required $n+1$-simplex. $\square$

Definition 3.4.12. Let $\mathbf{C}$ be a pointed model category. A *fibre sequence* in $\mathbf{C}$ is a pair of maps $F \xrightarrow{i} E \xrightarrow{p} B$ where p is a fibration of fibrant objects and F is the pull back in

$$\begin{tikzcd}
F \arrow[r, "i"] \arrow[d] & B \arrow[d, "p"] \\
\mathrm{pt} \arrow[r] & B.
\end{tikzcd}$$

Construction 3.4.13. Suppose $F \xrightarrow{i} E \xrightarrow{p} B$ is a fibre sequence, and $\sigma \in B$ is an element of $\pi_n(B, b_a)$. Then σ is represented by a map $s : \Delta[n] \to B$ sending $\partial\Delta[n]$ to b_a. The following diagram is commutative

$$\begin{tikzcd}
\Lambda^n[n] \arrow[r, "e_a"] \arrow[d] & E \arrow[d, "p"] \\
\Delta[n] \arrow[r, "s"] & B
\end{tikzcd}$$

and therefore we may produce a lift $\bar{s} : \Delta[n] \to E$. The boundary of $\Delta[n]$ is mapped to F, and most of it (all of $\Lambda^n[n]$) is mapped to e_a. The exception is the composite $\Delta[n-1] \xrightarrow{d^n} \Delta[n] \xrightarrow{\bar{s}} E$, which is a nontrivial map from $\Delta[n-1]$ to F, sending $\partial\Delta[n-1]$ to e_a, the basepoint of F. Therefore it allows us to define a class ∂s in $\pi_{n-1}(F, e_a)$.

The class of ∂s depends only on σ. There is a proof of this in [Hov99, Lemma 3.4.8] that is honest, in that it doesn't rely on having shown that **sSet** has a model structure. Let us give a dishonest argument.

Suppose we have two representative maps $s, s' : \Delta[n] \to B$, both representative of σ. There is a homotopy between them, which is to say, a map $H : \Delta[n] \times \Delta[1] \to B$ restricting to s and s' at either end. Let $\bar{s}, \bar{s}' : \Delta[n] \to E$ be lifts of s and s' as maps $\Lambda^n[n] \to B$. Then consider the commutative diagram

$$\begin{tikzcd}
\Lambda^n[n] \times \Delta[1] \coprod_{\Lambda^n[n] \times \partial\Delta[1]} \Delta[n] \times \partial\Delta[1] \arrow[r, "f"] \arrow[d] & E \arrow[d] \\
\Delta[n] \times \Delta[1] \arrow[r, "H"] & B
\end{tikzcd}$$

The map f sends $\Lambda^n[n] \times \Delta[1]$ to the basepoint, and restricts to $\bar{s}, \bar{s}'$ on $\Delta[n] \times \partial\Delta[1]$. The left vertical map is a trivial cofibration, so there is a lift, $\tilde{H} : \Delta[n] \times \Delta[1] \to E$, having image entirely in the fibre F, which yields the required homotopy between ∂s and $\partial s'$.

This argument is dishonest, because we used the model structure to produce $\tilde{H}$. In [Hov99, Lemma 3.4.8] (relying on a lot of prior work) this map is produced from scratch.

Remark 3.4.14. Again, to avoid getting entirely bogged down in technicalities, let us refer to [GJ99, Lemma 7.3] to tell us that the map ∂ is a group homomorphism when $n \geq 2$. In the case of $n = 1$, there is an action of the group $\pi_1(B, b)$ on $\pi_0(F, e)$ such that the map $\pi_0(F) \to \pi_0(E, e)$ identifies elements if and only if they lie in the same orbit.

Definition 3.4.15. Suppose $A \xrightarrow{f} B \xrightarrow{g} C$ are maps of pointed sets. Let $c_0 \in C$ be the basepoint of C. The sequence $g \circ f$ is *exact* if $g^{-1}(c_0) = \mathrm{im}(f)$.

Remark 3.4.16. A notable subcategory of the category of pointed sets is the category of groups, where the identity elements are the basepoints. A sequence $G \to G' \to G''$ of groups is exact in the sense of sets if and only if it is exact in the usual sense.

Proposition 3.4.17. *Suppose $F \xrightarrow{i} E \xrightarrow{p} B$ is a fibre sequence of pointed simplicial sets. Then there is a natural long exact sequence (of groups and pointed sets)*

$$\begin{aligned}
\pi_n(F, e) \xrightarrow{i_*} \pi_n(E, e) \xrightarrow{p_*} \pi_n(B, b) \xrightarrow{\partial} \pi_{n-1}(F, e) \to \dots \\
\to \pi_1(B, b) \to \pi_0(F, e) \to \pi_0(E, e) \to \pi_0(B, b)
\end{aligned}$$

Proof. There are a number of parts to this, and they are mostly uninteresting. There are three pairs of maps to consider:

1. $p_* \circ i_*$. This composite is constant, essentially by construction. To show that p_*^{-1} is exactly the image of i_*, consider a map $s : \Delta[n] \to E$ such that $i \circ s$ is homotopic to the trivial element. There is a lifting argument

$$\begin{tikzcd} \partial\Delta[n] \times \Delta[1] \coprod_{\partial\Delta[n]\times 1} \Delta[n] \times 1 \arrow[r, "e"] \arrow[d] & E \arrow[d] \\ \Delta[n] \times \Delta[1] \arrow[r] \arrow[ur, dotted, "\tilde{H}"] & B \end{tikzcd}$$

and at the endpoint 0, the lifted map $\tilde{H} : \Delta[n] \times 0 \to E$ gives us a representative for the class of s having image in F.

2. $\partial \circ p_*$. An analysis of the construction of $\partial(\sigma)$ shows that if σ is in the image of p_* then $\partial\sigma$ is trivial. To see that this is an if-and-only-if is a little more difficult. The argument here is shamelessly taken from [Hov99, Lemma 3.4.9]. Suppose $\partial(\sigma)$ is trivial. Choose a map $s : \Delta^n \to B$ representing the class of σ. There is a lift in

$$\begin{tikzcd} \Lambda^n[n] \arrow[r] \arrow[d] & E \arrow[d] \\ \Delta^n \arrow[r, "s"] \arrow[ur, dotted, "\gamma"] & B \end{tikzcd}$$

and there's a homotopy $H : \Delta[n-1] \times \Delta[1] \to F$ from $d_n\gamma$ to the constant map at e. We can use that homotopy to define the commutative diagram

$$\begin{tikzcd} \partial\Delta[n] \times \Delta[1] \coprod_{\partial\Delta[n]\times\{0\}} \Delta[n] \times \{0\} \arrow[r, "f"] \arrow[d] & E \arrow[d] \\ \Delta[n] \times \Delta[1] \arrow[r, "s \circ \mathrm{proj}_1"] & B. \end{tikzcd}$$

Define f to be γ on $\Delta[n] \times \{0\}$, to be $i \circ H$ on the n-th face of $\partial\Delta[n] \times \Delta[n]$ and constant on the rest of $\partial\Delta[n] \times \Delta[1]$. There is a lift in this diagram, and at $\Delta[n] \times \{1\}$ (the far end) gives a class τ in $\pi_n(E, e)$ such that $p_*(\tau) = \sigma$.

3. $i_* \circ \partial$. Showing this is null is exactly the "if" direction of Lemma 3.4.11. To show exactness, suppose $i^*(s)$ is trivial. Then by means of the "only-if" direction of Lemma 3.4.11, we can extend $s : \Delta[n] \to F$ to a map $t : \Delta[n+1] \to E$, that is constant on all but one face, where it agrees with s. Then compose to get a map $p \circ t : \Delta[n+1] \to B$, which represents a homotopy class. By construction, $\partial(p \circ t)$ is the class of s.

□

Remark 3.4.18. Similar arguments show that in the model category of spaces, a fibre sequence $F \xrightarrow{i} E \xrightarrow{p} B$ gives rise to a long exact sequence of homotopy groups:

$$\pi_{n+1}(B, b) \to \pi_n(F, e) \to \pi_n(E, e) \to \pi_n(B, b) \to \pi_{n-1}(F, e) \to \dots.$$

In this case, we also know that $\pi_n(X, x)$ is a group when $n \geq 1$ and an abelian group when $n \geq 2$. The morphisms appearing in the long exact sequence are group homomorphisms.

Corollary 3.4.19. *Let K be a pointed Kan complex. Then the map given by realization $\pi_n(K, k) \to \pi_n(|K|, |k|)$ is an isomorphism of groups.*

Hovey is careful enough to do this without yet having established a model structure. We will not be so careful. In fact, we will assume quite a number of results we haven't proved here. You can consult [Hov99, Proposition 3.6.3] if you want to see this done correctly—but beware you also will need the group structure on the homotopy groups from [GJ99, I.§7]

Dishonest proof. We can factor the inclusion of the basepoint $k \to K$ into a trivial cofibration followed by a fibration $PK \overset{\sim}{\twoheadrightarrow} K$. The object PK is fibrant and weakly equivalent to a point. It is not difficult to prove directly that the simplicial homotopy groups of a fibrant object weakly equivalent to a point are trivial. Let WK denote the fibre of $PK \to K$.

The realization of a fibration is a fibration, so $|PK| \to |K|$ is also a fibration, and $|PK|$ is contractible. Write $\Omega|K|$ for the fibre of this fibration. There are two long exact sequences and commuting maps between them, yielding a commuting square:

$$\begin{tikzcd}
\pi_1(K,k) \arrow[r, "\cong"] \arrow[d] & \pi_0(WK, pk) \arrow[d, "\cong"] \\
\pi_1(|K|, k) \arrow[r, "\cong"] & \pi_0(\Omega|K|)
\end{tikzcd}$$

(here the horizontal maps are bijections, due to the group action). The right vertical map is an isomorphism for elementary reasons. Therefore the left vertical map is also an isomorphism. The result now follows by iterating the construction, $\pi_2(K, k) = \pi_0(WWK)$. □

Example 3.4.20. The long exact sequence of a fibration is a very useful long exact sequence to have around. We haven't devoted a lot of effort to establishing fibrations yet, but here is an example.

Suppose $\tilde{X} \to X$ is a universal covering space. Covering space theory tells us that $\pi_n(X, x) \cong \pi_n(\tilde{X}, x)$ whenever $n \geq 2$. This means that we know $\pi_n(S^1, s)$ for all n. Since the universal cover is contractible, the groups are trivial unless $n = 1$, in which case we get an infinite cyclic group, as is well known.

There is a nontrivial fibration, the Hopf fibration, given by taking $\mathbb{C}^2 \setminus \{0\}$, and then taking a quotient by the $\mathbb{C}^\times$ action, to get $\mathbb{C}P^1 \simeq S^2$. In fact, we could restrict attention to the unit sphere in $\mathbb{C}^2$, and take a quotient by $S^1 \subseteq \mathbb{C}^\times$. In this case we get a fibration $S^3 \to S^2$ with fibre S^1: the *Hopf fibration*.

The long exact sequence of a fibration then tells us that $\pi_n(S^3) \xrightarrow{\cong} \pi_n(S^2)$ when $n \geq 3$, and in low dimensions there is an exact sequence

$$0 \to \pi_2(S^3) \to \pi_2(S^2) \to \pi_1(S^1) \to \pi_1(S^2) \to 0$$

It is well known that $\pi_i(S^n) = 0$ if $i < n$, and $\pi_n(S^n)$ is infinite cyclic provided $n \geq 1$. In particular, we see that $\pi_3(S^2) \cong \mathbb{Z}$.

Part II

Local homotopy theory

Chapter 4

Sites and Sheaves

4.1 Sheaves on spaces

One of the most important purposes of a topology on a space is to define a sheaf on that space. Unfortunately, this is often obscured in the disciplines of topology itself.

Definition 4.1.1. Let X be a topological space, and let $\mathbf{X}_{\text{ord}}$ denote the category having as objects the open subsets of X with the inclusion maps.

A *presheaf (of sets) on* X is a functor $\mathbf{X}_{\text{ord}}^{\text{op}} \to \mathbf{Set}$. One can vary the target category here to get a presheaf of e.g. groups, simplicial sets, rings etc.

Example 4.1.2. An example to keep in mind is the presheaf $\mathcal{B}$ on X that assigns to an open embedding $U \to X$ the set of bounded continuous functions $U \to \mathbb{R}$. Note that $\mathcal{B}(\emptyset) = *$.

Definition 4.1.3. If $\{f_i : U_i \to X\}_{i \in I}$ is a family of open embeddings such that $V = \bigcup_i \operatorname{im} f_i$, we say the f_i are a *covering of* V. Given such a family of maps, we can form all possible pullbacks

$$f_i \times f_j : U_i \times_X U_j \to X.$$

This is a categorical way of writing a space that we all know already: $U_i \cap U_j$. There are maps $U_i \cap U_j \to U_i$ and $U_i \cap U_j \to U_j$—the first and second inclusions. Therefore we can define two systematic maps

$$\prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I^2} \mathcal{F}(U_i \times_X U_j). \tag{4.1}$$

An element in the source here consists of an element of $\mathcal{F}(U_i)$ for each $i \in I$. The first rightward map produces an element in each $\mathcal{F}(U_i \cap U_j)$ (for all pairs i, j) by looking for the element we had in U_i and applying $\mathcal{F}(\cdot)$ to the first inclusion. The second rightward map does the same for the second inclusion.

Definition 4.1.4. If we start with an element of $\mathcal{F}(V)$, and use that to produce our elements of $\prod_{i \in I} \mathcal{F}(U_i)$, and then use either the first or second map in (4.1), the answer in each case will be the same. We will merely have restricted along the inclusions $U_i \cap U_j \to V$ in both cases. We say that $\mathcal{F}$ is a *sheaf* if the diagram

$$\mathcal{F}(V) \xrightarrow{q} \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I^2} \mathcal{F}(U_{i,j}) \tag{4.2}$$

is always an equalizer. That is, if the map q is an injection identifying $\mathcal{F}(V)$ with the subset of $\prod_{i\in I} \mathcal{F}(U_i)$ consisting of those I-tuples of elements for which the two ways of restricting further agree.

Example 4.1.5. For instance, take $\mathcal{B}$, bounded continuous functions, as before. Let $X = \mathbb{R}$ and let U_i denote the open cover of all of $\mathbb{R}$ given by the open sets $U_n := (n-1, n+1)$ as $n \in \mathbb{Z}$ varies. Then an element of $\prod_{n\in\mathbb{Z}} \mathcal{F}(U_n)$ means a family of bounded continuous functions $\psi_n : (n-1, n+1) \to \mathbb{R}$. The two maps we defined to $\prod_{n,m\in\mathbb{Z}^2} \mathcal{F}(U_n \cap U_m)$ are easy to figure out: for instance, the first one produces a function on $(n-1, n+1) \cap (m-1, m+1)$ by restricting ϕ_n. Note that most of the time, but not always, this is the unique function $\emptyset \to \mathbb{R}$.

Now consider an unbounded continuous function on $\mathbb{R}$, e.g. $\psi(x) = x^2$. Then we observe that ρ gives rise to a family of restrictions $\psi_n|_{(n-1,n+1)} : (n-1, n+1) \to \mathbb{R} \in \mathcal{B}(U_n)$, and the restrictions of ψ_n to intersections $U_n \cap U_m$ always agree, but the ψ_n do not assemble to give an element of $\mathcal{B}(\mathbb{R})$. Therefore, $\mathcal{B}$ is not a sheaf.

A little more thought shows that $\mathcal{B}(V) \to \prod_{i\in I} \mathcal{B}(U_i)$ will always be injective, but it is not generally the whole equalizer. Terminology fans may refer to $\mathcal{B}$ as a *separated presheaf*.

Remark 4.1.6. We will see later that there is a sheaf $\mathcal{L}$ that best approximates $\mathcal{B}$, in that any map of presheaves $\mathcal{B} \to \mathcal{F}$ where $\mathcal{F}$ is a sheaf must factor uniquely as $\mathcal{B} \to \mathcal{L} \to \mathcal{B}$. The sheaf $\mathcal{L}$ is the sheaf of *locally bounded functions* on $\mathbb{R}$. The idea to bear in mind is that sheaves are the right thing to think about when you are considering "local" behaviour, and it is not possible to answer the question "is this function bounded?" by looking locally.

4.2 Sheaves on a site

Our reference for all this is [MLM92, Chapter III].

Definition 4.2.1. A *presheaf* on a category $\mathbf{C}$ is a functor $\mathcal{F} : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$.

Definition 4.2.2. Let $\mathbf{C}$ be a category having all pullbacks (i.e., limits of diagrams $X \leftarrow Y \to Z$). A *basis for a Grothendieck topology* or a *pretopology* on $\mathbf{C}$ is an assignment to each object X of $\mathbf{C}$ of a collection $\mathrm{Cov}(X)$ of *covering families* with codomain X, satisfying the following axioms:

1. Any set consisting of a single isomorphism $\{f : X' \to X\}$ is in $\mathrm{Cov}(X)$.
2. if $\{f_i : X_i \to X\}_{i\in I}$ is in $\mathrm{Cov}(X)$, then for any $g : Y \to X$, the pullbacks $\{g \times f_i : Y \times_X X_i \to Y\}$ are in $\mathrm{Cov}(Y)$—coverings are closed under pullback.
3. If $\{f_i \in X_i \to X\}_{i\in I}$ is in $\mathrm{Cov}(X)$ and if, for all $i \in I$, we have elements $\{f_{ij} : X_{ij} \to X_i\}_{j\in J_i}$ in $\mathrm{Cov}(X_i)$, then the massive composite family $\{f_i \circ f_{ij}\}_{j\in J_i, i\in I}$ is in $\mathrm{Cov}(X)$—coverings are closed under refinement.

A category with a basis for a Grothendieck pretopology will be called a *site*.

Definition 4.2.3. If $\mathbf{C}$ is a site, then a *sheaf* (of sets) on $\mathbf{C}$ is a presheaf $\mathcal{F}$on $\mathbf{C}$ with the further property that for all $X \in \mathbf{C}$ and all $\{f_i : U_i \to X\} \in \mathrm{Cov}(X)$, the diagram

$$\mathcal{F}(X) \to \prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j\in I^2} \mathcal{F}(U_i \times XU_j) \tag{4.3}$$

is an equalizer.

Remark 4.2.4. There is a minimal pretopology you can define, where only the isomorphisms are covering. For this pretopology, all presheaves are sheaves.

Remark 4.2.5. It is possible to define a category of presheaves on $\mathbf{C}$, denoted $\mathbf{Pre}(\mathbf{C})$, where a morphism $\mathcal{F} \to \mathcal{F}'$ is a natural transformation of functors, i.e., maps $\mathcal{F}(U) \to \mathcal{F}'(U)$ for all objects U such that the obvious squares commute. The sheaves for a pretopology are a special kind of presheaf, and we can define a *category of sheaves* $\mathbf{Sh}(\mathbf{C})$ as the full subcategory where the objects are sheaves.

Remark 4.2.6. We already have one example of a site, $\mathbf{X}_{\text{ord}}$, and here the sheaves are as we defined them before. Here's another example of a site: fix a finite group G and let $G\mathbf{fSet}$ denote the category of finite sets with G action and G equivariant morphisms. Declare a family of maps $\{f_i : A_i \to B\}$ to be covering if B is the union of the images of the f_i.

We will not give the proof of the following long theorem, but it is not difficult.

Theorem 4.2.7. *Let $\mathbf{C}$ be a site. Then the inclusion of the category of sheaves in the category of presheaves $\iota : \mathbf{Sh}(\mathbf{C}) \to \mathbf{Pre}(\mathbf{C})$ has a left adjoint, a. Moreover, the functor a commutes with finite limits.*

You can find a proof in [MLM92, Theorem III.5.1]. The idea is: given a presheaf $\mathcal{F}$, define a new presheaf $\mathcal{F}^+$ by defining $\mathcal{F}^+(V)$ using equalizers. Somehow, doing this construction twice results in a sheaf.

Example 4.2.8. If $\mathcal{B}$ is the presheaf of bounded continuous functions on a topological space X, then $a\mathcal{B}$ is the sheaf of locally bounded functions.

Remark 4.2.9. When a category $\mathbf{C}$ is equipped with a pretopology, it is possible to endow it with something called a *Grothendieck topology*, and the category of sheaves on $\mathbf{C}$ depends only on the topology, not the pretopology. Grothendieck topologies themselves are hard to work with, and people seldom bother—instead we work with the pretopology.

Remark 4.2.10. A monomorphism (injection) or epimorphism (surjection) of presheaves is detected objectwise: e.g., $\mathcal{F} \to \mathcal{F}'$ is a monomorphism if and only if $\mathcal{F}(V) \to \mathcal{F}'(V)$ is an injection for all V. For sheaves, the story is more complicated. A map $\mathcal{F} \to \mathcal{F}'$ is a monomorphism of sheaves if it is a monomorphism of presheaves, but it is an epimorphism of sheaves if, for all V and all elements $x \in \mathcal{F}'(V)$, there exists some cover $\{U_i \to V\}_{i \in i}$ for which the restrictions to $\mathcal{F}'(U_i)$ of x are in the images of $\mathcal{F}(U_i) \to \mathcal{F}'(U_i)$—that is, epimorphisms are tested locally.

4.2.1 The Yoneda functor

Construction 4.2.11. Suppose $X \in \mathbf{C}$ is an object, then we can produce a presheaf $\eta_X \in \mathbf{Pre}(\mathbf{C})$ using the rule

$$\eta_X(Y) = \mathbf{C}(Y, X).$$

This yields a functor $\eta : \mathbf{C} \to \mathbf{Pre}(\mathbf{C})$, and it is a lemma, the Yoneda lemma, that η is a full and faithful functor, i.e., an embedding of categories. Consequently, we will frequently write X instead of η_X. You can prove the Yoneda lemma yourself as an exercise.

Notation 4.2.12. Let $\mathbf{C}$ be a site, and let $\mathbf{X}$ be an object. It may be the case that the presheaf η_X is actually a sheaf. We say that η_X is a *representable sheaf*. If all presheaves η_X are sheaves, we say the pretopology on $\mathbf{C}$ is *subcanonical*.

4.3 The étale and Nisnevich sites

Fix a field k. We assume you know about the category of smooth k-varieties. If k is a perfect field (e.g., an algebraically closed field) then this is the same as the category of regular k-varieties. It will be denoted $\mathbf{Sm}_k$.

4.3.1 Étale maps of varieties

There are a number of different ways of defining étale maps of varieties or of schemes. Our aim here is to be quick, and relatively explicit. This is not the best definition for all purposes.

Definition 4.3.1. A map of rings $\phi : R \to S$ is *standard étale* if it is isomorphic to the canonical map

$$R \to \left(\frac{R[x]}{(f)} \right)_g$$

where f' is a unit in $\left(\frac{R[x]}{(f(x))} \right)_g$.

Example 4.3.2. A good example to keep in mind is that $\mathbb{C}[t] \to \mathbb{C}[t,x]/(x^2 - t)$ fails to be standard étale because the derivative of $x^2 - t$ is $2x$, which is not invertible. If we draw the affine varieties (over $\mathbb{C}$) associated to this map of rings, we get a double cover of $\mathbb{A}^1$, with a ramification at the origin. If we discard that point, by inverting x, we get a standard étale map

$$\mathbb{C}[t] \to \left(\frac{\mathbb{C}[t,x]}{(x^2 - t)} \right)_x .$$

Definition 4.3.3. A map of schemes $f : Y \to X$ is *étale* if it is locally of finite presentation and, for each $x \in X$, there exists an open affine neighbourhood $\operatorname{Spec} R \ni f(x)$ and $\operatorname{Spec} S \ni x$ such that $f(\operatorname{Spec} S) \subseteq \operatorname{Spec} R$ and the induced map of rings $R \to S$ is standard étale.

Remark 4.3.4. If you don't care about schemes, and only about varieties, then you should ignore the phrase "locally of finite presentation".

Remark 4.3.5. The image of an étale map is open.

Remark 4.3.6. There are many alternate definitions of what it means for a map to be étale. I recommend looking at the wikipedia article on the topic to start with, then following the references there: [19].

We will give one alternative definition here, but we will not prove that the definitions are equivalent. If A is a commutative ring, then a *square*-0 *ideal* I is an ideal I such that $I^2 = 0$. A map $f : Y \to X$ of schemes is *formally étale* if it satisfies the r.l.p. uniquely w.r.t. all maps $\operatorname{Spec} A/I \to \operatorname{Spec} A$ for all rings A and all square-0 ideals I. That is, if a unique lift always exists in

$$\begin{tikzcd}
\operatorname{Spec} A/I \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

A map of schemes is étale if it is locally of finite type and formally étale.

If you don't like general schemes, that's fine. A map of affine varieties $f : \operatorname{Spec} S \to \operatorname{Spec} R$ is étale if you can always find a unique lift in

$$\begin{tikzcd} R \arrow[r] \arrow[d] & A \arrow[d] \\ S \arrow[r] \arrow[ur, dashed] & A/I \end{tikzcd}$$

where I is a square-0 ideal in A. This is just a condition on rings. A map of varieties $f : Y \to X$ is étale if you can cover the target with affine schemes $\operatorname{Spec} S$ and the source with affine schemes $\operatorname{Spec} R \subseteq f^{-1}(\operatorname{Spec} S)$ so that $\operatorname{Spec} S \to \operatorname{Spec} R$ corresponds to an étale map of rings.

We will not prove the equivalence of our two definitions of étale, since this would take us too long. A proof exists in de Jong's Stacks Project; start with [deJ17, Tag 025K] and work backwards from there.

Proposition 4.3.7. *If $f : Y \to X$ is an étale map of varieties and $g : X' \to X$ is a map, then the pullback map $Y \times_X X' \to X'$ is an étale map.*

Proof. We can reduce this to the affine case. The pull-back of a map of finite presentation is again of finite presentation, and pulling back preserves the lifting property. □

Proposition 4.3.8. *The composite of two étale maps is again étale.*

Proof. The composite of two ring maps of finite presentation is again of finite presentation, which is just an exercise in ring theory. This handles the "locally of finite presentation" part. If two maps are formally étale, then so is their composite. □

Proposition 4.3.9. *An open immersion (open embedding) of varieties is an étale map.*

Proof. Locally an embedding takes the form $\operatorname{Spec} R \to \operatorname{Spec} R_f$, which is standard étale. □

Definition 4.3.10. A family of maps of k-varieties, $\{f_i : Y_i \to X\}$, is an *étale covering* if each map is étale and if, for each $x \in X$, there exists some $f_i : Y_i \to X$ such that x is in the image of f_i, i.e., the family is jointly surjective.

Remark 4.3.11. We say a family of maps $\{f_i : Y_i \to X\}$ is a *Zariski covering* if it is jointly surjective and each map is an open immersion.

Definition 4.3.12. The *big étale site* $(\mathbf{Sm}_k, \text{ét})$ (resp. *big Zariski site* $(\mathbf{Sm}_k, \mathrm{Zar})$) of k is the site where the objects are the k-varieties and the covering maps are the étale coverings (resp. Zariski coverings).

Remark 4.3.13. One can also start with a different base scheme, S, and define the *big étale site* of $\mathbf{Sm}_S$, the category of finite type smooth S-schemes, in the obvious way, i.e., the covering maps are the jointly surjective families of étale maps. One could also do this for the Zariski topology.

Remark 4.3.14. Given a variety S, one can also define the *small étale site* of S. Here the objects are étale maps $f : X \to S$ of varieties. The covering families are defined the same way. This site is the one most often considered for calculating étale cohomology, but in fact either site can be used for that purpose: [Mil80].

4.3.2 Nisnevich covers

Here is a somewhat unusual definition of a Nisnevich covering.

Definition 4.3.15. Let X be a noetherian scheme (variety). A *Nisnevich covering* is a family of étale maps $\{f_i : Y_i \to X\}$ with the following property:

There exist closed subschemes $\emptyset = Z_0 \subseteq Z_1 \subseteq \cdots \subseteq Z_{n-1} \subseteq Z_n = X$ such that for each $U_i := Z_i \setminus Z_{i-1}$, there is some $f_i : Y_i \to X$ in the cover such that the pullback

$$f|_{U_i} : U_i \times_X Y \to U_i$$

has a section, i.e., a map $g : U_i \to U_i \times_X Y$ such that $f|_{U_i \times_X Y} \circ g = \mathrm{id}_{U_i}$.

Example 4.3.16. Here is an example of an étale covering that is not a Nisnevich covering:

$$\operatorname{Spec} \mathbb{C}[t, t^{-1}, x]/(t - x^2) \to \operatorname{Spec} \mathbb{C}[t, t^{-1}].$$

This is clearly standard étale, and you can verify that it is surjective.

To see that this is not a Nisnevich covering, argue as follows: any proper open subset of the target here is of the form $\operatorname{Spec} \mathbb{C}[t, t^{-1}]_f$, and, in particular, has the same fraction field as $\mathbb{C}[t, t^{-1}]$, i.e., $\mathbb{C}(t)$ (This is a general feature of irreducible schemes). On the other hand, the fraction field of $\mathbb{C}[t, t^{-1}, x]/(t - x^2)$ is $\mathbb{C}(x)$, where $x = t^2$. The covering map induces the map $\mathbb{C}(t) \to \mathbb{C}(x)$ on fraction fields, sending $t \mapsto x^2$. This map is not an isomorphism. If we could find a section on a nonempty open subscheme of the target, then this map of fields would have an inverse, a contradiction.

Remark 4.3.17. The usual definition of a Nisnevich covering is that it is an étale covering $\{f_i : Y_i \to X\}$ such that for all $x \in X$, there exists some i and some $y \in Y_i$ such that the induced map on residue fields $\kappa(x) \to \kappa(y)$ is an isomorphism.

Remark 4.3.18. A Zariski covering is a kind of Nisnevich covering, and a Nisnevich covering is a kind of étale covering. Therefore, it is the case that a sheaf for the étale topology is a sheaf for the Nisnevich topology, and a sheaf for the Nisnevich topology is a sheaf for the Zariski topology.

Remark 4.3.19. We allow our varieties to be disconnected.

Lemma 4.3.20. *Suppose a presheaf $\mathcal{F} : \mathbf{Sm}_k^{op} \to \mathbf{Set}$ satisfies the sheaf condition for étale (resp. Nisnevich, Zariski) covers $\{f : Y \to X\}$ consisting of a single map of varieties. Then $\mathcal{F}$ is an étale (resp. Nisnevich, Zariski) sheaf.*

Proof. A homework exercise asks you to show that if a presheaf satisfies the sheaf condition for an (étale) covering $\{f_i : Y_i \to X\}$ then it satisfies descent for a covering $\{f_i : Y_i \to X\} \cup \{g : W \to X\}$ (adding a single new map). In fact, this works however many new maps we add. The homework assignment was written the way it was to keep the notation in check.

Now, if you have an étale covering of a variety $\{f_i : Y_i \to X\}$, then to cover X (a topologically compact space), it suffices to take finitely many of the open images of the maps f_i. Therefore, there is a finite étale subcover $\{f_i : Y_i \to X\}_{i=1}^n$.

We then can engage in a notational swindle: write $Y = \coprod_{i=1}^n Y_i$ and $f = \coprod_{i=1}^n f_i$. This replaces the étale cover by a single surjective étale map. $\square$

4.3.3 The functor of points

The Yoneda embedding $\eta : \mathbf{Sm}_k \to \mathbf{Pre}(\mathbf{Sm}_k)$ is often called the *functor of points* in algebraic geometry.

We cut a corner by assuming the following proposition:

Proposition 4.3.21. *Let $f : Y \to X$ be a surjective étale map of varieties, and let $g : Y \to Z$ be a map of varieties with the property that the two composites $Y \times_X Y \rightrightarrows Y \to Z$ agree. Then there is a map of varieties $\tilde{g} : X \to Z$ such that $\tilde{g} \circ f = g$.*

Proof. This result is an application of FPQC descent for schemes. For the experts:

- an étale map is a flat map.
- a surjective flat map is faithfully flat
- a map of varieties is necessarily quasicompact.

and then we can apply FPQC descent. This constructs the map $\tilde{g}$ assuming f is faithfully flat (FP) and quasicompact (QC). I worked through a proof of this the last time I taught this course, so you can find an outline in the old notes, but it's not very relevant to the rest of the material, so I'll skip it. $\square$

Corollary 4.3.22. *Let X be a smooth variety. The presheaf η_X defined by $\eta_X(Y) = \mathbf{Sm}_k(Y, X)$ is an étale (and therefore a Nisnevich, Zariski) sheaf on $\mathbf{Sm}_k$.*

Proof. To test if something is an étale sheaf, you just have to look at surjective étale maps $f : Y \to X$. The presheaf η_X satisfies this due to the proposition. $\square$

Notation 4.3.23. If $\mathcal{F}$ is a presheaf and $\operatorname{Spec} R$ is an affine scheme, we will sometimes write $\mathcal{F}(R)$ instead of $\mathcal{F}(\operatorname{Spec} R)$. Recall that we will sometimes write X instead of η_X.

Lemma 4.3.24. *Suppose $\phi : \mathcal{F} \to \mathcal{G}$ is a map of Zariski sheaves on $\mathbf{Sm}_k$ such that $\phi : \mathcal{F}(\operatorname{Spec} R) \to \mathcal{G}(\operatorname{Spec} R)$ is a bijection for all affine varieties $\operatorname{Spec} R$, then ϕ is an isomorphism.*

Proof. First note that if $\{f_i : Y_i \to X\}$ is a Zariski cover of X by affine k-schemes, then $\mathcal{F}(X) \subset \prod_{i \in I} \mathcal{F}(Y_i)$.

Now, suppose $\{f_i : Y_i \to X\}$ is a Zariski cover of X by affine k-schemes, and for each pair $Y_i \times_X Y_j$, let $\{Z_{ij\ell}\}_{\ell \in L_{ij}}$ be an affine cover of $Y_i \times_X Y_j$. Then there is an equalizer diagram

$$\mathcal{F}(X) \to \prod_{i \in I} \mathcal{F}(Y_i) \rightrightarrows \prod_{i,j \in I^2} \prod_{\ell \in L_{ij}} \mathcal{F}(Z_{ij\ell})$$

from which we can deduce the result by means of uniqueness of equalizers. $\square$

This lemma implies that it's often enough to consider the values $\mathcal{F}(R)$ as R varies over the coordinate rings of affine varieties.

Example 4.3.25. Here are some examples of varieties and the (étale) sheaves they represent, indicated by applying to affine varieties:

1. $\mathbb{A}^1_k(R) = \mathbf{Sm}_k(\operatorname{Spec} R, \operatorname{Spec} k[t]) = k - \mathbf{Alg}(k[t], R) = R$.

2. $\mathbb{A}^n_k(R) = R^n$ (the Yoneda functor preserves products).
3. Define $\mathbb{G}_m = \operatorname{Spec} k[t, t^{-1}] = \mathbb{A}^1_k \setminus \{0\}$. Then $\mathbb{G}_m(R) = R^\times$.
4. $\mathbb{A}^n_k \setminus \{\mathbf{0}\}$ for $n \geq 2$ will appear in the homework.
5. $\mathbb{P}^n_k(R)$: this is slightly harder to figure out directly. An element of this set is an isomorphism class of exact sequences

$$R^{n+1} \longrightarrow L \longrightarrow 0$$

where L is a projective module of rank 1. The isomorphisms are diagrams of R-modules

$$\begin{tikzcd} R^{n+1} \arrow[r] \arrow[d, equal] & L \arrow[r] \arrow[d] & 0 \\ R^{n+1} \arrow[r] & L' \arrow[r] & 0 \end{tikzcd}$$

If R is a ring for which every rank-1 projective module is free, then we can replace L by R, and the surjective map $R^{n+1} \to R$ is actually the data of an $n+1$-tuple of elements $[r_0 : r_1 : \cdots : r_n]$ in R, such that the ideal these elements generate is all of R, taken up to the equivalence relation

$$[r_0 : r_1 : \cdots : r_n] \sim [\lambda r_0 : \lambda r_1 : \cdots : \lambda r_n], \qquad \lambda \in R^\times.$$

Example 4.3.26. The functor of points is a very useful way to think of group schemes. For instance, GL_n is a scheme, the open subscheme determined by the invertibility of the determinant polynomial in $\mathrm{Mat}_{n\times n} \cong \mathbb{A}^{n^2}$. But as a functor of points, it is very easy to understand. We all know what $\mathrm{GL}_n(R)$ is.

Since the Yoneda lemma gives us a full and faithful embedding and preserves products, the fact that $\mathrm{GL}_n(\cdot)$ is a sheaf of groups defined on $\mathbf{Sm}_{\mathbb{Z}}$ implies that GL_n is a group object in the category of schemes.

4.4 Nisnevich squares

The following idea shows up often in the literature about $\mathbb{A}^1$ homotopy, so we should devote a little time to it. For instance, it appears in [MV99], in Section 3.1. Be warned, in [MV99], the publishers made a mistake and numbered the propositions etc. within each chapter independently, but failed to include the chapter numbers. Therefore there are several things numbered 3.1 etc. When you refer to this paper, you should state the page number as well as the proposition number etc.

Definition 4.4.1. An *elementary Nisnevich square* in $\mathbf{Sm}_k$ is a caretesian square of the form

$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] & V \arrow[d, "p"] \\ U \arrow[r, "j"] & X \end{tikzcd}$$

where p is étale, j is an open embedding and such that $p^{-1}(X \setminus U) \to X \setminus U$ is an isomorphism (of closed varieties).

Proposition 4.4.2. *Suppose $\mathcal{F} : \mathbf{Sm}_k^{op} \to \mathbf{Set}$ is a presheaf. Then $\mathcal{F}$ is a Nisnevich sheaf if and only if it converts every elementary Nisnevich square into a pullback square of sets.*

Remark 4.4.3. Note that a presheaf satisfying this condition on elementary Nisnevich squares converts disjoint unions to products.

The proof is taken from [MV99, p 96–97].

Proof. There are two parts to this. In one direction, the "if" direction, the argument is by induction. Suppose $\mathcal{F}$ has the property for all elementary Nisnevich squares.

Recall that a Nisnevich cover $\{f_i : Y_i \to X\}_{i \in I}$ of X is an étale covering (with finitely many elements) such that there exists a stratification $Z_1 \subseteq \cdots \subseteq Z_n = X$ so that the cover has sections on the locally-closed complements of the strata. We argue by induction on the length of the stratification. If $n = 1$, then the cover is actually split and the cover contains an isomorphism as a subcover, so we can conclude by a homework assignment.

Suppose $n \geq 2$, and suppose $\mathcal{F}$ satisfies the sheaf condition for all covers having stratification-length $< n$. We may assume we have a stratification of minimal length. Write $f = \coprod f_i$ for convenience. Choose a splitting map s for $f^{-1}(Z_1) \to Z_1$. A split surjective étale map is an isomorphism of a component onto its image (alg. geo exercise). There is therefore decomposition $f^{-1}(Z_1) = \Im(s) \coprod W$ for some closed complement $W \subseteq \coprod_i Y_i$, Then set $U = X \setminus Z_1$ and $V = \coprod_i Y_i \setminus W$. This U and V form an elementary Nisnevich square over X, and the pullback of f to U has a splitting sequence of length $n - 1$. Therefore, by assumption on $\mathcal{F}$ and induction both the sequences

$$\mathcal{F}(X) \to \mathcal{F}(U) \times \mathcal{F}(V) \rightrightarrows \mathcal{F}(U \times_X V)$$
$$\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(Y_i \times_X U) \rightrightarrows \prod_{i \in I} \mathcal{F}(Y_i \times_X Y_j \times_X U)$$

are equalizer sequences. The rest is reasonably elementary, and can be done by chasing elements. For instance, we see that there is an inclusion $\mathcal{F}(X) \to \prod_{i \in I} \mathcal{F}(Y_i \times_X U) \times \mathcal{F}(V)$. But this map factors through $\mathcal{F}(X) \to \mathcal{F}(\coprod_{i \in I} Y_i)$ (by virtue of the definition of V) and so we see that $\mathcal{F}(X) \to \prod_{i \in I} \mathcal{F}(Y_i)$ is an inclusion. The other part of the equalizer condition is similar, but longer. We give an outline here so that the reader can (ideally) reconstruct it in private: Suppose we have an element of $\prod_{i \in I} \mathcal{F}(Y_i)$ for which the two maps to $\prod_{i,j \in I^2} \mathcal{F}(Y_i \times_X Y_j)$ agree. Then applying restriction maps gives us an element of $\prod_{i \in I} \mathcal{F}(Y_i \times_X U)$ satisfying exactly the gluing condition to yield an element of $\mathcal{F}(U)$. We also get an element of $\mathcal{F}(V)$ by restricting from $\mathcal{F}(\coprod_{i \in I} Y_i)$. Chase a diagram to see that the two ways of producing an element of $\mathcal{F}(U \times_X V)$ agree. Therefore we arrive at an element of $\mathcal{F}(X)$, as desired.

The "only if" direction goes as follows. Let $\mathcal{F}$ be a Nisnevich sheaf. Suppose U, V make up an elementary Nisnevich square, then $U \coprod V \to X$ is a Nisnevich covering, and so we expect

$$\mathcal{F}(X) \to \mathcal{F}(U) \times \mathcal{F}(V) \rightrightarrows \mathcal{F}(U \times_X U) \times \mathcal{F}(U \times_X V) \times \mathcal{F}(V \times_X U) \times \mathcal{F}(V \times_X V) \tag{4.4}$$

to be an equalizer. Some of the conditions imposed by the equalizer are unnecessary: $U \times_X U = U$, so that imposes no condition, and $U \times_X V$ and $V \times_X U$ impose the same condition. We want to conclude that

$$\mathcal{F}(X) \to \mathcal{F}(U) \times \mathcal{F}(V) \rightrightarrows \mathcal{F}(U \times_X V)$$

is an equalizer diagram, since this is the same as being a pullback. Therefore we have to conclude that the equalizer constraint imposed by $\mathcal{F}(V) \rightrightarrows \mathcal{F}(V \times_X V)$ is also redundant.

There is a diagonal map $\Delta : V \to V \times_X V$. This is actually an étale map, which is perhaps surprising, but is a fun argument (and on the homework). Also, the map $j : U \times_X V \times_X V \to V \times_X V$ is an open immersion. We also see that $\{\Delta, j\}$ forms a Nisnevich cover of $V \times_X V$, so there is an inclusion

$$\mathcal{F}(V \times_X V) \to \mathcal{F}(V) \times \mathcal{F}(U \times_X V \times_X V).$$

In particular, the two maps $\mathcal{F}(V) \rightrightarrows \mathcal{F}(V \times_X V)$ from (4.4) impose a condition on elements of $\mathcal{F}(V)$ that can be restated as two conditions: the equalizer of $\mathcal{F}(V) \rightrightarrows \mathcal{F}(V)$—both maps being the identity, so no condition—and the equalizer of $\mathcal{F}(V) \rightrightarrows \mathcal{F}(U \times_X V \times_X V)$ given by two projections onto $U \times_X V \subseteq V$.

Now suppose $(\alpha, \beta) \in \mathcal{F}(U) \times \mathcal{F}(V)$ are such that the restrictions to $\mathcal{F}(U \times_X V)$ of α and β agree. Then the two restrictions of β to $\mathcal{F}(U \times_X V \times_X V)$ that you get by projecting away from the second or third factors must agree with the pullbacks of α along the first projection $\mathcal{F}(U) \to \mathcal{F}(U \times_X V \times_X V)$. So they agree. This means that the equalizer condition $\mathcal{F}(V) \rightrightarrows \mathcal{F}(U \times_X V \times_X V)$ is vacuous in (4.4), and we conclude. $\square$

4.5 Stalks

Stalks are a useful way of working with sheaves on a site.

A good place to find more than you knew you wanted to know about stalks is [GK15]. We'll use this as a starting point.

Definition 4.5.1. Let $\mathbf{S}$ be a category. Then a functor $p^* : \mathbf{S} \to \mathbf{Set}$ is called a *fibre functor* if p commutes with finite limits and all colimits. If $\mathbf{S}$ is a category of sheaves on a site, then p^* will automatically have a right adjoint p_*, and the pair $p = (p^*, p_*)$ is called a *point* of $\mathbf{S}$, and $p^*\mathcal{F}$ is called the *stalk* of $\mathcal{F}$ at p.

Definition 4.5.2. A category of sheaves $\mathbf{Shv}_\tau(\mathbf{C})$ has *enough points* if there exists a set of points $\{p_i\}_{i \in I}$ for which the following condition holds:

"A map $\mathcal{F} \to \mathcal{G}$ of sheaves is an isomorphism if and only if $p_i^*\mathcal{F} \to p_i^*\mathcal{G}$ is a bijection for all $i \in I$."

The family $\{p_i\}_{i \in I}$ is said to be a *conservative family of points*.

Remark 4.5.3. You can detect both epimorphisms and monomorphisms with a conservative family of points. For instance, a map $\phi : \mathcal{F} \to \mathcal{G}$ is an epimorphism of sheaves if and only if the pushout map $\mathcal{G} \to \mathcal{G} \coprod_{\mathcal{F}} \mathcal{G}$ is an isomorphism. Applying p^* preserves pushouts, so that ϕ is an epimorphism if and only if $p^*\mathcal{G} \to p^*\mathcal{G} \coprod_{p^*\mathcal{F}} p^*\mathcal{G}$ is a bijection as p ranges over a conservative family of points, which is the case if and only if $p^*\mathcal{F} \to p^*\mathcal{G}$ is onto for all p.

Proposition 4.5.4 (Deligne). *Suppose $\mathbf{C}$ is a site having all fibre products and τ is a Grothendieck topology in which all covering families have finite subfamilies that are also covering. Then $\mathbf{Shv}_\tau(\mathbf{C})$ has enough points.*

This implies that the étale, Nisnevich and Zariski sites all have enough points. Regrettably, there are sites without enough points, but we will not encounter them in this course.

4.5.1 Calculating stalks

Definition 4.5.5. A category $\mathbf{I}$ is said to be *pseudofiltered* if it satisfies the following conditions:

1. For every two morphisms $f : i \to j$ and $g : i \to j'$ with common domain, there exists an object k and morphisms $u : j \to k$ and $v : j' \to k$ so that $u \circ f = v \circ g$.

2. For every two parallel morphisms $f : i \to j$ and $f' : i \to j$, there exists some morphism $w : i \to l$ such that $w \circ f = w \circ f'$.

A category is said to be *filtered* if it is filtered and has exactly one path component, i..e, it's not empty and for any two objects, there's a zig-zag of maps between them.

A category $\mathbf{I}$ is said to be *cofilitered* if the opposite category is filtered. Sometimes we mess up and say *filtered* instead of *cofiltered*.

Proposition 4.5.6. *Let $\mathbf{I}$ be a filtered small category and let $\mathbf{J}$ be a finite category. Let $X : \mathbf{I} \times \mathbf{J} \to \mathbf{Set}$ be a diagram of sets: objects in this diagram are indexed X_{ij} (for fixed i, the diagram X_{i*} is finite, and for fixed j, the diagram X_{*j} is filtered). Then there is a bijection*

$$\operatorname*{colim}_{i \in I} \lim_{j \in J} X_{ij} \to \lim_{j \in J} \operatorname*{colim}_{i \in I} X_{ij}.$$

This is [ML98, Theorem 1, p 211].

Proof. This can be reduced to checking two cases for $\mathbf{J}$. Either it is a very boring category with two objects and no non-identity morphisms (in which case the limit is just a finite product) or it is $\bullet \rightrightarrows \bullet$, in which case the limit is an equalizer.

These two cases are left as exercises. □

Construction 4.5.7. Here's a way of constructing stalks for topological spaces. If X is a topological space, then $\mathbf{X}_{\text{ord}}$ denotes the category where the objects are open subsets of X and the morphisms are inclusions. Let V be a nonempty object of this category, and let $v \in V$.

We will try to associate an adjoint pair of functors $p^* \dashv p_*$ between $\mathbf{Pre}(\mathbf{X}_{\text{ord}})$ and $\mathbf{Set}$ to v. We've deliberately made this a little harder for ourselves by specifying $v \in V$: if we just wanted to do this construction, we'd say "$v \in X$" and get on with it. But later, we'll be looking at cases where there's no perfect analogue for X, so we give ourselves extra flexibility.

First, let's try to construct the functor $p_* : \mathbf{Set} \to \mathbf{Shv}(\mathbf{X}_{\text{ord}})$. For a given set S, the name for the (pre)sheaf $p_* S$ is the *skyscraper sheaf* with value S at v. Here's a slightly longwinded way of constructing it:

Let $\mathbf{X}(V, v)$ denote a category where the objects are maps $i : U \to V$ in $\mathbf{X}_{\text{ord}}$ such that v is in the image of i. If there is such a map $i : U \to V$, then clearly there is a unique such map. There is also an obvious forgetful functor $\mathbf{X}(V, v) \to \mathbf{X}_{\text{ord}}$ where you take $i : U \to V$ and forget everything except for U.

Let us produce a presheaf $p_*^c S$ on $\mathbf{X}(V, v)$ by always declaring the value to be S and all restriction maps to be identities. Now we perform what is called a "right Kan extension" of p_*^c along the forgetful functor. That is, to define

$$p_* S(U) = \lim p_*^c S(V' \to V)$$

taking a limit over all $V' \to V$ such that there exists a map $V' \to U$.

The effect of all this work is that $p_*S(U) = *$ (a limit over an empty diagram) if there is no open set containing v and contained in U, i.e., if $v \notin U$, and $p_*S(U) = S$ if $v \in U$.

We didn't specify what p_* does on maps of sets, but it's obvious.

We can also construct a left adjoint $p^* : \mathbf{Pre}(\mathbf{X}_{\text{ord}}) \to \mathbf{Set}$. Here's how: given a presheaf $\mathcal{F}$ we can "restrict" $\mathcal{F}$ to $\mathbf{X}(V, v)$ in the obvious way. Then we take a colimit

$$p^*\mathcal{F} = \operatorname*{colim}_{V' \in \mathbf{X}(V,v)} \mathcal{F}(V').$$

It is an exercise to verify that p^* is a left adjoint to p_*. The main idea is as follows: suppose $p^*\mathcal{F} \to S$ is a map of sets, then for every $V' \to V$, a neighbourhood of v, we get a map $\mathcal{F}(V') \to S$, and these maps are all compatible with one another and restrictions. Then we get a natural transformation of functors $\mathcal{F} \to p_*^c S$ on the category $\mathbf{X}(V, v)$, and the Kan extension gives us a natural transformation $\mathcal{F} \to p_*S$. This construction is reversible, so that given a map $\mathcal{F} \to p_*S$, we get a map $p^*\mathcal{F} \to S$, and it's easy to verify that the two constructions are inverse to one another.

The skyscraper sheaf is actually a sheaf: you can verify directly it satisfies the gluing condition. It is therefore formal that $p^* \dashv p_*$, which we constructed as an adjoint between presheaves and sets actually restricts to an adjunction

$$p^* : \mathbf{Shv}(\mathbf{X}_{\text{ord}}) \leftrightarrows \mathbf{Set} : p_*$$

It is also formal that $p^*\mathcal{F} \to p^*a\mathcal{F}$ is a bijection (here a is the associated sheaf functor).

Since p^* is a left adjoint, it preserves all colimits. Since p^* is formed as a colimit over a filtered category $\mathbf{X}(V, v)$, it commutes with finite limits.

Definition 4.5.8. We may require the following general definition later. Let X be an object of $\mathbf{Sm}_k$ and let $Z \to X$ be a subscheme. Then a *Nisnevich neighbourhood* of Z in X is an étale map $f : Y \to X$ and a closed subvariety $Z' \to Y$ such that $f|_{Z'} : Z' \to Z$ is an isomorphism.

The most commonly arising case of this is when $Z = \{x\}$ is a single point—including possibly nonclosed points. Then $\overline{\{x\}}$ is an irreducible closed subvariety and a Nisnevich neighbourhood of (X, x) is an étale map $f : Y \to X$ and a section on a dense open subset of $\overline{\{x\}}$. More scheme-theoretically, the Nisnevich neighbourhood is a pair $(f : Y \to X, y)$ where f is étale and $f(y) = x$, and f induces an isomorphism on residue fields $\kappa(y) \to \kappa(x)$.

Notation 4.5.9. Let's say $(f : Y \to X, y)$ is a *simple Nisnevich neighbourhood* or SNN if it is a Nisnevich neighbourhood of $x = f(y)$ and $f^{-1}(x)$ consists only of y.

Construction 4.5.10. Now we can do something similar but for $\mathbf{Sm}_k$ with the Nisnevich topology. I'm not aware of anywhere where this construction is actually explicitly done: usually the "small Nisnevich site" is considered and perhaps some change-of-site arguments. At any rate, the big Nisnevich site is a little more challenging that $\mathbf{X}_{\text{ord}}$, but similar.

Let $X \in \mathbf{Sm}_k$ be a nonempty k-variety and let $x \in X$ be a point (of the Zariski topological space). Let us define a category $\mathbf{C}$ where the objects of $\mathbf{C}$ are simple Nisnevich neighbourhoods $(f : Y \to X, y)$ of (X, x).

The category $\mathbf{C}$ is filtered, and has an obvious forgetful functor to $\mathbf{Sm}_k$. Given a set S, we define $p_*^c S$ as before: it's the constant presheaf with value S. Then we Kan extend as before. It's helpful to do this Kan extension in two steps. First, let's extend to $\mathbf{Sm}_k/X$: we define $p_*S(h : U \to X)$ as a limit over all SNNs $Y \to X, y$ that map to U over X. One sees that $p_*S(h : U \to X) = \prod_{j \in h^{-1}(x)'} S$ where $h^{-1}(x)'$ is the subset of points in $h^{-1}(x)$ such that h induces an isomorphism of residue fields. Then we Kan extend along the forgetful functor $\mathbf{Sm}_k/X \to \mathbf{Sm}_k$. This means that

$$p_*S(U) = \lim_{U \leftarrow Y \xrightarrow{h} X} \prod_{h^{-1}(x)'} S.$$

This is right adjoint to a functor $p^*\mathcal{F}$ that takes a presheaf $\mathcal{F}$ to $\operatorname{colim}_{\mathbf{C}} \mathcal{F}(Y)$. As before, p_* produces sheaves—in this case, Nisnevich sheaves—so that p^* preserves colimits of presheaves and of sheaves, and $p^*\mathcal{F} \to p^*a\mathcal{F}$ is a bijection. Moreover, p^*, being defined as a filtered colimit, preserves finite limits.

Remark 4.5.11. As $x \in X$ ranges over all points in all objects of $\mathbf{Sm}_k$, this furnishes a conservative set of points. We won't prove this, but it's essentially just an exercise.

4.5.2 Hensel's lemma

Definition 4.5.12. Let $(A, \mathfrak{m})$ denote a local ring. We say A satisfies *Hensel's lemma* or is *henselian* if the following (known as Hensel's lemma, [Eis95, Theorem 7.3]) holds:

Given a polynomial $f(x) \in A[x]$ and an "approximate root" $a \in A$ such that $f(a) \equiv 0 \pmod{ f'(a)^2 m}$, then there exists $b \in A$ such that $f(b) = 0$ and $b \equiv a \pmod{m}$, and furthermore, if $f'(a)$ is not a zero divisor, then b is unique with this property.

Remark 4.5.13. Complete local rings are henselian. For instance, power series rings over fields are henselian. Given a noetherian local ring $(R, \mathfrak{m})$, one can produce a map of local rings $R \to R^\wedge$, the $\mathfrak{m}$-completion. Therefore every noetherian local ring R maps to a henselian local ring. In fact, there is an initial such ring, the *henselization* R^h of R (defined up to unique isomorphism). It is different from the completion in general, e.g., for reasons of cardinality. See [Eis95, Chapter 7] for more.

Remark 4.5.14. Suppose $\mathcal{F}$ is a representable sheaf, η_Z, written as simply Z. Then in the diagram over which we take the colimit,

$$p^*Z = \operatorname*{colim}_{\mathbf{C}} Z(Y)$$

we may restrict to a (cofinal) subcategory $\mathbf{C}'$ of $\mathbf{C}$, consisting of maps with affine source $\operatorname{Spec} R \to X$. Then we are calculating

$$p^*Z = \operatorname*{colim}_{\mathbf{C}'} Z(R).$$

It is proved in [Gro67, §18.6] that

$$p^*Z = Z(\mathcal{O}^h_{x,X}).$$

It is also the case that all sheaves are colimits of representable sheaves, and p^* commutes with colimits, so insofar as one can make sense of $\mathcal{F}(\mathcal{O}^h_{x,X})$, this symbol denotes $p^*\mathcal{F}$.

Notation 4.5.15. A local ring $(R, \mathfrak{m})$ is *strictly henselian* if $R/\mathfrak{m}$ is separably closed. Strictly henselian rings are related to stalks for the étale topology in the same way that henselian rings are related to the stalks for the Nisnevich topology.

Chapter 5

Local Homotopy Theory

5.1 Simplicial Presheaves

Definition 5.1.1. A *simplicial presheaf* $\mathcal{X}$ in a category $\mathbf{C}$ is a presheaf $\mathcal{X} : \mathbf{C}^{\mathrm{op}} \to \mathbf{sSet}$ or, equivalently, a simplicial object in the category of presheaves of sets. There is an obvious category of simplicial presheaves, $\mathbf{sPre}(\mathbf{C})$, endowed with all limits and all colimits. If $\mathbf{C}$ is a site, then there is a subcategory of simplicial sheaves, $\mathbf{sShv}(\mathbf{C})$, again having all limits and colimits.

Remark 5.1.2. There is an embedding $\mathbf{Pre}(\mathbf{C}) \to \mathbf{sPre}(\mathbf{C})$ where you view a presheaf as a simplicial presheaf having nondegenerate 0-simplicies only. There is an embedding of sheaves in simplicial sheaves as well.

Remark 5.1.3. There is an embedding $\mathbf{sSet} \to \mathbf{sPre}(\mathbf{C})$, sending K to the constant simplicial presheaf $U \mapsto K$.

Remark 5.1.4. As a special case of having all limits, $\mathbf{sPre}(\mathbf{C})$ has products. This has a right adjoint, $\mathrm{Map}(\mathcal{X}, \mathcal{Y})$. In order to determine what $\mathrm{Map}(\mathcal{X}, \mathcal{Y})$ actually is, we use the following corollary of the Yoneda lemma

$$\mathcal{Z}(U)_n = \mathbf{sPre}(\eta_U \times \Delta[n], \mathcal{Z}).$$

Therefore

$$\mathrm{Map}(\mathcal{X}, \mathcal{Y})(U)_n = \mathbf{sPre}(\eta_U \times \Delta[n] \times \mathcal{X}, \mathcal{Y}).$$

There is a simplicial structure on $\mathbf{sPre}$. The action $\mathcal{X} \otimes K$ is given by $\mathcal{X} \times K$; the simplicial mapping object $\mathcal{S}(\mathcal{X}, \mathcal{Y})$ has n-simplices $\mathbf{sPre}(\mathcal{X} \times \Delta[n], \mathcal{Y})$. The cotensor object $\mathcal{X}^K$ is given by $\mathrm{Map}(K, \mathcal{X})$.

Remark 5.1.5. There are also pointed versions of all the above. A (global) basepoint of a simplicial sheaf $\mathcal{X}$ is the same as a map $* \to \mathcal{X}$ where $*$ is the terminal object. The terminal object $* \cong \Delta[0]$, and when we are dealing with $\mathbf{sPre}(\mathbf{Sm}_k)$, the terminal object is isomorphic to $\mathrm{Spec}\, k$.

Example 5.1.6. In contrast to what happens with simplicial sets, there can be nonempty simplicial presheaves without globally defined points. For example, consider the $\mathbb{R}$-variety $X = \mathrm{Spec}\, \mathbb{R}[x^2]/(x^2 + 1) = \mathrm{Spec}\, \mathbb{C}$, viewed as a discrete simplicial set. If R is a ring, then $X(R)$ is the set of square roots of -1 in R, viewed as a simplicial set. Since $\mathbb{R}$ itself has no square root of -1, there is no map $* = \mathrm{Spec}\, R \to X$, but $X(\mathbb{C}) = \{\pm i\}$, so X is not empty.

5.1.1 Weak equivalences

Definition 5.1.7. A *global* or*objectwise weak equivalence* is a map $f : \mathcal{X}_\bullet \to \mathcal{Y}_\bullet$ such that for all $U \in \mathbf{C}$, the map $f(U) : \mathcal{X}_\bullet(U) \to \mathcal{Y}_\bullet(U)$ is weak equivalence of simplicial sets.

Definition 5.1.8. Let $\mathbf{C}$ be a site and $\mathcal{X}_\bullet$ a simplicial presheaf. It is possible to define $\pi_0^{\mathrm{pre}}(\mathcal{X}_\bullet)$, a presheaf of sets on $\mathbf{C}$, by means of the rule

$$\pi_0^{\mathrm{pre}}(\mathcal{X}_\bullet)(U) = \pi_0^{\mathrm{pre}}(\mathcal{X}_\bullet(U))$$

Then we define $\pi_0(\mathcal{X})$ as the associated sheaf.

You might rush headlong at this point to try to define $\pi_n(\mathcal{X}_\bullet)$, but there's a problem with basepoints. For instance, in $\mathbf{sPre}(\mathbf{Sm}_k)$, the object $\operatorname{Spec}\mathbb{C} \times \Delta[1]/\partial\Delta[1]$ should have nontrivial π_1, but it has no basepoints.

Construction 5.1.9. Fix an integer $n \geq 1$. Suppose $\mathcal{X}$ is a simplicial presheaf on $\mathbf{C}$, and $U \in \mathbf{C}$, and $x_0 \in \mathcal{X}(U)$. Let $\mathbf{C}/U$ denote the category where the objects are maps $j : V \to U$ in $\mathbf{C}$. For any such map, there is a basepoint $j^*(x_0) \in \mathcal{X}(V)$. This allows us to define a presheaf on $\mathbf{C}/U$ given by

$$\pi_n^{\mathrm{pre}}(\mathcal{X}, x_0) : V \mapsto \pi_n(\mathcal{X}(V), j^* x_0)$$

The category $\mathbf{C}/U$ inherits a pretopology from $\mathbf{C}$, and so it makes sense to take an associated sheaf $\pi_n(\mathcal{X}, x_0)$ on $\mathbf{C}/U$. The construction of $\pi_n(\mathcal{X}, x_0)$ is functorial in $\mathcal{X}$.

Definition 5.1.10. Let $\mathbf{C}, \tau$ be a site. A morphism of simplicial sheaves $f : \mathcal{X}_\bullet \to \mathcal{Y}_\bullet$ is a *local weak equivalence* (sometimes called a *simplicial weak equivalence* if

1. $f_* : \pi_0(\mathcal{X}_\bullet) \to \pi_0(\mathcal{Y}_\bullet)$ is an isomorphism and
2. for all $n > 0$, all $U \in \mathbf{C}$ and all $x_0 \in \mathcal{X}(U)$, the morphism $f_* : \pi_n(\mathcal{X}, x_0) \to \pi_n(\mathcal{Y}, f(x_0))$ is an isomorphism of sheaves on $\mathbf{C}/U$.

Proposition 5.1.11. *A global weak equivalence is a local weak equivalence.*

To prove the next major result, we need some technical facts about filtered colimits simplicial sets:

Lemma 5.1.12. *Let $\mathbf{I}$ be a small filtered category and let X and Y be $\mathbf{I}$-shaped diagrams of simplicial sets (having X_i, Y_i as objects), and let $f : X \to Y$ be a natural transformation. If $f : X_i \to Y_i$ is a fibration (resp. trivial fibration) of simplicial sets for each $i \in I$, then $f : \operatorname{colim}_i X_I \to \operatorname{colim}_i Y_i$ is a fibration (resp. trivial fibration) of simplicial sets.*

A proof is to be found in the notes of Hirschhorn on Homotopy Colimits.

Lemma 5.1.13. *Let $\mathbf{I}$ be a small filtered category and let X be a $\mathbf{I}$-shaped diagram of simplicial sets (having X_i as objects) and let K be a simplicial set having finitely many nondegenerate simplices. Then* $\operatorname{colim}_{i \in I} \mathbf{sSet}(K, X_i) = \mathbf{sSet}(K, \operatorname{colim}_{i \in I})$. *Consequently,* $\operatorname{colim}_{i \in I} \operatorname{Map}(K, X_i) = \operatorname{Map}(K, \operatorname{colim}_{i \in I} X_i)$.

The proof follows from the analogous fact in the category of sets.

Proposition 5.1.14. *Let p be a point of $\mathbf{C}$, and $\mathcal{X}$ a simplicial presheaf taking values in Kan complexes. Then the natural map*

$$\pi_0(p^*\mathcal{X}) \to p^*\pi_0(\mathcal{X})$$

is an isomorphism.

The proof assumes that p^* is formed as a filtered colimit over neighbourhoods. Fortunately, this is always the case, but unfortunately we haven't established this. In the case we care about (the Nisnevich site of $\mathbf{Sm}_k$), we know this is the case.

Proof. Note that since $\mathcal{X}(U)$ is Kan for every U, it follows from Lemma 5.1.12 that $p^*\mathcal{X}$ is a Kan complex.

Since $\mathcal{X}(U)$ is a Kan complex, we can define $\pi_0(\mathcal{X}(U))$ as the coequalizer of $\mathcal{X}_1(U) \rightrightarrows \mathcal{X}_0(U)$. Then $p^*\pi_0(\mathcal{X})$ is naturally isomorphic to the coequalizer of $p^*\mathcal{X}_1 \rightrightarrows p^*X_0$, which is $\pi_0(p^*\mathcal{X})$. $\square$

Proposition 5.1.15. *Let $\mathcal{X}$ be a simplicial presheaf taking values in Kan complexes, and let $x_0 \in \mathcal{X}(U)_0$ be an element. Let p be a point of $\mathbf{C}/U$. Then for all n, the natural map $\pi_n(p^*\mathcal{X}, x_0) \to p^*\pi_n(\mathcal{X}, x_0)$ is an isomorphism.*

Again, we assume points are given by filtered colimits over neighbourhoods.

Proof. First of all, we see that

$$\begin{aligned} \Omega^n p^*\mathcal{X} &= \mathrm{Map}_+(S^n, \operatorname*{colim}_U \mathcal{X}(U)) \\ &= \operatorname*{colim}_U \mathrm{Map}_+(S^n, \mathcal{X}(U)) = p^*\Omega^n\mathcal{X}. \end{aligned}$$

Next, observe that $\pi_0(\Omega^n p^*\mathcal{X}) = \pi_n(p^*\mathcal{X}, p^*x_0)$, whereas $\pi_0(p^*\Omega^n\mathcal{X}) = p^*\pi_0(\Omega^n\mathcal{X}) = p^*\pi_n(\mathcal{X}, x_0)$. $\square$

Proposition 5.1.16. *Let $\mathbf{C}, \tau$ be a site such that for all $U \in \mathbf{C}$, the slice category $\mathbf{C}/U$ has enough points and let $f : \mathcal{X} \to \mathcal{Y}$ be a map of simplicial presheaves. Then f is a local weak equivalence if and only if $p^*f : p^*\mathcal{X} \to p^*\mathcal{Y}$ is a weak equivalence of simplicial sets for all points $p \in P$.*

Proof. First we can replace $\mathcal{X}$ and $\mathcal{Y}$ by globally equivalent simplicial presheaves taking values in Kan complexes.

The map $\pi_0(\mathcal{X}) \to \pi_0(\mathcal{Y})$ is an isomorphism of sheaves if and only if $\pi_0(p^*\mathcal{X}) \to \pi_0(p^*\mathcal{Y})$ is an isomorphism for all points.

Moreover, if $U \in \mathbf{C}$ is an object, then the induced site $\mathbf{C}/U$ has enough points, and any point of this category induces a point of $\mathbf{C}$ by composition. Let $x_0 \in \mathcal{X}(U)$, then $\pi_n(\mathcal{X}, x_0) \to \pi_n(\mathcal{Y}, f(x_0))$ is an isomorphism of sheaves on $\mathbf{C}/U$ if and only if for all points p, the map $\pi_n(p^*\mathcal{X}, p^*x_0) \to \pi_n(p^*\mathcal{Y}, p^*f(x_0))$ is an isomorphism. Therefore if p^*f is an equivalence for all p, the map f is a local weak equivalence. Conversely, if f is a local weak equivalence, then for any basepoint $x_0 \in p^*\mathcal{X}$, we can find some U such that x_0 lifts to $\tilde{x}_0 \in \mathcal{X}(U)$, and since the maps of sheaves $\pi_n(\mathcal{X}, \tilde{x}_0) \to \pi_n(\mathcal{Y}, f(\tilde{x}_0))$ are isomorphisms, the same holds for the stalks. $\square$

Corollary 5.1.17. *The associated sheaf functor $\mathcal{X} \to a\mathcal{X}$ is a natural local weak equivalence.*

Some sources prefer to work with $\mathbf{sPre}(\mathbf{Sm}_k)$ and others with $\mathbf{sShv}(\mathbf{Sm}_k)$. This corollary tells us that either will lead us to the same homotopy category.

Here's something that local weak equivalences do for us.

Construction 5.1.18. Suppose $\mathbf{C}, \tau$ is a site and that $\mathcal{U} = \{f_i : U_i \to X\}_{i \in I}$ is a covering family. Then we can make a simplicial object, the *nerve* of $\mathcal{U}$ in $\mathbf{sPre}(\mathbf{Sm}_k)$ as follows. If $\mathbf{i} = (i_0, \dots, i_n) \in I^n$ is an n-tuple of elements in the indexing set, then define

$$\mathcal{U}_{\mathbf{i}} = U_{i_0} \times_X U_{i_1} \times \cdots \times_X U_{i_n}.$$

Define

$$N\mathcal{U}_n = \coprod_{\mathbf{i} \in I^n} \mathcal{U}_{\mathbf{i}}.$$

This coproduct is taken in the category of presheaves on $\mathbf{C}$. These presheaves assemble to make a simplicial presheaf $N\mathcal{U}_\bullet$ in which the face maps are given by projections and the degeneracy maps by diagonals.

The object X yields a simplicial presheaf, having X in each dimension and having only identity maps. There is a map $N\mathcal{U}_\bullet \to X$, since each object appearing in the disjoint unions of each level of $N\mathcal{U}_\bullet$ are objects over X.

Proposition 5.1.19. *If $\mathcal{U} = \{f_i : U_i \to X\}_{i \in I}$ is a covering family, then the map $N\mathcal{U}_\bullet \to X$ is a local weak equivalence.*

The proof of this is on the homework.

5.1.2 The local injective structure

Theorem 5.1.20. *Let $\mathbf{C}, \tau$ be a site. Then there is a cofibrantly generated simplicial model structure, called the injective model structure on $\mathbf{sPre}(\mathbf{C})$ in which:*

- *the weak equivalences are the local weak equivalences,*
- *the cofibrations are the monomorphisms,*
- *the fibrations are defined by their lifting properties.*

We will not give the proof of this. It appears as [JSS15, Theorem 5.8]. I have also been negligent and have not defined something:

Definition 5.1.21. A model category $\mathbf{C}$ is *left proper* if the pushout of a weak equivalence along a cofibration is a weak equivalence. Dually, it is *right proper* if the pullback of a weak equivalence along a fibration is a weak equivalence. It is *proper* if it is both.

Remark 5.1.22. If every object in a model category is cofibrant, then Ken Brown's lemma implies that the structure is left proper.

Remark 5.1.23. The injective model structure on $\mathbf{sPre}(\mathbf{C})$ is proper.

Notation 5.1.24. A fibration in the injective local model structure will be called an *injective fibration*. The injective fibrations are somewhat difficult to work with.

Chapter 6

Left Bousfield Localization

Bousfield localization is a very useful idea in homotopy theory, to the extent that there is a whole book about it [Hir03]. The big idea is the following: suppose we have been given a model category $\mathbf{M}$ equipped with a class of weak equivalences $\mathbf{W}$, cofibrations $\mathbf{Cof}$ and fibrations $\mathbf{Fib}$. Suppose $\mathcal{A}$ is a set of maps in $\mathbf{M}$ that we would like to make into weak equivalences. Then there is a universal model category, the *left Bousfield localization* of $\mathbf{M}$ at $\mathcal{A}$, which is another model structure on $\mathbf{M}$, having the same category of cofibrations, such that the maps in $\mathcal{A}$ have been turned into weak equivalences (along with the maps in $\mathbf{W}$) and with a smaller category of fibrations. Write $L_{\mathcal{A}}\mathbf{M}$ for this model category—it has the same underlying category as $\mathbf{M}$. Then the identity functors give us a Quillen adjunction

$$\mathrm{id} : \mathbf{M} \rightleftarrows L_{\mathcal{A}}\mathbf{M} : \mathrm{id}$$

and any Quillen adjunction $\phi : \mathbf{M} \to \mathbf{N}$ such that the maps in $\mathcal{A}$ are sent to weak equivalences in $\mathbf{N}$ factors through $L_{\mathcal{A}}\mathbf{M}$.

6.1 Definitions

Definition 6.1.1. Let $\mathbf{M}$ be a simplicial model category and A a set of maps in $\mathbf{M}$. An object $m \in \mathbf{M}$ is *A-local* if, for all maps $f : a \to a'$ in A, the induced map

$$\mathcal{S}(Qa', Rm) \to \mathcal{S}(Qa, Rm)$$

is a weak equivalence.

Remark 6.1.2. Variations on this definition exist. For instance, [Hir03] requires m to be fibrant, but [MV99] do not. We can also use other mapping spaces to make this definition: this allows [Hir03] to make the definition for non-simplicial model categories by constructing a different kind of mapping object.

Definition 6.1.3. Let $\mathbf{M}$ be a simplicial model category and A a set of maps in $\mathbf{M}$. A map $f : x \to y$ is an *A-equivalence* if the induced map

$$\mathcal{S}(Qy, Rm) \to \mathcal{S}(Qx, Rm)$$

is a weak equivalence for all A-local objects m.

Remark 6.1.4. The A-equivalences include the weak equivalences, have the 2-out-of-3 property and are closed under retracts.

Definition 6.1.5. Let $\mathbf{M}$ be a simplicial model category and A a set of maps in $\mathbf{M}$. We can define the *left Bousfield localization* of $\mathbf{M}$ at A to be a model category $\mathrm{L}_A\mathbf{M}$ having the same underlying category as $\mathbf{M}$, and the same cofibrations. The weak equivalences of $\mathrm{L}_A\mathbf{M}$ are the A-equivalences. The fibrations are defined to be the maps having the r.l.p. w.r.t. trivial cofibrations.

Proposition 6.1.6. *In $\mathrm{L}_A\mathbf{M}$, the morphisms of A are weak equivalences.*

The proof is obvious.

Proposition 6.1.7. *Let $\mathbf{M}$ be a simplicial model structure and A a set of maps. Then the identity functor in each direction yields a Quillen adjunction*

$$\mathrm{id} : \mathbf{M} \rightleftarrows \mathrm{L}_A\mathbf{M} : \mathrm{id}.$$

The proof is obvious.

Proposition 6.1.8. *Let $\mathbf{M}$ be a simplicial model structure and A a set of maps. Then $\mathrm{L}_A\mathbf{M}$ is a simplicial model structure, the tensor, cotensor and simplicial mapping object being the same as in $\mathbf{M}$.*

Proof. The underlying categories of $\mathbf{M}$ and $\mathrm{L}_A\mathbf{M}$ are the same, so the required adjunctions all hold. What remains to be proved is that the various adjunctions are Quillen adjunctions.

Fix a cofibrant object $X \in \mathrm{L}_A\mathbf{M}$. We wish to show that the functors

$$\cdot \otimes X : \mathbf{sSet} \rightleftarrows \mathrm{L}_A\mathbf{M} : \mathcal{S}(X, \cdot)$$

form a Quillen adjoint pair. To do this, it suffices to show that $\cdot \otimes X$ preserves cofibrations and trivial cofibrations. There is nothing to do here.

Fix a simplicial set K. We wish to show that the functors

$$K \otimes \cdot : \mathrm{L}_A\mathbf{M} \rightleftarrows \mathrm{L}_A\mathbf{M} : (\cdot)^K$$

form a Quillen adjunction. To do this, it suffices to show that $K \otimes \cdot$ preserves cofibrations (already done) and trivial cofibrations. This last step is not already done, because trivial cofibrations are not the same in $\mathrm{L}_A\mathbf{M}$ as they were in $\mathbf{M}$, there are more. Therefore, suppose $X \to Y$ is a trivial cofibration in $\mathrm{L}_A\mathbf{M}$. We want to show that $K \otimes X \to K \otimes Y$ is an A-equivalence. To do this, we test (a cofibrant replacement of) it against (the fibrant replacement of) an arbitrary A-local object RZ.

$$\begin{gathered} \mathcal{S}(K \otimes QY, RZ) \to \mathcal{S}(K \otimes QX, RZ) \\ \mathrm{Map}(K, \mathcal{S}(QY, RZ)) \to \mathrm{Map}(K, \mathcal{S}(QX, RZ)) \end{gathered}$$

where the two lines are equivalent by adjunction. The second line consists of mapping spaces in $\mathbf{sSet}$, and by hypothesis the target objects are weakly equivalent simplicial sets. They are Kan complexes by a diagram chase we probably should have considered long ago, but we leave here as an exercise. $\square$

The difficulty with localizations is that we lose control over the fibrant objects.

Proposition 6.1.9. *Suppose* $\mathbf{M}$ *is a simplicial model category, and* A *is a set of maps in* $\mathbf{M}$*. Then an object* X *of* $\mathrm{L}_A\mathbf{M}$ *is fibrant if and only if* X *is fibrant in* $\mathbf{M}$ *and* A*-local.*

Proof. First of all, suppose W is fibrant in $\mathrm{L}_A\mathbf{M}$. Then X is fibrant in $\mathbf{M}$. Let us show it is A-local. Suppose $f : X \to Y$ is a map in A. This is a weak equivalence in $\mathrm{L}_A\mathbf{M}$, and so is $Qf : QX \to QY$, the cofibrant replacement. Since W is fibrant in $\mathrm{L}_A\mathbf{M}$, and the model structure is simplicial, the functor $\mathcal{S}(\cdot, W)$ sends trivial cofibrations to trivial fibrations. By Ken Brown's lemma, it sends weak equivalences between cofibrant objects to weak equivalences. So $\mathcal{S}(QY, W) \to \mathcal{S}(QX, W)$ is a weak equivalence, indicating that W is A-local.

Conversely, suppose W is A-local and fibrant in $\mathbf{M}$. We wish to show $W \to \mathrm{pt}$ has the r.l.p. w.r.t. maps $X \to Y$ that are cofibrations and A-equivalences. An unassigned exercise (a diagram) implies it's sufficient to establish this when $X \to Y$ is a cofibration of cofibrant objects and an A-equivalence. But then $\mathcal{S}(Y, W) \to \mathcal{S}(X, W)$ is a fibration and, since W is A-local, also a weak equivalence of simplicial sets. In particular, it is surjective on 0-simplices. That implies that any map $Y \to W$ admits a lift to a map $X \to W$, as required. $\square$

Remark 6.1.10. Many other things can be said about left Bousfield localization. All of [Hir03, Chapter 3] is on this topic, as is much of the book.

6.2 Homotopy limits and colimits

Example 6.2.1. Consider the diagram $\mathrm{pt} \to I \leftarrow \mathrm{pt}$ including a point at each end of the closed unit interval. A moment's thought shows that the limit of this diagram is the empty space. Now, replace either of the two inclusions by the identity map $\mathrm{id} : I \to I$, which is isomorphic in the homotopy category. Now the limit consists of a single point.

This shows that the homotopy type of a limit can depend on more than the homotopy classes of the objects.

Notation 6.2.2. If $\mathbf{I}$ is a category, and $i \in \mathbf{I}$ an object, the notation $\mathbf{I}/i$ denotes the *slice category* of $\mathbf{I}$ over i, where the objects of $\mathbf{I}/i$ are morphisms $f : j \to i$ and the morphisms of $\mathbf{I}/i$ are commuting triangles.

Notation 6.2.3. If $\mathbf{C}$ is a small category, then $N\mathbf{C}$ denotes the nerve of the category. This is a simplicial set where the n simplicies consist of n-tuples $c_0 \to c_1 \to \cdots \to c_n$.

Example 6.2.4. For instance, if $\mathbf{C}$ is the cospan $\bullet \to \bullet \leftarrow \bullet$, then $N\mathbf{C}$ is a simplicial set with three 0-simplices and two nondegenerate 1-simplices.

Remark 6.2.5. Suppose $F : \mathbf{C} \to \mathbf{D}$ is a functor between small categories. Then F induces a map, NF, of simplicial sets.

You can prove that $N(\mathbf{C} \times \mathbf{C}') \approx N\mathbf{C} \times N\mathbf{C}'$. There is no unexpected step.

If $\nu : F \to G$ is a natural transformation of functors, then ν can be conceptualized as a functor $\nu : catC \times (0 \to 1) \to \mathbf{D}$. Here $\nu(x, 0) = F(x)$ and $\nu(x, 1) = G(x)$ and $\nu(f : x \to y, 0 \to 1)$ is the morphism $\nu_y \circ F(f)$ (or $G(f) \circ \nu_x$, these agree by naturality). Therefore ν induces a homotopy between NF and NG.

Corollary 6.2.6. *If* $\mathbf{C}$ *is a small category with an initial or a final object, then* $N\mathbf{C}$ *is contractible.*

Proof. Since $N\mathbf{C} \approx N\mathbf{C}^{\mathrm{op}}$, it suffices to do one case. Let us suppose $\emptyset$ is an initial object of $\mathbf{C}$. Then the constant functor $\emptyset : \mathbf{C} \to \mathbf{C}$ has a natural transformation to $\mathrm{id} : \mathbf{C} \to \mathbf{C}$. Therefore $N\mathrm{id} = \mathrm{id}$ is homotopic to the constant map at $\emptyset$. $\square$

Construction 6.2.7. This construction shows how to produce the *homotopy limit* of a diagram in a simplicial model category. It is a kind of derived limit, as can be made precise (but will not be made precise here, because we haven't got forever).

Let $\mathbf{M}$ be a simplicial model category and let $\mathbf{I}$ be a small category. Let $X : \mathbf{I} \to \mathbf{M}$ be a diagram. The *homotopy limit*, $\operatorname{holim}_{\mathbf{I}} X$ is defined as an equalizer of two maps:

$$\prod_{i \in \mathbf{I}} (X_i)^{N(\mathbf{I}/i)} \rightrightarrows \prod_{i \to j} (X_j)^{N(\mathbf{I}/i)}$$

The two maps are reasonably obvious: given any $i \to j$, we can get two maps, the first by composing $X_i \to X_j$:

$$X_i^{N(\mathbf{I}/i)} \to X_j^{N(\mathbf{I}/i)}$$

and the second because there is an evident "forgetful" functor $\mathbf{I}/i \to \mathbf{I}/j$, and therefore a contravariant map

$$X_j^{N(\mathbf{I}/j)} \to X_j^{N(\mathbf{I}/i)}.$$

Then we can assemble these to get an equalizer diagram as asserted.

Remark 6.2.8. If you replace $N(\mathbf{I}/i)$ in this definition by pt, you get a construction of the ordinary limit. This gives us a natural transformation for any diagram $\lim D \to \operatorname{holim} D$. This map is generally not an equivalence.

Construction 6.2.9. The *homotopy colimit* is exactly dual. That is

Let $\mathbf{M}$ be a simplicial model category and let $\mathbf{I}$ be a small category. Let $X : \mathbf{I} \to \mathbf{M}$ be a diagram. The *homotopy colimit*, $\operatorname{hocolim}_{\mathbf{I}} X$ is defined as a coequalizer of two maps:

$$\coprod_{i \to j} (X_i) \otimes N(j/\mathbf{I})^{\mathrm{op}} \rightrightarrows \coprod_{i \in \mathbf{I}} (X_j) \otimes N(i/\mathbf{I})^{\mathrm{op}}$$

Remark 6.2.10. Dually to the case of limits, there is a map from $\operatorname{hocolim} D \to \operatorname{colim} D$.

Remark 6.2.11. These definitions have been taken from [Hir03, Chapter 18]. They are based on [BK72, Ch XI], from what is probably the most influential book in homotopy theory.

Some sources require all the objects in a homotopy limit construction to be fibrant, and all objects in a hocolim to be cofibrant, but we do not.

Example 6.2.12. The majority of homotopy limits and colimits encountered, like the majority of ordinary limits and colimits, are pushouts and pullbacks.

Here is an unwinding of the homotopy limit of a diagram $X \xrightarrow{f} Y \xleftarrow{g} Z$. For concreteness, assume this is a diagram of simplicial sets, but the idea works in any simplicial category. This is a functor from a category $0 \to 1 \leftarrow 2$. Now let us consider the nerves of $\mathbf{I}/i$ for each of these objects. For 0 and 2, the nerve is $\Delta[0]$, whereas for 1 it consists of the simplicial set $L := 0 \to 1 \leftarrow 2$. (We remark in passing that these are all contractible).

The homotopy pullback of $X \to Y \leftarrow Z$ is an equalizer. That is, it is a subobject of the triple $X \times Y^L \times Z$. The precise subobject is not hard to explain. Fix an n. An n-simplex $x \in X$ maps to $f(x) \in Y$ and a simplex $z \in Z$ maps to $g(z) \in Z$. An n-simplex of Y^L is a kind of 'path' of n-simplices in Y, parametrized by $0 \to 1 \leftarrow 2$, in particular it has a 0-end and a 2-end. An n-simplex of the homotopy pullback is the data of an n-simplex x of X, an n-simplex z of Z and an L-paramatrized path from x to z.

Example 6.2.13. Entirely dually, the homotopy colimit of

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \\ Y & \end{tikzcd}$$

i.e., the *homotopy pushout*, is given by a two-sided mapping cylinder.

Definition 6.2.14. If X and Y are two objects in a simplicial model category, the *join* $X * Y$ of X and Y is the homotopy pushout of the diagram

$$\begin{tikzcd} X \times Y \arrow[r] \arrow[d] & Y \\ X & \end{tikzcd}$$

in which both maps are projection maps.

Here is an important feature of homotopy (co)limits that we will not prove.

Theorem 6.2.15. *Let $D, D' : \mathbf{I} \to \mathbf{M}$ be two diagrams of fibrant objects in a simplicial model category, and let $\Phi : D \to D'$ be a natural transformation between them that is an objectwise weak equivalence. Then the induced map $\operatorname{holim} D \to \operatorname{holim} D'$ is a weak equivalence.*

You can find this in [Hir03, Section 18.5].
Here is another useful result.

Proposition 6.2.16. *Suppose $\mathbf{M}$ is a simplicial model category. Suppose $X : \mathbf{I} \to \mathbf{M}$ is a small diagram and $Y \in \mathbf{M}$. Then there is a natural isomorphism of simplicial sets*

$$\mathcal{S}(\operatorname*{hocolim}_{\mathbf{I}} X_i, Y) \xrightarrow{\cong} \operatorname*{holim}_{\mathbf{I}^{op}} \mathcal{S}(X_i, Y)$$

Proof. The proof is an exercise in adjunctions and unravelling the definition. □

Notation 6.2.17. Let $\mathbf{M}$ be a (simplicial) model category. Say that a diagram D in $\mathbf{M}$ is *homotopy commutative* if the image of D in $\operatorname{Ho}\mathbf{M}$ is commutative. Other sources may use this term differently, so beware.

Example 6.2.18. Suppose

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

is a homotopy commutative diagram such that D is fibrant and A is cofibrant. Then the homotopy commutativity condition amounts to the statement that there is a (left) homotopy $\phi : A \otimes \Delta[1] \to D$ from $A \to B \to D$ to $A \to C \to D$. Fix ϕ.

Then we have maps $A \to B$ and $A \to C$ and a map $A \otimes \Delta[1] \to D$. This third map can easily be extended to a map $A \otimes N(\bullet \leftarrow \bullet \to \bullet) = A \otimes \Lambda^0[2]$ that is ϕ on one of the legs and a degeneracy of the constant map $A \to B \to D$ on the other (i.e., it's a trivial homotopy).

We have exactly the data now to construct a map $A \to \mathrm{holim}(B \to D \leftarrow C)$. If this map is a weak equivalence, then we say the original diagram is *homotopy cartesian*.

The theory of *homotopy cocartesian* diagrams is entirely dual.

Remark 6.2.19. The geometric realization of a homotopy cartesian diagram of simplicial sets is homotopy cartesian.

Proposition 6.2.20. *Suppose*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d, "f"] \\ C \arrow[r, "g"] & D \end{tikzcd}$$

is a homotopy cartesian diagram of pointed spaces (or pointed simplicial sets). Then there is a homotopy Mayer–Vietoris sequence

$$\dots \longrightarrow \pi_i(A, a_0) \longrightarrow \pi_i(B, b_0) \oplus \pi_i(C, c_0) \xrightarrow{f_* \oplus -g_*} \pi_{i-1}(\Omega D, d_0) \longrightarrow \dots$$

Proof. This is in the homework. □

6.3 Fibrancy conditions

6.3.1 Global injective, objectwise, and local fibrations

For concreteness, everything in this section is about $\mathbf{Sm}_k$ with the Nisnevich topology, unless otherwise stated.

There are four concepts of "fibrancy". They are listed here in strictly increasing order of strength. Any map having a property in the list has all preceding properties:

1. local fibration (see Definition 6.3.5)

2. objectwise fibration (see Definition 6.3.3)

3. global injective fibration (see Definition 6.3.1)

4. (local) injective fibration (see Notation 5.1.24)

While no two of these classes of map agree, the lower two classes are somewhat obscure, and examples are hard to come by.

Definition 6.3.1. A map $f : \mathcal{X} \to \mathcal{Y}$ is a *global injective fibration* (as distinct from a local injective fibration, simply called an injective fibration) if it has the right lifting property w.r.t. maps that are cofibrations (i.e., monomorphisms) and global weak equivalences.

Proposition 6.3.2. *A local injective fibration is a global injective fibration.*

Definition 6.3.3. A map $f : \mathcal{X} \to \mathcal{Y}$ is a *sectionwise*, *projective* or *objectwise fibration* if $f(U) : \mathcal{X}(U) \to \mathcal{Y}(U)$ is a fibration for all U.

Proposition 6.3.4. *Suppose $f : \mathcal{X} \to \mathcal{Y}$ is a global injective fibration, and $U \in \mathbf{Sm}_k$. Then f is an objectwise fibration.*

Proof. We want to show that $f(U) : \mathcal{X}(U) \to \mathcal{Y}(U)$ has the r.l.p. w.r.t. horn inclusions.

The Yoneda lemma implies that $\mathcal{S}(\eta_U, \mathcal{X}) = \mathcal{X}(U)$. By adjunction, any diagram

$$\begin{tikzcd}
\Lambda^r[n] \arrow[r] \arrow[d] & \mathcal{X}(U) \arrow[d] \\
\Delta[n] \arrow[r] & \mathcal{Y}(U)
\end{tikzcd}$$

is equivalent to

$$\begin{tikzcd}
\Lambda^r[n] \times \eta_U \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\
\Delta[n] \times \eta_U \arrow[r] \arrow[ur, dashed] & \mathcal{Y}
\end{tikzcd}$$

where we have the lifting because the left hand arrow is an objectwise equivalence and a cofibration. Therefore by adjunction the hoped-for lift exists. $\square$

Definition 6.3.5. Suppose $f : \mathcal{X} \to \mathcal{Y}$ is a map of simplicial presheaves. We say f is a *local fibration* if, for any covering $\{g_i : U_i \to Z\}_{i \in I}$, and any horns $\Lambda^j[n] \to \Delta[n]$, you can find lifts

$$\begin{tikzcd}
\Lambda^j[n] \arrow[r] \arrow[d] & \mathcal{X}(Z) \arrow[r, "g_i^*"] & \mathcal{X}(U_i) \arrow[d] \\
\Delta[n] \arrow[r] \arrow[urr, dashed] & \mathcal{Y}(Z) \arrow[r, "g_i^*"] & \mathcal{Y}(U_i)
\end{tikzcd} .$$

We say it f is a *local trivial fibration* if it has the analogous lifting property as above with $\Lambda^j[n] \to \Delta[n]$ replaced by $\partial\Delta[n] \to \Delta[n]$.

Remark 6.3.6. These are weaker condition than being objectwise fibrations or objectwise trivial fibrations respectively.

Here is a theorem that is somewhat surprising, since it says that a condition you might regard as weak (local trivial fibration) implies local weak equivalence.

Theorem 6.3.7. *A map $q : \mathcal{X} \to \mathcal{Y}$ of simplicial presheaves is a local weak equivalence and a local fibration if and only if it is a local trivial fibration.*

You can find this topic covered extensively in [JSS15, Section 4.2]. For instance, the Theorem above is Theorem 4.32.

6.3.2 The Brown–Gersten property

As I will explain later, there are variations on this idea.

Definition 6.3.8. A simplicial presheaf $\mathcal{Y}$ has the *Brown–Gersten property* or *BG property* if $\mathcal{Y}(\emptyset) \simeq *$ and for any elementary distinguished square

$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] & V \arrow[d, "\text{ét}"] \\ U \arrow[r] & X \end{tikzcd}$$

the diagram

$$\begin{tikzcd} \mathcal{Y}(X) \arrow[r] \arrow[d] & \mathcal{Y}(V) \arrow[d] \\ \mathcal{Y}(U) \arrow[r] & \mathcal{Y}(X) \end{tikzcd}$$

is homotopy cartesian.

Remark 6.3.9. If $\mathcal{Y}$ is a simplicial sheaf, then the first condition is automatically satisfied.

Remark 6.3.10. If $\mathcal{Y}$ above is a simplicial sheaf, then $\mathcal{Y}(\cdot)$ applied to an elementary Nisnevich square yields a cartesian (but not necessarily homotopy cartersian) diagram. If, however, $\mathcal{Y}$ has the property that it is a simplicial sheaf and also that $\mathcal{Y}(X) \to \mathcal{Y}(U)$ is a fibration of fibrant objects whenever $U \to X$ is a Zariski open embedding, then the cartesian square in question is homotopy cartesian and so $\mathcal{Y}$ has the BG property (note that for a sheaf, $\mathcal{Y}(\emptyset) = *$).

The map $U \to X$ (or $\eta_U \to \eta_X$, if you prefer) is a monomorphism, and therefore a cofibration. Consequently if $\mathcal{Y}$ is injective fibrant, then the induced map $\mathcal{Y}(X) = \mathcal{S}(\eta_X, \mathcal{Y}) \to \mathcal{S}(\eta_U, Y) = \mathcal{Y}(U)$ is a fibration of fibrant objects. In particular, we see that injective fibrant objects have the BG property.

For the following proposition, we need a small technical lemma, which is essentially [Jar87, Corollary 2.7].

Lemma 6.3.11. *There is a model structure on the category of simplicial Nisnevich sheaves on $\mathbf{Sm}_k$ such that the associated sheaf functor $a_{Nis} : \mathbf{sPre}(\mathbf{Sm}_k) \to \mathbf{sShv}(\mathbf{Sm}_k)$ is a left Quillen equivalence, and the forgetful functor v is the right Quillen adjoint equivalence.*

Remark 6.3.12. This implies that we can use a composite functor vRa_{Nis} as a fibrant replacement for our simplicial presheaves, so that we may assume that the fibrant replacements are actually simplicial sheaves.

Proposition 6.3.13. *A simplicial presheaf $\mathcal{Y}$ has the BG property if and only if any local weak equivalence $\mathcal{Y} \to \mathcal{Y}'$, where $\mathcal{Y}'$ has the BG property, induces an objectwise equivalence $\mathcal{Y} \to \mathcal{Y}'$.*

Remark 6.3.14. This is a very useful result because the BG property does arise in practice and can be checked. To say $\mathcal{Y} \to R\mathcal{Y}$ is an objectwise weak equivalence is the next best thing to understanding $R\mathcal{Y}$ entirely. While for a particular model of $R\mathcal{Y}$, we might not know $R\mathcal{Y}(U)$ explicitly, we know it is weakly equivalent to $\mathcal{Y}(U)$.

Proof of Proposition 6.3.13. For the 'if' direction: Note that we can always find a local weak equivalence $\mathcal{Y} \to R\mathcal{Y}$ where the target has the BG property—just take fibrant replacement and refer to Remark 6.3.10. If this is an objectwise equivalence, then because $R\mathcal{Y}$ has the BG property, so too does $\mathcal{Y}$.

The 'only if' direction takes work, and the full proof will only be sketched here. The full proof is drawn from [MV99, pp. 100-102] and [BG73, Theorem 1'].

It is necessary to work from time to time in $\mathbf{Sm}_k/S$ where S is a variety. This category carries a Nisnevich topology, of course. The BG property can also be defined here, and will be called the "BG property over a base S".

Suppose we have a weak equivalence $\mathcal{Y} \to \mathcal{Y}'$ of simplicial presheaves where both source and target have the BG property. We want to show that this yields an objectwise equivalence after sheafification.

First, we may assume $\mathcal{Y} \to \mathcal{Y}'$ is objectwise a Kan fibration of fibrant simplicial sets, and that both have the property that $\mathcal{Y}(\emptyset) = * = \mathcal{Y}'(\emptyset)$. The argument is to replace $\mathcal{Y}'$ by the functor

$$U \mapsto \operatorname{Sing} |\mathcal{Y}'(U)|$$

which is Kan, and takes $\emptyset$ to pt if $\mathcal{Y}'$ does. So we may assume $\mathcal{Y}'(U)$ is a Kan complex for all U.

For each U, consider a functorial factorization of the maps $\mathcal{Y}(U) \overset{\sim}{\rightarrowtail} \mathcal{Y}''(U) \twoheadrightarrow \mathcal{Y}'(U)$. Then replace $\mathcal{Y}$ by $\mathcal{Y}''$.

Lemma 6.3.15. *Suppose $\mathcal{F}$ is a simplicial presheaf having the BG property over a base S and such that $\mathcal{F}$ is Nisnevich locally contractible. Then $\mathcal{F}(X)$ is contractible provided it is not empty.*

Proof. The proof requires a detour into scheme theory; that is, I spent a lot of time trying to explain it using varieties exclusively, and I failed.

The proof is by induction on the dimension of X. If X has dimension 0, then X is a (disjoint union of) fields. The scheme $\operatorname{Spec} E$ where E is a field is a point for the Nisnevich topology, and so $\mathcal{F}(X) \simeq \prod * = *$.

Then an argument in [MV99, Lemma 1.17, p. 101] says that $\mathcal{F}$ is Zariski-locally contractible. This is where the scheme theory is needed, since it refers to $\operatorname{Spec} \mathcal{O}_{X,x} \setminus \{x\}$, which is not a variety as a rule. But the main idea is induction on the dimension of X and the use of the BG property.

Once we know $\mathcal{F}$ is Zariski-locally contractible, an argument of [BG73, Theorem 1'] finishes the proof. The argument is not difficult, but it is long. □

Lemma 6.3.16. *Suppose $\mathcal{F}$ is a simplicial presheaf having the BG property over a base S, such that $\mathcal{F}(\emptyset) = *$ and such that $\mathcal{F}$ is Nisnevich locally contractible. Then $\mathcal{F}(X)$ is contractible for all X.*

Proof. In light of the previous lemma, it suffices to show that $\mathcal{F}(X)_0$ is not empty.

Let $U \subseteq X$ be a maximal open subvariety such that $\mathcal{F}(U) \neq \emptyset$. Suppose for the sake of contradiction that s is the generic point of an irreducible component of $X \setminus U$. Since the stalk of $\mathcal{F}$ at s is contractible, there must be some Nisnevich neighbourhood V of $s \in X$ such that $\mathcal{F}(V)_0 \neq \emptyset$. That is, $V \to X$ is an étale map that has a section on some open subvariety W of $\overline{\{s\}}$. Consider $U \cup W$

and V', the restriction of V to $U \cup W$. Now there is an elementary distinguished square

$$\begin{tikzcd} U \times_{U \cup W} V' \arrow[r] \arrow[d] & V' \arrow[d] \\ U \arrow[r] & U \cup W \end{tikzcd}$$

and applying $\mathcal{F}$ gives us (up to homotopy)

$$\begin{tikzcd} \mathcal{F}(U \times_{U \cup W} V') \arrow[r] \arrow[d] & \mathcal{F}(V') \simeq * \arrow[d] \\ \mathcal{F}(U) \simeq * \arrow[r] & \mathcal{F}(U \cup W) \end{tikzcd}$$

Note that $\mathcal{F}(U \times_{U \cup W} V') \simeq *$, since it cannot be empty (it admits a map from a nonempty object). Then the homotopy-pullback square implies that $\mathcal{F}(U \cup W) \simeq *$, contradicting the maximality of U. So $U = X$. □

A more honest account than I have given here of all this can be found in [JSS15, Theorem 5.37].

We can finish the proof of the proposition. Suppose $f : \mathcal{Y} \to \mathcal{Y}'$ is an objectwise Kan fibration of Kan complexes, and suppose both source and target have the BG property. Let $y \in \mathcal{Y}'(S)_0$ for some S and consider the pullback diagram

$$\begin{tikzcd} \mathcal{F}_y \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d] \\ * \arrow[r, "y"] & \mathcal{Y}' \end{tikzcd}$$

of simplicial presheaves defined on $\mathbf{Sm}_k/S$. A diagram chase shows that $\mathcal{F}(\cdot)$ satisfies the BG property over S, and an argument at points shows that $\mathcal{F}_y$ is Nisnevich-locally contractible. Therefore $\mathcal{F}_y(S) \simeq *$ by Lemma 6.3.16—notably, $\mathcal{F}_y(S) \neq \emptyset$. First, this implies that the induced map $\pi_0\mathcal{Y}(S) \to \pi_0\mathcal{Y}'(S)$ is surjective, since for any basepoint $y \in \mathcal{Y}'(S)$ of the target, there is some basepoint in the fibre.

If you fix any basepoint $x \in \mathcal{Y}(S)$, then the homotopy long exact sequence of a fibration $\mathcal{F}_{f(x)}(S) \to \mathcal{Y}(S) \to \mathcal{Y}'(S)$ shows that the induced map on all homotopy groups $\pi_n(\mathcal{Y}(S), x) \to \pi_n(\mathcal{Y}'(S), f(x))$ (and π_0) is an isomorphism, which is what we wanted to show.

□

Corollary 6.3.17. *Let $\mathcal{X}$ be a simplicial presheaf on $\mathbf{Sm}_k$. Then the following are equivalent:*

- *$\mathcal{X}$ is globally injective fibrant and satisfies the Brown–Gersten condition;*
- *$\mathcal{X}$ is (locally) injective fibrant.*

Proof. We already know the second implies the first.

Now suppose $\mathcal{X}$ is globally injective fibrant, and satisfies BG. Then $\mathcal{X} \to R_{Nis}\mathcal{X}$ is an objectwise equivalence.

Suppose $i : \mathcal{A} \to \mathcal{B}$ is a locally trivial cofibration. Since i is a cofibration, the induced map $\mathcal{S}(\mathcal{B}, \mathcal{X}) \to \mathcal{S}(\mathcal{A}, \mathcal{X})$ is a fibration (use the simplicial structure for the global injective model structure).

Similarly the maps $\mathcal{S}(A, \mathcal{X}) \to \mathcal{S}(A, \mathcal{R}_{Nis} X)$ and similarly for $\mathcal{B}$ are weak equivalences, since $\mathcal{X} \to \mathcal{R}_{Nis}\mathcal{X}$ is an equivalence of globally injective fibrant objects. Finally, $\mathcal{S}(\mathcal{B}, R_{Nis}\mathcal{X}) \to \mathcal{S}(\mathcal{A}, R_{Nis}\mathcal{X})$ is an equivalence, because $\mathcal{A} \to \mathcal{B}$ is a local equivalence and $R_{Nis}\mathcal{X}$ is locally fibrant.

From the diagram

$$\begin{tikzcd}
\mathcal{S}(\mathcal{B}, \mathcal{X}) \arrow[r, "\sim"] \arrow[d, two heads] & \mathcal{S}(\mathcal{B}, R\mathcal{X}) \arrow[d, two heads, "\sim"] \\
\mathcal{S}(\mathcal{A}, \mathcal{X}) \arrow[r, "\sim"] & \mathcal{S}(\mathcal{A}, R\mathcal{X})
\end{tikzcd}$$

we see that $\mathcal{S}(shB, \mathcal{X}) \to \mathcal{S}(\mathcal{A}, \mathcal{X})$ is a trivial fibration, and consequently induces a surjection on 0-simplices. Therefore any map $\mathcal{A} \to \mathcal{X}$ admits an extension to $\mathcal{B} \to \mathcal{X}$, so $\mathcal{X}$ is injective fibrant. $\square$

Remark 6.3.18. The BG condition has variations. What we described here is a BG condition for any elementary Nisnevich square, and the conclusion we came to that $\mathcal{X} \to R_{Nis}\mathcal{X}$ induces an equivalence $\mathcal{X}(U) \to R_{Nis}\mathcal{X}(U)$ for all objects.

In [MV99], they actually define BG for what they call "Brown–Gersten" classes of objects. For instance, one can restrict attention to diagrams where all objects are Zariski open subvarieties of affine varieties (*quasi-affine varieties*). If a sheaf $\mathcal{X}$ has the quasi-affine BG property, then the map $\mathcal{X}(U) \to R_{Nis}\mathcal{X}(U)$ is an equivalence provided U is quasi-affine.

A similar result appears in [AHW17, Theorem 3.3.4]. The terminology used there is different from the terminology we use here, but in our language they prove:

Suppose $\mathcal{X}$ satisfies the Brown–Gersten condition for elementary Nisnevich squares where all terms are affine. Then $\mathcal{X}(\operatorname{Spec} R) \to R_{Nis}\mathcal{X}(\operatorname{Spec} R)$ is an equivalence for all affine varieties $\operatorname{Spec} R$. This is more delicate than what we proved, because they can't freely pass to open subobjects.

Example 6.3.19. The original example of the Brown–Gersten condition is in [BG73, Proposition 4] (there they call it "pseudo-flasque"), where it is applied in the Zariski topology to a simplicial sheaf P with the property that $\pi_i(P(X)) = K_{i-1}(X)$ where X is a smooth variety (they consider nonsmooth objects as well, so they work with G-theory).

The main point of Nisnevich's seminal paper [Nis89] is to show that P, as considered by Brown–Gersten, has the BG property for what is now called the Nisnevich topology, but was there called the *cd-topology*. The reference is to [Nis89, Example 4.5]. By 1989, when this paper was written, people had moved on from sheaves of spaces to define K-theory to sheaves of spectra, but the result is really the same.

Remark 6.3.20. There is an objectwise injective model structure on $\mathbf{sPre}(\mathbf{Sm}_k)$, as we observed above. It's the "local" model structure for the trivial topology.

The reason all this material has been placed in the chapter on Bousfield localization is because of the following result [Isa05, Theorem 4.9].

The injective local model structure on $\mathbf{sPre}(\mathbf{Sm}_k)$ is the left Bousfield localization of the injective objectwise model structure at the class of maps

$$U \coprod_{U \times_X V} V \to X$$

where $U \to X$ and $V \to X$ form an elementary Nisnevich square.

Chapter 7

The Dold–Kan Correspondence

Our references for this material are [JSS15] and [Wei94, Chapter 8.4]. A pleasant account of the case of abelian groups is to be found in [GJ99, Section III.2].

7.1 Generalities

At first, what we have to say works for any site, $\mathbf{C}$. I will explain the theory for $\mathbb{Z}$-modules, i.e., abelian groups, but variations exist for any ring R in place of $\mathbb{Z}$, or even any presheaf R of rings on $\mathbf{C}$.

Notation 7.1.1. Let $\mathbf{Ab}(\mathbf{C})$ denote the category of presheaves of $\mathbb{Z}$-modules on $\mathbf{C}$. This is an abelian category.

Notation 7.1.2. Let $\mathbf{sAb}(\mathbf{C})$ denote the category of simplicial presheaves of abelian groups. This category admits two descriptions: either as simplicial presheaves where all presheaves of n-simplices are abelian groups and the face and degeneracy maps are homomorphisms, or as abelian-group objects in $\mathbf{sPre}(\mathbf{C})$.

There is also a sheaf version of this, $\mathbf{Sh}(\mathbf{C})$.

Notation 7.1.3. Suppose $\mathbf{A}$ is an abelian category Let $\mathbf{Ch}_+(\mathbf{A})$ denote the category of non-negatively graded chain complexes in $\mathbf{A}$.

Construction 7.1.4. There is a free abelian group functor

$$\mathbb{Z} : \mathbf{sPre}(\mathbf{C}) \to \mathbf{sAb}(\mathbf{C})$$

taking a simplicial presheaf X to the presheaf $\mathbb{Z}X$ where the n-simplices are the free abelian group generated by X_n.

Write $\tilde{\mathbb{Z}}X$ for the associated sheaf to $\mathbb{Z}X$.

Construction 7.1.5. Suppose $\mathbf{A}$ is an abelian category. Consider a simplicial object $X_\bullet \in \mathbf{sA}$.

We can form a chain complex $NX \in \mathbf{Ch}_+(\mathbf{A})$. called the *normalized chain complex* of $X_\bullet$ in the following way:

First, form the *unnormalized chains* CX by setting $CX_n = X_n$ for all n, and defining a differential

$$\hat{d}_n = \sum_{i=0}^{n} (-1)^i d_i : X_n \to X_{n-1}$$

You may wonder how you know this is a chain complex (i.e., that $\hat{d}_n \circ \hat{d}_{n+1} = 0$, but this is an elementary exercise. What's more, you've already seen or done this, in defining singular homology.

Then define $NX_n \subseteq CX_n$ by

$$NX_n = \bigcap_{i=0}^{n-1} \ker d_i$$

Note that a simplicial identity $d_i d_j = d_{j-1} d_i$ for $i < j$ shows that this indeed defines a sub-chain-complex of CX_n. Since most of the d_i vanish on NX_n, observe that the differential on $\hat{d}_n : NX_n \to NX_{n-1}$ is given by $\hat{d}_n = (-1)^n d_n$.

The construction of N is functorial, so we obtain a functor

$$N : \mathbf{sA} \to \mathbf{Ch}_+(\mathbf{A})$$

Proposition 7.1.6. *Let X be a nonnegatively graded chain complex of abelian groups. Let DX_i denote the subgroup of X_i generated by degenerate simplices. Then DX_* forms a subcomplex of CX_* and the natural map*

$$NX_* \to CX_* \to CX_*/DX_*$$

is an isomorphism of complexes.

See [GJ99, Theorem III.2.1].

Construction 7.1.7. There is an inverse functor, denoted K. On objects, it is defined as follows. Given a chain complex $C_*, \hat{d}$, define $K_n(C)$ to be the direct sum

$$K_n(C) := \bigoplus_{\text{surjections } [n] \to [p]} C_p$$

(the direct sum contains one copy of C_p for every surjective map $\eta : [n] \to [p]$ in the simplex category Δ)

Now we have $K_n(C)$ for each $n \geq 0$, we define simplicial structure maps. Suppose $\alpha : [m] \to [n]$ is a map in Δ. We must define $\alpha^* : K_n(C) \to K_m(C)$. Each summand $C_p[\eta]$ of the source corresponds to a surjection $\eta : [n] \to [p]$. Consider the composite $\eta \circ \alpha : [m] \to [p]$. We can factor this as $\epsilon \circ \eta'$ where η' is surjective and ϵ is injective.

- If ϵ is the identity map, then define α^* on the summand $C_p[\eta]$ to be the identification of $C_p[\eta]$ with $C_p[\eta']$.
- If ϵ is the inclusion $[p-1] \subset [p]$ (what we long-ago called d^p) then define α^* to be the map $\hat{d}_p : C_p \to C_{p-1}$ in the chain complex.
- In all other cases, define the map on $C_p[\eta]$ to be 0.

You can verify that all this is really functorial in C.

You can also check that this really does define a simplicial object, i.e. that $(\alpha \circ \beta)^* = \beta^* \circ \alpha^*$. This contains few surprises (you do have to use the fact that $\hat{d}^2 = 0$ at one point, of course), so we don't write it out here.

Remark 7.1.8. In an abelian category, a finite direct sum is isomorphic to the corresponding finite direct product.

Proposition 7.1.9. *The functors K and N form an inverse pair of functors, and K is left adjoint to N. In particular, the categories $\mathbf{sA}$ and $\mathbf{Ch}_+(\mathbf{A})$ are equivalent.*

Proof. The verification that $NKC \cong C$ is routine.

It is somewhat technical to show that $KNX_\bullet \cong X_\bullet$, and we refer to [Wei94, Section 8.4.4] for the proof.

Proving that the functors are adjoint is then very easy. □

Remark 7.1.10. A situation like this, where we have adjoint functors $K \dashv N$ that are an equivalence is called an *adjoint equivalence*. It is an easy exercise to show the functors have the other adjunction relation as well: $N \dashv K$.

Remark 7.1.11. Here comes a bunch of homological algebra I don't want to cover in detail.

Suppose $\mathbf{A}$ is a category of sheaves on a site—including the case of presheaves when the topology is trivial. Then there is a tensor product on chain complexes (defined in the usual way) and there is a tensor product on $\mathbf{sA}$, defined levelwise.

There are natural transformations

$$\Delta_{K,J} : N(K \otimes J) \to N(K) \otimes N(J) \quad \text{the } \textit{Alexander–Whitney map}$$

(see [Wei94, Section 8.5]) and

$$\nabla_{K,J} : N(K) \otimes N(J) \to N(K \otimes K) \quad \text{the } \textit{Eilenberg–Zilber map}.$$

(see [May92, Def. 29.7], where he calls it the "Eilenberg–MacLane map").

These two maps, ∇ and Δ, are somewhat inverse to each other: $\Delta \circ \nabla$ is the identity, while $\nabla \circ \Delta$ is chain-homotopic to the identity.

Proposition 7.1.12. *Suppose $f, g : K \to J$ are two maps of simplicial sheaves of abelian groups, and suppose there is a (left) homotopy $f \simeq g$. Then there is a chain homotopy $N(f) \simeq N(g)$.*

Proof. Write $\phi : K \times \Delta[1] \to J$ for the homotopy. Since J is an abelian group object, this extends to a map $\phi : K \otimes \mathbb{Z}\Delta[1] \to J$. Then apply N and the Eilenberg–Zilber map to obtain $N(\phi) : N(K) \otimes N\mathbb{Z}\Delta[1] \to NJ$. But $N\mathbb{Z}\Delta[1]$ is a chain complex of constant sheaves $0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to 0$ (with the expected differential $1 \mapsto (-1, 1)$) and so $N(\phi)$ induces the desired chain homotopy. □

Proposition 7.1.13. *If $f \simeq g : C \to D$ are chain-homotopic maps between chain complexes of sheaves of abelian groups, then $K(f) \simeq K(g)$ are left homotopic.*

We will not give the proof here: a construction is given in [Wei94, Section 8.4].

Remark 7.1.14. Taken together, these two propositions imply that, not only are the two categories of $\mathbf{sAb}(\mathbf{C})$ and $\mathbf{Ch}_+(\mathbf{Ab}(\mathbf{C}))$ equivalent, but that the two notions of homotopy in each induce the equivalence relations on maps.

In particular, if $A_\bullet$ is an objectwise fibrant simplicial sheaf of abelian groups on $\mathbf{C}$ (pointed at 0), then for any $U \in \mathbf{C}$:

$$\begin{aligned}\pi_n(A_\bullet(U)) &= \mathbf{sSet}(\Delta[i]/\partial\Delta[n], A_\bullet(U))/\sim = \mathbf{sAb}(\Sigma^n\mathbb{Z}, A_\bullet(U))/\sim \\ &= \mathbf{Ch}_+(\mathbb{Z})(N\Sigma^n\mathbb{Z}, NA_\bullet(U))/\sim = \mathrm{H}_n(NA_\bullet(U))\end{aligned}$$

Remark 7.1.15. This goes some way to explaining why simplicial (pre)sheaves are not more popular; in the 'abelian group' case, they are equivalent to (nonnegatively graded) chain complexes. Whenever you see a nonnegatively-graded chain complex of sheaves, as is frequently seen in algebraic geometry, you now know this is a special case of simplicial sheaves. The simplicial objects are more flexible, since they can describe non-abelian situations.

Part III

$\mathbb{A}^1$-homotopy

Chapter 8

The $\mathbb{A}^1$-homotopy theory

8.1 Definitions

Definition 8.1.1. The *injective* $\mathbb{A}^1$*-model structure* is the left Bousfield localization of the injective local model structure on $\mathbf{Sm}_k$ at the set of maps $p_2 : \mathbb{A}^1 \times U \to U$. The homotopy category is called the *motivic homotopy category* or $\mathbb{A}^1$*-homotopy category* of k, and is denoted $\mathcal{H}(k)$. There is also a pointed version of the model structure and a pointed homotopy category $\mathcal{H}(k)_\bullet$.

Remark 8.1.2. This is automatically a simplicial model structure. It also happens to be proper: [MV99, Theorem 2.7, p. 71].

Notation 8.1.3. An object $\mathcal{X} \in \mathbf{sPre}(\mathbf{Sm}_k)$ is $\mathbb{A}^1$*-local* if, for all $U \in \mathbf{Sm}_k$, the map

$$p_2^* : \mathcal{S}(U, R_{Nis}\mathcal{X}) \to \mathcal{S}(\mathbb{A}^1 \times U, R_{Nis}\mathcal{X})$$

is a weak equivalence

This is different from the definition in [MV99], where they instead ask for

$$i_0^* : [\mathcal{Y} \times \mathbb{A}^1, \mathcal{X}] \to [\mathcal{Y}, \mathcal{X}]$$

to be an isomorphism for all $\mathcal{Y}$, the maps being calculated in the local homotopy category (denoted $\mathcal{H}_s$ there). Note that they use $i_0 : \mathrm{pt} \to \mathbb{A}^1$, rather than the more canonical projection, because they want to work with inclusions (i.e., cofibrations). But since $\mathrm{pt} \to \mathbb{A}^1 \to \mathrm{pt}$ is a retraction, the 2-out-of-3 property means their definition is equivalent to the obvious analogue using projection maps.

Remark 8.1.4. The n-simplices of the simplicial mapping space $\mathcal{S}(U, R_{Nis}\mathcal{X})$, are naturally isomorphic to $\mathbf{sPre}(\mathbf{Sm}_k)(U \times \Delta[n], R_{Nis})$. View the simplicial presheaf as a presheaf on $\mathbf{Sm}_k \times \Delta$, and apply the Yoneda lemma to the representable object $U \times \Delta[n]$ to deduce that the n-simplices of $\mathcal{S}(U, R_{Nis}\mathcal{X})$ are naturally isomorphic to $R_{Nis}\mathcal{X}(U \times \Delta[n]) = R_{Nis}\mathcal{X}(U)_n$. That is, $\mathcal{S}(U, R_{Nis}\mathcal{X}) = R_{Nis}\mathcal{X}(U)$.

Therefore, $\mathbb{A}^1$-locality can be recast as saying

$$R_{Nis}\mathcal{X}(U) \to R_{Nis}\mathcal{X}(U \times \mathbb{A}^1)$$

is a weak equivalence for all U.

The following result reconciles the two definitions of “$\mathbb{A}^1$-local”.

Proposition 8.1.5. *The two definitions of $\mathbb{A}^1$-local objects agree.*

Proof. Our definition is that $\mathcal{X}$ is $\mathbb{A}^1$-local if $R_{Nis}\mathcal{X}(U) \to R_{Nis}\mathcal{X}(\mathbb{A}^1 \times U)$ is a weak equivalence for all objects $U \in \mathbf{Sm}_k$. Here the map is that induced by projection $\mathbb{A}^1 \times U \to U$. This map has a section, given by inclusion at 0, denoted $i_0 : U \to \mathbb{A}^1 \times U$. Then by 2-out-of-3, the object $\mathcal{X}$ is $\mathbb{A}^1$-local (in our sense) if and only if $i_0^* : R_{Nis}\mathcal{X}(\mathbb{A}^1 \times U) \to R_{Nis}\mathcal{X}(U)$ is a weak equivalence. Note that this map is always a fibration, since $R_{Nis}\mathcal{X}$ is fibrant and i_0 is a cofibration. The result now follows from [MV99, Lemma 2.8, p72]. □

Remark 8.1.6. From the general theory of left Bousfield localizations, we know that an object is $\mathbb{A}^1$-fibrant if and only if it is locally fibrant and $\mathbb{A}^1$-local.

Remark 8.1.7. Write $R_{\mathbb{A}^1}$ (sometimes $L_{\mathbb{A}^1}$) for a fibrant replacement functor in the $\mathbb{A}^1$-model category. In our current presentation of the subject, the “$\mathbb{A}^1$-homotopy type” of an object $\mathcal{X}$ is really the information of the local homotopy type of $R_{\mathbb{A}^1}\mathcal{X}$, which in turn is actually the information of the homotopy types of the simplicial sets $R_{\mathbb{A}^1}\mathcal{X}(U)$ as U varies, and the maps $R_{\mathbb{A}^1}\mathcal{X}(U) \to R_{\mathbb{A}^1}\mathcal{X}(V)$.

The benefit of having an $\mathbb{A}^1$-local object $\mathcal{X}$ is that $R_{\mathbb{A}^1}\mathcal{X} \simeq R_{Nis}\mathcal{X}$ in that case.

Notation 8.1.8. We'll write $[\mathcal{X}, \mathcal{Y}]_{\mathbb{A}^1}$ for the set of maps $\mathcal{X} \to \mathcal{Y}$ in the (unpointed) $\mathbb{A}^1$-homotopy category. We will use $[\mathcal{X}, \mathcal{Y}]_{\mathbb{A}^1,\bullet}$ for the pointed case.

Example 8.1.9. There are relatively few $\mathbb{A}^1$-local objects that admit elementary description. Here's one: the Nisnevich sheaf $\mathbb{G}_m$ viewed as a discrete simplicial sheaf. The proof of this will be an exercise.

Example 8.1.10. The concept of $\mathbb{A}^1$-equivalence is of major importance. Strictly speaking, it applies to maps of simplicial presheaves, but we're often most interested in the case of representable presheaves (i.e., varieties).

The following are examples of $\mathbb{A}^1$-equivalences:

1. All projection maps $\mathbb{A}^1 \times U \to U$ (by definition)

2. All sections of projection maps $U \to \mathbb{A}^1 \times U$ (by 2-out-of-3)

3. Both the above examples, but with $\mathbb{A}^1$ replaced by $\mathbb{A}^n$ (by induction).

4. Suppose $p : Y \to X$ is a map of varieties such that X has a finite Zariski cover $\{f_i : U_i \to X\}$ such that $p|_{p^{-1}(U_i)} : p^{-1}(\bigcap U_i) \to \bigcap U_i$ is an $\mathbb{A}^1$-equivalence for all nontrivial intersections of the U_i. Then p is an $\mathbb{A}^1$-equivalence.

   In the case of two open sets, U, V, use the (homotopy) pushout square in the category of simplicial Nisnevich sheaves

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & U \arrow[d] \\ V \arrow[r] & X \end{tikzcd}$$

   (since $U \to X$ is a monomorphism, this pushout square is a homotopy pushout square as well).

Then do the same with

$$\begin{tikzcd} p^{-1}(U \cap V) \arrow[r] \arrow[d] & p^{-1}(U) \arrow[d] \\ p^{-1}(V) \arrow[r] & Y \end{tikzcd}$$

The map p induces a map of homotopy pushout squares that is an objectwise $\mathbb{A}^1$-equivalence. Therefore $p : Y \to X$ is an $\mathbb{A}^1$-equivalence.

5. You can modify this argument to apply when the covering is a Nisnevich covering. We leave the details as an exercise.

6. The above theory applies when $p : Y \to X$ is a map such that there is a (finite) Zariski cover $\{U_i\}$ of X such that for all U_i, the map $p^{-1}(U_i) \to U_i$ is isomorphic to a projection $U_i \times \mathbb{A}^n \to U_i$. In particular when $p : Y \to X$ is the map of a vector bundle map, it is an $\mathbb{A}^1$-equivalence.

Notation 8.1.11. The object $\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\}$ (given the basepoint $1 \in \mathbb{G}_m$) is called the *Tate circle*. Sometimes we will write S^α for this.

Example 8.1.12. Work in the pointed $\mathbb{A}^1$-model structure. There is a (homotopy) pushout diagram

$$\begin{tikzcd} \mathbb{G}_m \arrow[r] \arrow[d] & \mathbb{A}^1 \arrow[d] \\ \mathbb{A}^1 \arrow[r] & \mathbb{P}^1 \end{tikzcd}$$

This arises from a Zariski covering, and the same argument as before says it is both a pushout and a homotopy pushout diagram in the local model structure on simplicial sheaves. (In presheaves, by the way, the pushout $\mathbb{A}^1 \coprod_{\mathbb{G}_m} \mathbb{A}^1$ is not isomorphic to $\mathbb{P}^1$, but it is locally weakly equivalent to it).

In $\mathbb{A}^1$ model structures, however, $\mathbb{A}^1 \simeq *$, so we see that $\mathbb{P}^1$ is the homotopy pushout of $* \leftarrow \mathbb{G}_m \to *$. But this is a construction for the suspension $\Sigma\mathbb{G}_m = \mathbb{G}_m \wedge S^1$. We conclude that in $\mathbb{A}^1$-homotopy theory, $\Sigma\mathbb{G}_m = \mathbb{P}^1$.

Construction 8.1.13. Suppose X and Y are cofibrant objects in a simplicial model category. Then the homotopy pushout of the diagram $Y \leftarrow X \times Y \to X$ (the maps being projections) is called the *join* of X and Y and is denoted $X * Y$.

Suppose further that X and Y are pointed objects. Then consider the following diagram

$$\begin{tikzcd} X & X \times Y \arrow[l] \arrow[r] & Y \\ X \arrow[u] \arrow[d] & X \vee Y \arrow[l] \arrow[r] \arrow[u] \arrow[d] & Y \arrow[u] \arrow[d] \\ * & * \arrow[l] \arrow[r] & *. \end{tikzcd}$$

(all the maps being obvious ones). Take the homotopy colimits of the columns, to arrive at the diagram:

$$* \longleftarrow X \wedge Y \longrightarrow * .$$

The homotopy pushout of this diagram is $\Sigma(X \wedge Y)$.

On the other hand, if we take the homotopy colimits of the rows, we get

$$X * Y \longleftarrow CX \vee CY \longrightarrow * .$$

The term $CX \vee CY$, the wedge sum of the cones on X and Y, is contractible. Therefore the homotopy pushout of this diagram is equivalent to $X * Y$.

There is a 'Fubini' theorem for homotopy colimits, [CCS02, Theorem 24.9], which implies that $X * Y \simeq \Sigma(X \wedge Y)$.

Here is another classical examples of an $\mathbb{A}^1$-equivalence.

Example 8.1.14. Give $\mathbb{A}^n \setminus 0$ the basepoint $(1, \ldots, 1)$.

There is a pushout diagram (in the category of (simplicial) Nisnevich sheaves)

$$\begin{tikzcd} \mathbb{A}^1 \setminus 0 \times \mathbb{A}^{n-1} \setminus 0 \arrow[r] \arrow[d] & \mathbb{A}^1 \times \mathbb{A}^{n-1} \setminus 0 \arrow[d] \\ \mathbb{A}^1 \setminus 0 \times \mathbb{A}^{n-1} \arrow[r] & \mathbb{A}^n \setminus 0 \end{tikzcd}$$

which is also a homotopy pushout diagram. Therefore $\mathbb{A}^n \setminus 0$ is $\mathbb{A}^1$-equivalent to the homotopy pushout of the diagram:

$$\begin{tikzcd} \mathbb{A}^1 \setminus 0 \times \mathbb{A}^{n-1} \setminus 0 \arrow[r] \arrow[d] & \mathbb{A}^1 \setminus 0 \\ \mathbb{A}^{n-1} \setminus 0 & \end{tikzcd}$$

where both maps are projections. Therefore $\mathbb{A}^n \setminus 0 \simeq_{\mathbb{A}^1} (\mathbb{A}^1 \setminus 0) * (\mathbb{A}^{n-1} \setminus 0) \simeq \Sigma((\mathbb{A}^1 \setminus 0) \wedge (\mathbb{A}^{n-1} \setminus 0))$. By induction, we deduce that

$$\mathbb{A}^n \setminus 0 \simeq_{\mathbb{A}^1} S^{n-1} \wedge (\mathbb{G}_m)^{\wedge n}.$$

This kind of object ($\mathbb{A}^n \setminus 0$ or $\mathbb{P}^1$) is called a *motivic sphere*, since it is a smash product of simplicial and Tate circles.

Definition 8.1.15. An (simplicial) presheaf $\mathcal{X}$ is $\mathbb{A}^1$-*invariant* if $\mathcal{X}(U) \to \mathcal{X}(\mathbb{A}^1 \times U)$ is a weak equivalence for all U—if $\mathcal{X}$ is just a presheaf, then 'weak equivalence' means 'isomorphism'. Note that $\mathbb{A}^1$-invariance is not comparable to $\mathbb{A}^1$-locality for simplicial presheaves.

Example 8.1.16. Suppose $\mathcal{X}$ is a simplicial presheaf satisfying the following two conditions:

- $\mathcal{X}$ has the Brown-Gersten condition.
- $\mathcal{X}$ is $\mathbb{A}^1$-invariant.

By virtue of the Brown–Gersten condition, the natural map $\mathcal{X} \to R_{Nis}\mathcal{X}$ is an objectwise equivalence. Therefore, for all U, the map $R_{Nis}\mathcal{X}(U \times \mathbb{A}^1) \to R_{Nis}\mathcal{X}(U)$ is a weak equivalence (compare with $\mathcal{X}$). Consequently, $\mathcal{X}$ is $\mathbb{A}^1$-local, and so $R_{\mathbb{A}^1}\mathcal{X}$ is locally equivalent to $R_{Nis}\mathcal{X}$. In fact, by using the simplicial structure for instance, we see that $\mathcal{S}(U, R_{Nis}\mathcal{X}) \to \mathcal{S}(U, R_{\mathbb{A}^1}\mathcal{X})$, i.e., $R_{Nis}\mathcal{X} \to R_{\mathbb{A}^1}\mathcal{X}$ is an objectwise equivalence. But $R_{Nis}\mathcal{X}$ was objectwise equivalent to $\mathcal{X}$ itself. In conclusion, $\mathcal{X}(U) \to R_{\mathbb{A}^1}\mathcal{X}(U)$ is a weak equivalence for all U.

Example 8.1.17. The example above applies, in particular, to K-theory. Suppose $\mathcal{K}$ is a simplicial presheaf with the property that $\pi_0(\operatorname{sk} K(U)) = *$ and $\pi_i(\mathcal{K}(U)) \simeq K_{i-1}(U)$ for all i, for all smooth k-varieties U. Then Brown & Gersten and Nisnevich proved that $\mathcal{K}$ has the Brown–Gersten condition, and it is a well-known fact that K-theory of smooth schemes is $\mathbb{A}^1$-invariant. In particular, we may assume $\mathcal{K}$ is $\mathbb{A}^1$-fibrant.

Fix a global basepoint for $\mathcal{K}$. Then in the pointed $\mathbb{A}^1$-homotopy category, we have

$$[U_+ \wedge S^n, \mathcal{K}]_{\mathbb{A}^1,\bullet} = \pi_0\mathcal{S}_+(U_+ \wedge S^n, \mathcal{K}) = \pi_0(\Omega^n\mathcal{K}(U)) = K_{n-1}(U)$$

Definition 8.1.18. Let $\mathcal{A}$ be a sheaf of abelian groups. We say $\mathcal{A}$ is *strictly* $\mathbb{A}^1$*-invariant* if the cohomology group functors

$$\mathrm{H}^n_{Nis}(\cdot, \mathcal{A}) : \mathbf{Sm}_k \to \mathbf{Ab}$$

are $\mathbb{A}^1$-invariant for all n.

Proposition 8.1.19. *Let $\mathcal{A}$ be a sheaf of abelian groups. Then the following are equivalent:*

- *$\mathcal{A}$ is strictly $\mathbb{A}^1$-invariant.*
- *The spaces $K(\mathcal{A}, n)$ are $\mathbb{A}^1$-local for all n.*

Proof. The objects $R_{Nis}K(\mathcal{A}, n)$ are simplicial presheaves of groups. Each path component of $R_{Nis}K(\mathcal{A}, n)(U)$ is isomorphic to each other path component. Therefore $\mathbb{A}^1$-locality of $K(\mathcal{A}, n)$ is equivalent to the combined assertions that

$$\mathrm{H}^n_{Nis}(U, \mathcal{A}) = \pi_0(R_{Nis}K(\mathcal{A}, n)(U)) \to \pi_0(R_{Nis}K(\mathcal{A}, n)(\mathbb{A}^1 \times U) \simeq \mathrm{H}^n_{Nis}(U \times \mathbb{A}^1, \mathcal{A})$$

is a bijection and

$$\mathrm{H}^{n-i}_{Nis}(U, \mathcal{A}) = \pi_i(R_{Nis}K(\mathcal{A}, n)(U), 0) \to \pi_i(R_{Nis}K(\mathcal{A}, n)(\mathbb{A}^1 \times U), 0) = \mathrm{H}^{n-i}_{Nis}(\mathbb{A}^1 \times U, \mathcal{A})$$

is an isomorphism for all $i > 0$. This holds for all n if and only if $\mathcal{A}$ is strictly $\mathbb{A}^1$-invariant. □

Chapter 9

Naive $\mathbb{A}^1$-homotopy

9.1 Naive $\mathbb{A}^1$-homotopy

Definition 9.1.1. Suppose $f, g : \mathcal{X} \to \mathcal{Y}$ are two maps of simplicial presheaves. An *elementary $\mathbb{A}^1$-homotopy* from f to g is a map $H : \mathcal{X} \times \mathbb{A}^1 \to \mathcal{Y}$ such that $\mathcal{X} \times \{0\} \to \mathcal{X} \times \mathbb{A}^1 \xrightarrow{H} \mathcal{Y}$ is f and $\mathcal{X} \times \{1\} \to \mathcal{X} \times \mathbb{A}^1 \xrightarrow{H} \mathcal{Y}$ is g.

Remark 9.1.2. This gives us a relation on maps that is symmetric and reflexive, but not generally transitive. We say f and g are *naively $\mathbb{A}^1$-homotopic* if there exists some finite sequence of elementary $\mathbb{A}^1$ homotopies starting at f and ending at g.

Proposition 9.1.3. *If f and g are naively $\mathbb{A}^1$-homotopic, then they induce the same morphism in the $\mathbb{A}^1$-homotopy category.*

Proof. It suffices to prove this in the case of a single elementary homotopy. In the $\mathbb{A}^1$-homotopy category, the inclusion of $\mathcal{X} \to \mathcal{X} \times \mathbb{A}^1$ at 0 (or at 1) is an isomorphism with inverse given by the projection $p : \mathcal{X} \times \mathbb{A}^1 \to \mathcal{X}$. In particular, in this category, both f and g admit description as p^{-1}, so they agree. Therefore so too do $f = H \circ p^{-1} = g$. $\square$

Remark 9.1.4. We will see examples later where naïve homotopy classes of maps $\mathcal{X} \to \mathcal{Y}$ do not account for all maps $[\mathcal{X}, \mathcal{Y}]_{\mathbb{A}^1}$, but it might take a few lectures.

9.2 The $\mathbb{A}^1$-Singular functor

Definition 9.2.1. Let Δ^n_{alg} be the variety

$$\Delta^n_{\text{alg}} = \operatorname{Spec} \frac{k[x_0, \dots, x_n]}{1 - \sum_{i=0}^n x_i} \cong \mathbb{A}^n_k.$$

The varieties Δ^n_{alg} assemble to form a cosimplicial variety, i.e., a functor $\mathbf{\Delta} \to \mathbf{Sm}_k$. On objects, $[n] \mapsto \Delta^n_{\text{alg}}$. For morphisms, consider a map $f : [n] \to [m]$. Define a map of rings

$$k[x_0, \dots, x_m] \to k[y_0, \dots, y_n]$$

by $x_i \mapsto \sum_{j \in f^{-1}(i)} y_j$. The empty sum is taken to be 0. You can see that $\sum_{i=0}^m x_i \mapsto \sum_{j=0}^n y_j$ under this rule, so that the map descends to one on quotient rings

$$\frac{k[x_0, \ldots, x_m]}{\left(1 - \sum_{i=0}^m x_i\right)} \to \frac{k[y_0, \ldots, y_n]}{\left(1 - \sum_{j=0}^n y_j\right)}$$

and from there to a (covariant) map of varieties. This produces a cosimplicial smooth variety, denoted $\Delta^\bullet_{\text{alg}}$.

We can use $\Delta^\bullet_{\text{alg}}$ to produce an adjoint pair of functors

$$| \cdot |_{\text{alg}} \dashv \text{Sing}^{\mathbb{A}^1}$$

directly modelled on the realization and singular functors from simplicial sets.

Construction 9.2.2. Specifically, $|\mathcal{X}|_{\text{alg}}$ is defined as a coequalizer:

$$\coprod_{[n] \to [m]} \mathcal{X}_m \times \Delta^n_{\text{alg}} \rightrightarrows \coprod_{[n]} \mathcal{X}_n \times \Delta^n_{\text{alg}} \to |\mathcal{X}|_{\text{alg}}$$

and $\mathrm{Sing}^{\mathbb{A}^1} \mathcal{Y}$ is defined to be the right adjoint:

$$\mathrm{Sing}^{\mathbb{A}^1} \mathcal{Y}_n(U) = \mathrm{Map}(\Delta^n_{\text{alg}}, \mathcal{Y})(U)_n = \mathrm{Mor}(\Delta^n_{\text{alg}} \times U \times \Delta^n, \mathcal{Y}) = \mathcal{Y}(\Delta^n_{\text{alg}} \times U)$$

(the last step uses the Yoneda lemma).

We leave it as an exercise in category theory to prove that this really is an adjoint pair.

Remark 9.2.3. Let $\mathcal{X}$ be simplicial presheaf and U a variety. The set $\pi_0(\mathrm{Sing}^{\mathbb{A}^1} \mathcal{X}(U))$ admits the following description: It is the quotient of $\mathrm{Sing}^{\mathbb{A}^1} \mathcal{X}(U)_0$ by the equivalence relation generated by the two face maps: $d_0, d_1 : \mathrm{Sing}^{\mathbb{A}^1} \mathcal{X}(U)_1 \to \mathrm{Sing}^{\mathbb{A}^1} \mathcal{X}(U)_0$. What are these sets?

$$\mathrm{Sing}^{\mathbb{A}^1} \mathcal{X}(U)_0 = \mathcal{X}(U \times \Delta^0_{\text{alg}})_0 = \mathcal{X}(U)_0$$

and

$$\mathrm{Sing}^{\mathbb{A}^1} \mathcal{X}(U)_1 = \mathcal{X}(U \times \Delta^1_{\text{alg}})_1 \simeq \mathcal{X}(U \times \mathbb{A}^1)_1.$$

The two face maps d_0 and d_1 are given as composites of other face maps denoted by the same letters. For instance,

$$d_0 = \mathcal{X}(U \times \mathbb{A}^1)_1 \xrightarrow{e_0} \mathcal{X}(U)_1 \xrightarrow{d_0} \mathcal{X}(U)_0.$$

A similar statement is true about d_1.

A particularly interesting case is when $\mathcal{X}$ is concentrated in dimension 0, so all higher simplices are degenerate. For instance, suppose $\mathcal{F} = \mathcal{X}_0 = \mathcal{X}_1 = \ldots$. In this case $\pi_0(\mathrm{Sing}^{\mathbb{A}^1} \mathcal{F}(U))$ consists of the quotient of $\mathcal{F}(U) = \mathbf{sSet}(U, \mathcal{F})$ by naive $\mathbb{A}^1$-homotopy.

The $\mathbb{A}^1$-singular functor, $\mathrm{Sing}^{\mathbb{A}^1}$ has the following properties:

1. $\mathrm{Sing}^{\mathbb{A}^1}$ preserves limits—this is by virtue of its being a right adjoint.

2. $\mathrm{Sing}^{\mathbb{A}^1}$ converts the map $i_0 : * \to \mathbb{A}^1$ into a (left) homotopy equivalence.

3. There is a natural monomonorphism $\mathcal{X} \to \mathrm{Sing}^{\mathbb{A}^1}\mathcal{X}$ that is $\mathbb{A}^1$-equivalence for all $\mathcal{X}$.

4. $\mathrm{Sing}^{\mathbb{A}^1}$ preserves $\mathbb{A}^1$-fibrations.

These properties are enumerated on p87 of [MV99].

Remark 9.2.4. A trivial, but useful, consequence of the preservation-of-limits is that $\mathrm{Sing}^{\mathbb{A}^1}(*) = *$. You could prove this directly, of course.

Proposition 9.2.5. *Suppose $f, g : \mathcal{X} \to \mathcal{Y}$ are two morphisms and H is an elementary $\mathbb{A}^1$-homotopy between them. There exists a left (simplicial) homotopy from $\mathrm{Sing}^{\mathbb{A}^1}(f)$ to $\mathrm{Sing}^{\mathbb{A}^1}(g)$.*

Proof. Apply $\mathrm{Sing}^{\mathbb{A}^1}(\cdot)$ to the diagram

$$\begin{tikzcd}
\mathcal{X} \arrow[dr, "\mathrm{id}\times i_0"] & & \\
& \mathcal{X}\times\mathbb{A}^1 \arrow[r, "H"] & \mathcal{Y} \\
\mathcal{X} \arrow[ur, "\mathrm{id}\times i_1"] & &
\end{tikzcd}$$

to get

$$\begin{tikzcd}
\mathrm{Sing}^{\mathbb{A}^1}\mathcal{X} \arrow[dr, "\mathrm{id}\times \mathrm{Sing}^{\mathbb{A}^1} i_0"] & & \\
& \mathrm{Sing}^{\mathbb{A}^1}\mathcal{X}\times \mathrm{Sing}^{\mathbb{A}^1}\mathbb{A}^1 \arrow[r, "\mathrm{Sing}^{\mathbb{A}^1} H"] & \mathrm{Sing}^{\mathbb{A}^1}\mathcal{Y} \\
\mathrm{Sing}^{\mathbb{A}^1}\mathcal{X} \arrow[ur, "\mathrm{id}\times \mathrm{Sing}^{\mathbb{A}^1} i_1"'] & &
\end{tikzcd}.$$

Suppose we show that $\mathrm{Sing}^{\mathbb{A}^1} i_0$ is left homotopic to $\mathrm{Sing}^{\mathbb{A}^1} i_1$, i.e., that there is a map H' so the diagram below commutes:

$$\begin{tikzcd}
* \arrow[dr, "0"'] \arrow[drr, "i_0"] & & \\
& \Delta[1] \arrow[r, "H'"] & \mathrm{Sing}^{\mathbb{A}^1}\mathbb{A}^1 \\
* \arrow[ur, "1"] \arrow[urr, "i_1"'] & &
\end{tikzcd} \tag{9.1}$$

Then we may multiply this diagram by $\operatorname{Sing}^{\mathbb{A}^1}\mathcal{X}$ and compose with H to get

$$\begin{tikzcd}
\operatorname{Sing}^{\mathbb{A}^1}\mathcal{X} \arrow[dr] \arrow[drr, bend left, "\mathrm{id}\times\operatorname{Sing}^{\mathbb{A}^1} i_0"] & & & \\
& \operatorname{Sing}^{\mathbb{A}^1}\mathcal{X}\times\Delta[1] \arrow[r, "\mathrm{id}\times H'"] & \operatorname{Sing}^{\mathbb{A}^1}\mathcal{X}\times\operatorname{Sing}^{\mathbb{A}^1}\mathbb{A}^1 \arrow[r, "H"] & \operatorname{Sing}^{\mathbb{A}^1}\mathcal{Y} \\
\operatorname{Sing}^{\mathbb{A}^1}\mathcal{X} \arrow[ur] \arrow[urr, bend right, "\mathrm{id}\times\operatorname{Sing}^{\mathbb{A}^1} i_1"'] & & &
\end{tikzcd}$$

The composite $H\circ(\mathrm{id}\times H')$ provides the required homotopy.

It suffices then to treat the case in diagram (9.1), i.e., that of $i_0, i_1 : * \to \mathbb{A}^1$. We need to find a left homotopy between $\operatorname{Sing}^{\mathbb{A}^1}(i_0), \operatorname{Sing}^{\mathbb{A}^1}(i_1) : * \to \operatorname{Sing}^{\mathbb{A}^1}(\mathbb{A}^1)$. What are these two maps from a point to $\operatorname{Sing}^{\mathbb{A}^1}(\mathbb{A}^1)$? They are $* \mapsto 0, 1 \in \operatorname{Sing}^{\mathbb{A}^1}(\mathbb{A}^1)_0 = \mathbb{A}^1$. What is a left homotopy? It's a map $\Delta[1] \to \operatorname{Sing}^{\mathbb{A}^1}(\mathbb{A}^1)$, i.e., a global section of $\operatorname{Sing}^{\mathbb{A}^1}(\mathbb{A}^1)_1$, i.e., a map $H : \mathbb{A}^1 \to \mathbb{A}^1$. This map gives a homotopy between $d_0\circ H$ and $d_1\circ H$. In this case, taking $H = \mathrm{id}$ does the trick. □

Corollary 9.2.6. *For any $\mathcal{X}$, the map $j := \operatorname{Sing}^{\mathbb{A}^1}(i_0) : \operatorname{Sing}^{\mathbb{A}^1}(\mathcal{X}) \to \operatorname{Sing}^{\mathbb{A}^1}(\mathbb{A}^1\times\mathcal{X})$ is a homotopy equivalence.*

Proof. That is, we want to produce a homotopy inverse map. Produce p by applying $\operatorname{Sing}^{\mathbb{A}^1}$ to the projection $\mathbb{A}^1\times\mathcal{X}\to\mathcal{X}$. Now we show that this is a homotopy inverse.

By functoriality, $j\circ p = \mathrm{id}$, so there is nothing to check here. We want to show that $p\circ j$ admits a simplicial (left) homotopy to $\mathrm{id}_{\operatorname{Sing}^{\mathbb{A}^1}(\mathbb{A}^1\times\mathcal{X})}$.

In light of the proposition, it's sufficient to produce an elementary $\mathbb{A}^1$-homotopy between $0, \mathrm{id} : \mathbb{A}^1\times\mathcal{X}\to\mathbb{A}^1\times\mathcal{X}$. Such a homotopy is given by $\mathbb{A}^1\times\mathcal{X}\times\mathbb{A}^1\to\mathbb{A}^1\times\mathcal{X}$ given by $((r,x),t)\mapsto (r(1-t),x)xs$. □

Remark 9.2.7. The obvious map $\mathcal{X}\to\operatorname{Sing}^{\mathbb{A}^1}\mathcal{X}$ is a monomorphism, since it is levelwise split.

Proposition 9.2.8. *The map $\mathcal{X}\to\operatorname{Sing}^{\mathbb{A}^1}\mathcal{X}$ is an $\mathbb{A}^1$-equivalence.*

Sketch. We claim that for any $\mathcal{X}$ and n, the canonical map $j : \mathcal{X} = \operatorname{Map}(*,\mathcal{X}) \to \operatorname{Map}(\mathbb{A}^n n,\mathcal{X})$ is an $\mathbb{A}^1$-homotopy equivalences (and therefore $\mathbb{A}^1$-weak equivalences). There is a map e_0 given by evaluation at $\mathbf{0}$ which gives a left inverse for j, so it suffices to prove that $j\circ e_0$ is elementary $\mathbb{A}^1$-homotopic to the identity map. An elementary $\mathbb{A}^1$ homotopy is given by

$$H : \operatorname{Map}(\mathbb{A}^n,\mathcal{X})\times\mathbb{A}^1\to\operatorname{Map}(\mathbb{A}^n,\mathcal{X}),\quad H(f,t)(\mathbf{u}) = f(t\mathbf{u}).$$

Observe that $\operatorname{Sing}^{\mathbb{A}^1}\mathcal{X}_n \cong \operatorname{Hom}(\mathbb{A}^n,\mathcal{X}_n)$.

Next, we exploit a well-known homotopy theory trick: Suppose $\mathcal{X}$ is a simplicial presheaf, then we may consider $\mathcal{X}$ as a functor $\mathcal{X} : \mathbf{\Delta}^{\mathrm{op}}\to\mathbf{Pre}(\mathbf{Sm}_k)\subseteq\mathbf{sPre}(\mathbf{Sm}_k)$. In particular, it is possible to view this as a diagram of (simplicial) presheaves. Then we may construct the homotopy colimit:

$$\operatorname*{hocolim}_{\Delta^{\mathrm{op}}}\mathcal{X}_n$$

and this turns out to be equivalent to $\mathcal{X}$ itself [BK72, XII 3.4].

Using this trick, the map $\mathcal{X} \to \mathrm{Sing}^{\mathbb{A}^1} \mathcal{X}$ is equivalent to a map between homotopy colimits

$$\operatorname*{hocolim}_{\Delta^{\mathrm{op}}} \mathcal{X}_n \to \operatorname*{hocolim}_{\Delta^{\mathrm{op}}} \mathrm{Map}(\mathbb{A}^n, \mathcal{X}_n)$$

which is objectwise an $\mathbb{A}^1$-equivalence, so it's an $\mathbb{A}^1$-equivalence as required. □

Proposition 9.2.9. *The functor* $\mathrm{Sing}^{\mathbb{A}^1}$ *is a right Quillen endofunctor.*

Sketch of sketch. Show that the adjoint $|\cdot|_{\mathrm{alg}}$ is left Quillen. To show it preserves cofibrations (monomorphisms) is a direct calculation ([MV99, Lemma 3.10, p90]). Then to show that it preserves $\mathbb{A}^1$-equivalences is [MV99, Lemma 3.12, p90]. Again, one uses a hocolim trick to reduce to a special case of $\mathcal{F} \times \Delta[n]$ where $\mathcal{F}$ is a simplicial sheaf. Then it's a calculation. □

Example 9.2.10. The following appears as [MV99, Example 2.7, p107]. Take $U_0 = \mathbb{A}^1 \setminus 0$, $U_1 = \mathbb{A}^0 \setminus 1$ and $U_{01} = \mathbb{A}^1 \setminus 0, 1$. Choose a closed embedding $j : U_{01} \to \mathbb{A}^2$, and define $\mathcal{F} = (U_0 \times \mathbb{A}^2) \cup_{U_{01}} (U_1 \times \mathbb{A}^2)$, a pushout using j in the category of Nisnevich sheaves. Morel & Voevodsky use this example because this pushout is relatively easy to calculate: if X is a connected smooth scheme, then

$$\mathcal{F}(X) = \mathbf{Sm}_k(X, U_0 \times \mathbb{A}^2) \cup_{\mathbf{Sm}_k(X, U_{01})} \mathbf{Sm}_k(X, U_1 \times \mathbb{A}^2).$$

Since $\mathcal{F}$ is a sheaf in the Nisnevich topology, if we view it as a simplicial sheaf entirely concentrated in dimension 0, then it is a Nisnevich-fibrant object.

Now consider $\mathrm{Sing}^{\mathbb{A}^1} \mathcal{F}$. Evaluated at a connected X, in level n this gives us

$$\mathbf{Sm}_k(X \times \mathbb{A}^n, U_0 \times \mathbb{A}^2) \cup_{\mathbf{Sm}_k(X \times \mathbb{A}^n, U_{01})} \mathbf{Sm}_k(X \times A^n, U_1 \times \mathbb{A}^2).$$

You can check $\mathbf{Sm}_k(X \times \mathbb{A}^n, U_0)$ is equal to $\mathbf{Sm}_k(X, U_0)$, and similarly for U_1 and U_{01}. Furthermore, $\mathbf{Sm}_k(X \times \mathbb{A}^n, U_0 \times \mathbb{A}^2) = \mathbf{Sm}_k(X \times \mathbb{A}^n, U_0) \times \mathbf{Sm}_k(X \times \mathbb{A}^n, \mathbb{A}^2)$. Taking all this together, one obtains an isomorphism

$$\mathrm{Sing}^{\mathbb{A}^1} \mathcal{F} = (U_0 \cup_{U_{01}} U_1) \times \mathrm{Sing}^{\mathbb{A}^1}(\mathbb{A}^2) \times \mathrm{Sing}^{\mathbb{A}^1}(\mathbb{A}^2)$$

The right hand side is locally equivalent to $\mathbb{A}^1$, since $\mathrm{Sing}^{\mathbb{A}^1}(\mathbb{A}^2) \simeq *$.

On the other hand $\mathcal{S}(*, \mathbb{A}^1) = k$ while $\mathcal{S}(\mathbb{A}^1, \mathbb{A}^1) = k[x]$, so that $\mathbb{A}^1$ itself is far from being $\mathbb{A}^1$-local. This shows that even if $\mathcal{F}$ is Nisnevich fibrant, $\mathrm{Sing}^{\mathbb{A}^1} \mathcal{F}$ may not be $\mathbb{A}^1$-fibrant.

Remark 9.2.11. In [MV99, Lemma 2.6, p107], it is shown that $R_{Nis} \circ (R_{Nis} \circ \mathrm{Sing}^{\mathbb{A}^1})^{\mathbb{N}} \circ R_{Nis}\mathcal{X}$ is an $\mathbb{A}^1$-local object for all $\mathcal{X}$. Here you have to take a colimit to make sense of the 'infinitely iterated' functor. You might hope that in some cases, some finite and possibly small iteration of $\mathrm{Sing}^{\mathbb{A}^1}$ and R_{Nis} will lead to an $\mathbb{A}^1$-local object.

Example 9.2.12. In [AHW18], it is proved that if G is a finitely presented smooth k-group scheme such that $\mathrm{H}^1_{Nis}(\cdot, G)$ is $\mathbb{A}^1$-invariant, then $R_{Nis} \mathrm{Sing}^{\mathbb{A}^1} G$ is $\mathbb{A}^1$-fibrant (Theorem 2.3.2).

It is also proved that (under the same hypothesis on G) that if $B_{Nis}G$ denotes the simplicial presheaf that assigns to U the nerve of the category of (Nisnevich) G-torsors on U, then $R_{Nis} \mathrm{Sing}^{\mathbb{A}^1} B_{Nis}G$ is $\mathbb{A}^1$-fibrant (Theorem 2.2.5).

Remark 9.2.13. In fact, [AHW18, Theorem 2.2.5] asserts even more. If $U = \mathrm{Spec}\, R$ is affine, then

$$[U, B_{Nis}G]_{\mathbb{A}^1} = \pi_0(B_{Nis}G(U)).$$

The left hand side admits a description as $\pi_0(R_{\mathbb{A}^1} B_{Nis}G(U))$.

9.3 Jouanolou's device

Example 9.3.1. This example is a case of Jouanolou's device. The variety $\mathbb{P}^1$ represents a functor that sends a scheme X to isomorphism classes of exact sequences

$$\mathcal{O}_X^2 \to \mathcal{L} \to 0$$

where $\mathcal{L}$ is a line bundle on X, i.e., a locally free module of constant rank 1. An "isomorphism" between such sequences is a diagram:

$$\begin{tikzcd} \mathcal{O}_X^2 \arrow[r] \arrow[d, equal] & \mathcal{L} \arrow[r] \arrow[d, "\phi", "\cong"'] & 0 \\ \mathcal{O}_X^2 \arrow[r] & \mathcal{L}' \arrow[r] & 0 \end{tikzcd}$$

Now consider the subvariety Q of $\mathrm{Mat}_{2\times 2}$ consisting of matrices A such that $A^2 = A$ and $\mathrm{Tr}(A) = 1$. This is a closed affine subvariety of $\mathrm{Mat}_{2\times 2}$. In light of the condition $A^2 = A$, the condition $\mathrm{Tr}(A) = 1$ implies that A is a nontrivial idempotent ($A \neq I_2$ and $A \neq 0$).

The variety Q represents the functor sending X to the set of nontrivial idempotents $A : \mathcal{O}_X^2 \to \mathcal{O}_X^2$. Associated to such an idempotent, there is a short exact sequence

$$0 \to \ker A \to \mathcal{O}_X^2 \to \operatorname{im} A \to 0$$

and this is split by the inclusion $\operatorname{im} A \to \mathcal{O}_X^2$. By forgetting the splitting and $\ker A$, we obtain a morphism $Q \to \mathbb{P}^1$.

The variety $\mathbb{P}^1$ can be covered by two open subvarieties isomorphic to $\mathbb{A}^1$. Any map $\mathbb{A}^1 \to \mathbb{P}^1$ is given by the data of an exact sequence up to isomorphism

$$k[t]^2 \xrightarrow{\phi} F \to 0$$

where F is a projective module of rank 1 over $\mathbb{A}^1 = \operatorname{Spec} k[t]$. Since every projective module over a PID is free, we may fix an isomorphism $F = k[t]$. Then ϕ can be written as a unimodular pair (f, g) of elements in $k[t]$, and two such pairs are isomorphic if they differ by multiplication by $\lambda \in k[t]^\times = k^\times$. The kernel of ϕ is generated by $(-g, f)^t \in k[t]^2$.

An open affine cover of $\mathbb{P}^1$ is given by the two maps corresponding to $(1, t)$ and $(t, 1)$.

Let i denote the first of these two maps and form the pullback square:

$$\begin{tikzcd} Q_{\mathbb{A}^1} \arrow[r] \arrow[d] & Q \arrow[d] \\ \mathbb{A}^1 \arrow[r, "i"] & \mathbb{P}^1 \end{tikzcd}.$$

A map $\operatorname{Spec} R \to Q_{\mathbb{A}^1}$ leads to a distinguished element $t \in R$ by composition with $Q_{\mathbb{A}^1} \to \mathbb{A}^1$. The map $\operatorname{Spec} R \to Q_{\mathbb{A}^1}$ then parametrizes matrices $A \in \mathrm{Mat}_{2\times 2}(R)$ along with elements $t \in R$ such that $A^2 = A$, $\mathrm{Tr}(A) = 1$ and $(-t1)^T \in \ker(A)$. For a given choice of t, such a matrix takes the form

$$A = \begin{bmatrix} 1 + ty & -t - t^2 y \\ y & -ty \end{bmatrix}$$

where $y \in R$ is arbitrary. This is by direct calculation. Applying to the identity map $\mathbb{A}^1 = \operatorname{Spec} k[t] \to \mathbb{A}^1$, we see that $Q_{\mathbb{A}^1} \to \mathbb{A}^1$ is isomorphic to a projection $\mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$. In particular, $Q_{\mathbb{A}^1} \to \mathbb{A}^1$ is an $\mathbb{A}^1$-equivalence.

A similar calculation applies to the second map as well. It follows that $Q \to \mathbb{P}^1$ is an $\mathbb{A}^1$-equivalence. The technical term for this is that Q is an *affine vector bundle torsor* over $\mathbb{P}^1$.

Remark 9.3.2. A similar story applies to $\mathbb{P}^n$. Take Q to be the space of $(n+1) \times (n+1)$ matrices satisfying $A^2 = A$ and $\operatorname{Tr}(A) = 1$. Again you can cover $\mathbb{P}^n$ by $\mathbb{A}^n$s. The only thing that's a little more intricate is the determination of the fibres of $Q_{\mathbb{A}^n} \to \mathbb{A}^n$, but again they work out to be isomorphic to $\mathbb{A}^n$ again.

Example 9.3.3. According to Asok-Hoyois-Wendt, you can calculate the set $\mathrm{H}^1(\operatorname{Spec} R, \mathrm{GL}_n)$ of isomorphism classes of (Nisnevich) vector bundles of rank n on a smooth affine k-variety using $\mathbb{A}^1$ homotopy theory. These are equivalent to rank-n projective modules. The calculation is

$$[\operatorname{Spec} R, B_{\mathit{Nis}}\,\mathrm{GL}_n]_{\mathbb{A}^1} = \mathrm{H}^1(\operatorname{Spec} R, \mathrm{GL}_n)$$

In fact, again by Asok–Hoyois–Wendt, you can calculate this set using naive $\mathbb{A}^1$-homotopy':

$$\pi_0(\operatorname{Sing}^{\mathbb{A}^1} B_{\mathit{Nis}}(\operatorname{Spec} R)) = \mathrm{H}^1(\operatorname{Spec} R, \mathrm{GL}_n).$$

Grothendieck has calculated the isomorphism classes of rank-n vector bundles on $\mathbb{P}^1$. Note that $\mathbb{P}^1$ is not affine. Every rank-2 bundle is isomorphic to exactly one $\mathcal{O}(a) \oplus \mathcal{O}(b)$ where $a \geq b$. On the other hand, over Q, the pullback of $\mathcal{O}(1) \oplus \mathcal{O}(-1)$ is free of rank 2. In particular, the induced map

$$\pi_0(B_{\mathit{Nis}}\,\mathrm{GL}_2(\mathbb{P}^1)) \to \pi_0(B_{\mathit{Nis}}\,\mathrm{GL}_2(Q))$$

is not a bijection.

Therefore $B_{\mathit{Nis}}\,\mathrm{GL}_2$ is not $\mathbb{A}^1$-local.

Chapter 10

The Purity Theorem

10.1 Preliminaries

Definition 10.1.1. Suppose X is a k-variety and $n \geq 1$ is a natural number. A *vector bundle of rank* n on X is a map $p : V \to X$ such that there exists a (Zariski) open cover $\{f_i : U_i \to X\}_{i \in I}$, not part of the data of the bundle, such that

1. For each $i \in I$, the inverse image $U_i \times_X V \to U_i$ is isomorphic to the projection $U_i \times \mathbb{A}^n \to U_i$ via an isomorphism $\phi_i : U_i \times_X V \to U_i \times \mathbb{A}^n$
2. On a double overlap $(U_i \cap U_j)$, the composite map $\phi_i \circ \phi_j^{-1} : (U_i \cap U_j) \times \mathbb{A}^n \to (U_i \cap U_j) \times \mathbb{A}^n$ over $U_i \cap U_j$ is linear, i.e., a map $U_i \cap U_j \to \mathrm{GL}_n$.

Remark 10.1.2. Strictly, what we've defined is a Zariski-local vector bundle. There are other definitions, based on Nisnevich or étale covers, but it's a theorem—Hilbert's Theorem 90—that these definitions are equivalent.

Remark 10.1.3. A *trivial vector bundle* is one isomorphic to the projection $\mathbb{A}^n \times X \to X$ for some n. Every vector bundle is (Zariski) locally trivial, but may not be globally trivial. As a consequence of the local triviality, if $p : V \to X$ is a vector bundle map, then p is an $\mathbb{A}^1$-equivalence.

Remark 10.1.4. A vector bundle of rank n on X is a parametrized version of a vector space. There are addition and scalar multiplication operations: $+ : V \times_X V \to V$ and $\cdot : \mathbb{A}^1 \times V \to V$ and a zero-section $z : X \to V$.

Remark 10.1.5. Suppose $X = \operatorname{Spec} R$ is an affine variety, and $p : V \to X$ is a vector bundle of rank n. Then the set of sections $s : X \to V$ of p carries the structure of an R-module M. What's more, restricting to any affine open $\operatorname{Spec} R[1/f]$ and doing the same construction yields $M[1/f] := R[1/f] \otimes_R M$. Since V is locally free, in the geometric sense, it follows that there exists a set of elements $\{f_1, \ldots, f_r\}$ of R, generating the unit ideal and such that $M[1/f_i]$ is free of rank n for all i. This is equivalent to the statement that M is a projective module of rank n.

It is a theorem that this leads to an equivalence of categories:

$$\text{projective } R\text{-modules of rank } n \equiv \text{rank-}n \text{ vector bundles on } \operatorname{Spec} R.$$

Definition 10.1.6. If $p : V \to X$ is a vector bundle, then define the *Thom space* $Th_X(V)$ of V over X as

$$Th_X(V) := \frac{V}{V \setminus z(X)}$$

in the category of (simplicial) presheaves.

Example 10.1.7. Consider $\mathbb{P}^{n+1}$. This is a k-variety, and for any field E/k, one can describe $\mathbb{P}^{n+1}(E)$ as the set of $n+2$-tuples $(r_0, \ldots, r_{n+1}) \in E^{n+2} \setminus \mathbf{0}$ considered up to multiplication by $\lambda \in R^\times$. The notation for such an equivalence class is $[r_0 : \cdots : r_{n+1}]$.

There is an open embedding $\mathbb{A}^{n+1} \to \mathbb{P}^{n+1}$ given by sending $(r_1, \ldots, r_n) \mapsto [1 : r_1 : \cdots : r_n]$— this is what the map does on field-valued points. Justifying this map more rigorously takes work which is left as an exercise.

There is also an open subvariety W of $\mathbb{P}^{n+1}$ where the E-points look like $[r_0 : r_1 : \cdots : r_n]$ for which at least one of $r_1, \ldots, r_n$ is a unit (i.e., not 0). There is a map $p : W \to \mathbb{P}^n$ given by forgetting r_0. Letting W_i denote the open subvariety of W where r_i is not 0, we see that p is locally isomorphic to $\mathbb{A}^1 \times W_i \to W_i$ (normalize to $[r_0/r_i : r_1/r_i : \cdots : 1 : \cdots : r_n/r_i]$ then read off r_0/r_i.

The two open subvarieties W and the image of $\mathbb{A}^{n+1}$ inside $\mathbb{P}^{n+1}$ intersect to give the open subvariety of $\mathbb{A}^{n+1}$ consisting of points that can be described as $[1 : r_1 : \cdots : r_n]$ where $(r_1, \ldots, r_n) \neq \mathbf{0}$. In summary, there is a (homotopy) pushout diagram:

$$\begin{tikzcd}
\mathbb{A}^{n+1} \setminus \{0\} \arrow[r, hook] \arrow[d, hook] & \mathbb{A}^{n+1} \arrow[d] \\
W \arrow[r] & \mathbb{P}^{n+1}
\end{tikzcd}$$

It's a formal consequence of such a pushout diagram that there is an isomorphism of quotients

$$\mathbb{A}^{n+1}/(\mathbb{A}^{n+1} \setminus \mathbf{0}) \cong \mathbb{P}^{n+1}/W$$

but the quotient $\mathbb{P}^{n+1}/W$ is a pushout in

$$\begin{tikzcd}
W \arrow[r] \arrow[d] & \mathbb{P}^{n+1} \arrow[d] \\
* \arrow[r] & \mathbb{P}^{n+1}/W
\end{tikzcd}$$

which is also a homotopy pushout by virtue of $W \to \mathbb{P}^{n+1}$ being an inclusion. We can replace $W \to \mathbb{P}^{n+1}$ by the $\mathbb{A}^1$-equivalent closed inclusion $\mathbb{P}^n \to \mathbb{P}^{n+1}$ and we deduce that

$$\mathbb{A}^{n+1}/(\mathbb{A}^{n+1} \setminus \{0\}) \simeq_{\mathbb{A}^1} \mathbb{P}^{n+1}/\mathbb{P}^n.$$

In fact, since $\mathbb{A}^{n+1} \simeq_{\mathbb{A}^1} *$, the quotient on the left is $\mathbb{A}^1$-equivalent to the homotopy pushout of

$$\begin{tikzcd}
{\mathbb{A}^1}^{n+1} \setminus \mathbf{0} \arrow[r] \arrow[d] & * \\
*
\end{tikzcd}$$

which is equivalent to $\Sigma {\mathbb{A}^1}^{n+1} \setminus 0 \simeq_{\mathbb{A}^1} S^{n+1} \wedge \mathbb{G}_m^{\wedge n+1}$.

In other words, $\mathbb{P}^{n+1}/\mathbb{P}^n \simeq_{\mathbb{A}^1} (\mathbb{P}^1)^{\wedge n+1}$.

Remark 10.1.8. In [MV99], they define the Thom space of V to be

$$Th_X(V) = \frac{\mathbb{P}(V \times \mathbb{A}^1)}{\mathbb{P}(V)}.$$

This is a parametrized version of the previous example (see [AE16, Section 4.7]).

Definition 10.1.9. Suppose $X = \operatorname{Spec} R$ is a smooth affine k-variety and I is an ideal defining a closed smooth subvariety, Z. Then I/I^2 has the structure of an R/I-module, and if I is smooth over k this is a projective R/I-module. Then $N_Z(X)$, the *normal bundle* of Z in X is the vector bundle associated to the dual module: $\operatorname{Hom}_{R/I}(I/I^2, R/I)$.

If X and Z are not affine, then the vector bundle $N_Z(X)$ can still be defined, by defining it on each term of an affine cover and proving it glues together properly.

Remark 10.1.10. The normal bundle of Z in X has a geometric meaning: for a $\mathbb{C}$-variety, for instance, it is the vector bundle that assigns to a point $z \in Z$ the vector space of all normal vectors to Z in X at the point z.

Even outside of this, and importantly, the bundle $N_Z(X)$ on Z depends on the structure of X 'near' Z only. For instance: Suppose $I \subseteq R$ is an ideal and $f \in R$ is an element such that f is a unit in R/I. Then f defines an affine open subvariety $U \subset \operatorname{Spec} R = X$. It is the case that $U \cap Z = Z$, and there is a map $N_Z(U) \to N_Z(X)$, and this map is an isomorphism. This is elementary: it's because $R/1 \to R/I \otimes_R R[1/f]$ is an isomorphism.

10.2 Pushouts

We're going to be using (homotopy) pushouts extensively.

Remark 10.2.1. Recall that if $A \to B$ is a cofibration in a (simplicial) proper model category, then the colimit and the homotopy colimit of a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \\ C & \end{tikzcd}$$

are weakly equivalent.

Example 10.2.2. In a simplicial pointed proper model category, the homotopy colimit of $* \leftarrow A \to *$ (where A is cofibrant) is weakly equivalent to $A \wedge S^1$.

Definition 10.2.3. In any category with a terminal object and all pushouts we can define the *cofibre* of a map $A \to B$ as the pushout

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ * \arrow[r] & B/A \end{tikzcd}.$$

Similarly, the *homotopy cofibre* is defined as a homotopy pushout.

Remark 10.2.4. The formation of (homotopy) cofibres is functorial, so that if

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d] & B_1 \arrow[d] \\ A_2 \arrow[r] & B_2 \end{tikzcd}$$

commutes, then there is an induced map $B_1/A_1 \to B_2/A_2$.

Proposition 10.2.5. *If a square*

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d] & B_1 \arrow[d] \\ A_2 \arrow[r] & B_2 \end{tikzcd}$$

is a pushout square, then $B_1/A_1 \to B_2/A_2$ is an isomorphism. If the category is pointed, then the converse is also true.

Proof. This is an exercise in chasing diagrams. □

Corollary 10.2.6. *In a simplicial proper model category, given a homotopy pushout*

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d] & B_1 \arrow[d] \\ A_2 \arrow[r] & B_2 \end{tikzcd}$$

the induced map on homotopy cofibres $B_1/_h A_1 \to B_2/_h A_2$ is a weak equivalence. If the category is pointed, then the converse also holds.

10.3 Blowing up

In the next section, we will be making use of the blowup of a variety at a closed subvariety. An account of the theory of blowing up would take too long. We refer to [Har77, Section II.7] or [Vak15, Chapter 22] for the general theory. For us, the following will have to suffice:

Proposition 10.3.1. *If (X, Z) is a smooth pair, then $\mathrm{Bl}_Z X$ is smooth variety.*

Proposition 10.3.2. *Let Z be a closed subvariety of X. There is a map $f : \mathrm{Bl}_Z X \to X$ such that $f^{-1}(X \setminus Z) \to X \setminus Z$ is an isomorphism and such that $f^{-1}(Z)$ is of codimension 1 (it is an effective Cartier divisor).*

Construction 10.3.3. This construction is called "deformation to the normal cone". Suppose $Z \subseteq X$ is a smooth pair (where Z and X are connected, although the disconnected case can be handled quite easily). Then it is possible to form $\mathrm{Bl}_{Z\times 0}(X \times \mathbb{A}^1)$. This is equipped with a map to $X \times \mathbb{A}^1$ and we view the $\mathbb{A}^1$ as a parameter space:

$$\begin{tikzcd} \mathrm{Bl}_{Z\times 0}(X \times \mathbb{A}^1) \arrow[rr, "f"] \arrow[dr] & & X \times \mathbb{A}^1 \arrow[dl] \\ & \mathbb{A}^1 & \end{tikzcd} .$$

Lying over all of $\mathbb{A}^1$ except 0, the map f is an isomorphism. Over 0, however, the fibre of $\mathrm{Bl}_{Z\times 0}(X\times\mathbb{A}^1)\to\mathbb{A}^1$ consists of two irreducible components. One of these is $\mathrm{Bl}_Z X$. The other is the (fibrewise) projectivization of the normal bundle of Z in X, denoted $\mathbb{P}(N_Z X)$.

10.4 The Purity Theorem

The following proof is adapted from the presentation in [AE16].

Notation 10.4.1. A *smooth pair* of varieties (X,Z) consists of smooth k-variety X and a closed smooth subvariety Z. A morphism of pairs $f:(X,Z)\to(X',Z')$ is a map $f:X\to X'$ for which $f(Z)\subseteq Z'$ and such that the square

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d] \\ Z' \arrow[r] & X' \end{tikzcd}$$

is a pullback square.

A morphism is a *Nisnevich map of smooth pairs* if $f:X\to X'$ is an étale map and $f(Z)\to Z'$ is an isomorphism.

Definition 10.4.2. A morphism $f:(X,Z)\to(X',Z')$ of smooth pairs is *weakly excisive* if the induced square

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X/(X\setminus Z) \arrow[d] \\ Z' \arrow[r] & X'/(X'\setminus Z') \end{tikzcd}$$

is a homotopy pushout in the $\mathbb{A}^1$-model structure.

Example 10.4.3. If $(X,Z)\to(X',Z')$ is a map of smooth pairs such that the maps $X\to X'$, $Z\to Z'$ and $X\setminus Z\to X'\setminus Z'$ are all $\mathbb{A}^1$-equivalences, then the map is weakly excisive. This is the case if $X\to X'$ is the structure map of a vector bundle, or a section of a vector bundle.

Example 10.4.4. If

$$\begin{tikzcd} U\times_X V \arrow[r] \arrow[d] & V \arrow[d] \\ U \arrow[r] & X \end{tikzcd}$$

is an elementary Nisnevich square, and if we write $Z=X\setminus U$ (and also $Z=V\setminus(U\times_X V)$), then $(V,Z)\to(U,Z)$ is weakly excisive. In fact, the square to be checked is the ENS itself, which is a homotopy pushout.

Remark 10.4.5. Let $(X,Z)\xrightarrow{f}(Y,W)\xrightarrow{g}(U,V)$ be composable maps of smooth pairs. If f is weakly excisive, then g is weakly excisive if and only if $g\circ f$ is. This is an easy exercise in homotopy pushout squares.

If g and $g\circ f$ are weakly excisive and g induces an $\mathbb{A}^1$-equivalence $W\to V$, then f is weakly excisive. This is a little more complicated, but still left as an exercise.

We will use the following version of [GR02, Exposé II, Théorème 4.10] without proof:

Theorem 10.4.6. *Suppose (X, Z) is a smooth pair where Z is of codimension c in X. Then there exists a Zariski cover $\{U_i \to X\}_{i \in I}$ and a set of étale morphisms $\{g_i : U_i \to \mathbb{A}^{n_i}\}_{i \in I}$ such that for all i the smooth pair $(U_i, Z \cap U_i)$ is isomorphic to the pullback of an inclusion of a linear subspace $(\mathbb{A}^{n_i}, \mathbb{A}^{n_i - c})$ along g_i.*

Lemma 10.4.7. *Suppose* **P** *is a property of smooth pairs (X, Z) such that:*

1. *if $\{U_i \to X\}_{i \in I}$ is a Zariski cover such that* **P** *holds for $\{(\bigcap_{j \in J} U_j, \bigcap_{j \in J} U_j \cap Z)\}$ for all nonempty subsets $J \subseteq I$, then* **P** *holds for (X, Z);*

2. *if $(V, Z) \to (X, Z)$ is a Nisnevich morphism of smooth pairs, then (V, Z) has* **P** *if and only if (X, Z) has* **P**.

3. *All pairs $(\mathbb{A}^n \times Z, Z)$, given by inclusion at* **0**, *have* **P**.

Then all smooth pairs have **P**.

Proof. Using property 1, we may suppose the pair (X, Z) is equipped with an étale map $g : X \to \mathbb{A}^n$ such that $Z = g^{-1}(\mathbb{A}^m)$ for some embedded $\mathbb{A}^m$, where Z has codimension $n - m$ in X.

Write $c = n - m$, and produce the product $X \times_{\mathbb{A}^n} (Z \times \mathbb{A}^c)$ where $Z \times \mathbb{A}^c \to \mathbb{A}^n$ is given by $g|_Z$ on the first component, and $\mathbb{A}^c$ is included as the last c components of $\mathbb{A}^n$.

There are étale maps as indicated

$$\begin{tikzcd} & X \times_{\mathbb{A}^n} (Z \times \mathbb{A}^c) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ Z \times \mathbb{A}^c & & X. \end{tikzcd}$$

That these are étale can be checked fairly easily, once you remember that the product of two étale maps is again étale. We would be done if $Z \times_{\mathbb{A}^n} Z$ mapped isomorphically down to Z under p_1 and p_2, but unfortunately, it need not.

It is the case that we can identify $Z \times_{\mathbb{A}^n} Z = Z \times_{\mathbb{A}^{n-c} \times \mathbf{0}} Z$, which has the advantage that we are taking a fibre product of étale maps

Similar to a homework exercise, however, $Z \times_{\mathbb{A}^{n-c}} Z \to Z$ is an étale map that has the diagonal $\Delta : Z \to Z \times_{\mathbb{A}^{n-c}} Z$ as a section. Therefore there is some closed $Y \subseteq Z \times_{\mathbb{A}^{n-c}} Z$ that such that $Z \times_{\mathbb{A}^{n-c}} Z = Z \coprod Y$. If we modify $X \times_{\mathbb{A}^n} (Z \times \mathbb{A}^c)$ by discarding the closed subvariety Y, we are left with two Nisnevich maps of smooth pairs:

$$((X \times_{\mathbb{A}^1} (Z \times \mathbb{A}^c)) \setminus Y, Z) \to (X, Z)$$
$$((X \times_{\mathbb{A}^1} (Z \times \mathbb{A}^c)) \setminus Y, Z) \to (Z \times \mathbb{A}^c, Z).$$

Therefore using 2, we have reduced the general case to the case of 3. □

Theorem 10.4.8. *Let (X, Z) be a smooth pair. There is a natural isomorphism in the $\mathbb{A}^1$-homotopy category*

$$\frac{X}{X \setminus Z} \simeq Th(N_Z X).$$

Proof. We construct a zig-zag of maps. Construct

$$D_Z X := \mathrm{Bl}_{Z\times\{0\}}(X \times \mathbb{A}^1) \setminus \mathrm{Bl}_{Z\times\{0\}}(X \times \{0\})$$

which is natural in smooth pairs. This is equipped with a map to $X \times \mathbb{A}^1$ and therefore to $\mathbb{A}^1$. The fibre at $\{0\}$ is $\mathbb{P}(N_Z X \oplus \mathcal{O}_Z) \setminus \mathbb{P}(N_Z X) \cong N_Z X$. We can find an embedded $Z \times \mathbb{A}^1$ as a closed subvariety of $D_Z X$; most fibres of $D_Z X$ are just X again, while the fibre at 0 is $\cong N_Z X$ and here the copy of Z is embedded as the 0-section.

Consider the smooth pair $(D_Z X, Z \times \mathbb{A}^1)$. There is a diagram

$$(X, Z) \xrightarrow{i_1} (D_Z X, Z \times \mathbb{A}^1) \xleftarrow{i_0} (N_Z X, Z)$$

and we claim that each of these two arrows is weakly excisive.

If we can prove this claim, then we can say that $X/(X \setminus Z) \to D_Z X/(D_Z X \setminus Z \times \mathbb{A}^1)$ is an $\mathbb{A}^1$-equivalence, and so too is $N_Z X/(N_Z X \setminus Z) \to D_Z X/(D_Z X \setminus Z \times \mathbb{A}^1)$. Here, $N_Z/(N_Z X \setminus Z)$ is a presentation of $Th(N_Z X)$, establishing the result.

Now we wish to show that both maps i_0 and i_1 are weakly excisive. Say a pair (X, Z) has property **P** if this holds. We show that property **P** satisfies the condition of Lemma 10.4.7.

1. If $\{U_i \to X\}_{i\in I}$ is a Zariski cover of X such that for each intersection $U_{\mathbf{i}}$, the pair $(U_{\mathbf{i}}, Z \cap U_{\mathbf{i}})$ has property **P**, then (X, Z) has property **P**. The proof of this is straightforward, but we sketch it here anyway.

   Let $\mathbf{i}$ be an n-tuple of elements in I. Let $U_{\mathbf{i}} = U_1 \cap U_2 \cap \cdots \cap U_n$ and similarly for $Z_{\mathbf{i}}$, $D_{Z_{\mathbf{i}}} X$. Then there is a (homotopy) pushout square

$$\begin{tikzcd}
Z_{\mathbf{i}} \arrow[r] \arrow[d] & U_{\mathbf{i}}/(U_{\mathbf{i}} \setminus Z_{\mathbf{i}}) \arrow[d] \\
Z_{\mathbf{i}} \times \mathbb{A}^1 \arrow[r] & D_{Z_{\mathbf{i}}} U_{\mathbf{i}}/D_{Z_{\mathbf{i}}} U_{\mathbf{i}} \setminus (Z_{\mathbf{i}} \times \mathbb{A}^1)
\end{tikzcd}$$

   and these different squares assemble to produce a (homotopy) pushout square of simplicial presheaves

$$\begin{tikzcd}
Z_\bullet \arrow[r] \arrow[d] & U_\bullet/U_\bullet \setminus Z_\bullet \arrow[d] \\
Z_\bullet \times \mathbb{A}^1 \arrow[r] & D_{Z_\bullet} U_\bullet/D_{Z_\bullet} U_\bullet \setminus (Z_\bullet \times \mathbb{A}^1)
\end{tikzcd}$$

   which is homotopy equivalent to

$$\begin{tikzcd}
Z \arrow[r] \arrow[d] & X/(X \setminus Z) \arrow[d] \\
Z \times \mathbb{A}^1 \arrow[r] & D_Z X/(D_Z X \setminus (Z \times \mathbb{A}^1))
\end{tikzcd}$$

   This establishes weak descent for i_1. The argument for i_0 is similar.

2. We want to show that $(V,Z) \to (X,Z)$ is a Nisnevich map of smooth pairs such that (V,Z) has property **P**, then (X,Z) has property **P**. Looking at the diagram

$$\begin{tikzcd}
(V,Z) \arrow[r, "i_1"] \arrow[d] & (D_Z V, Z \times \mathbb{A}^1) \arrow[d] & (N_Z V, Z) \arrow[l, "i_0"'] \arrow[d] \\
(X,Z) \arrow[r, "i_1"] & (D_Z X, Z \times \mathbb{A}^1) & (N_Z X, Z) \arrow[l, "i_0"']
\end{tikzcd}$$

and using Remark 10.4.5, we see that it's enough to show that the vertical maps are weakly excisive. The vertical maps are Nisnevich maps of smooth pairs, so it's sufficient to verify that a Nisnevich map of smooth pairs is weakly excisive. This is example 10.4.4.

3. The last thing we have to show is that a pair $(Z \times \mathbb{A}^n Z, Z)$, where Z is embedded at **0**, has property **P**. This is not very difficult. The variety Z comes along for the ride, all the argument is already inherent in the case of $(\mathbb{A}^n, 0)$. Recall the definition of $D_Z(Z \times \mathbb{A}^n)$ as a difference of two blow-ups:

$$\mathrm{Bl}_{Z\times 0}(Z \times \mathbb{A}^n \times \mathbb{A}^1) \setminus \mathrm{Bl}_{Z\times 0}(Z \times \mathbb{A}^n \times 0),$$

The larger blow up here is the total space of an $\mathbb{A}^1$-bundle over $Z \times \mathbb{P}^n$, and the smaller is the total space of the sub-bundle over $Z \times \mathbb{P}^{n-1}$. We may identify $D_Z(Z \times \mathbb{A}^n)$ as the total space of an $\mathbb{A}^1$-bundle over $Z \times \mathbb{A}^n$—in fact, this bundle is trivial.

The maps $i_1 : (Z \times \mathbb{A}^n, Z) \to (D_Z Z \times \mathbb{A}^n, Z \times \mathbb{A}^1)$ and $i_0 : (N_Z Z \times \mathbb{A}^n, Z) \to (D_Z Z \times \mathbb{A}^n, Z \times \mathbb{A}^1)$ are both sections of this bundle, and are therefore weakly excisive. This concludes the proof.

□

Remark 10.4.9. This is a kind of excision result, since it says that the homotopy type of $X/(X \setminus Z)$ depends only on the structure of X 'near Z'.

Chapter 11

Motivic Cohomology

11.1 Ω-spectra

Throughout work in a pointed proper simplicial model category, $\mathbf{M}$.

Definition 11.1.1. We say a sequence of fibrant objects $\{E_n\}_{n\in\mathbb{N}}$ and maps $s_n : E_n \to \Omega E_{n+1}$ is an Ω-*spectrum* if each of the maps s_n is a weak equivalence.

Notation 11.1.2. Suppose E is an Ω-spectrum. Define a functor $E^i : \mathbf{M} \to \mathbf{Ab}$ by

$$E^i(X) = [X, E_i]$$

Superficially, this takes values in sets, but in fact $[X, E_i] = [X, \Omega^2 E_{i+2}]$, which carries an abelian group structure.

Remark 11.1.3. We see directly that there are suspension isomorphisms $E^i(X) \cong E^{i+j}(\Sigma^j X)$. This allows us to extend the definition to $E^{-i}(X) = E^0(\Sigma^i X)$ for negative values of i.

Proposition 11.1.4. *Let E be an Ω-spectrum as above. Suppose $f : X \to Y$ is a map with homotopy cofibre $g : Y \to C_f$. Then there is a natural long exact sequence*

$$\cdots \to E^i(C_f) \to E^i(Y) \to E^i(X) \to E^{i+1}(C_f) \to \ldots$$

Proof. For simplicity, we assume all objects appearing are cofibrant. If they are not, replace f by $Qf : QX \to QY$ and proceed from there.

First of all, the homotopy cofibre of g is equivalent to a map $h : C_f \to \Sigma X$, and iterating this, we get a map equivalent to $-\Sigma f : \Sigma X \to \Sigma Y$.

The homotopy cofibre sequences we obtain in this way are called *rotations* of the original

Applying the simplicial mapping functor $\mathcal{S}(\cdot, RE_i)$ to a homotopy cofibre sequence to get a homotopy fibre sequence of simplicial sets:

$$\mathcal{S}(C_f, RE_i) \to \mathcal{S}(Y, RE_i) \to \mathcal{S}(X, RE_i).$$

Apply π_0 to get an exact sequence $[C_f, E_i] \to [Y, E_i] \to [X, E_i]$. Applying this to the various rotations of the original sequence gives us the long exact sequence. $\square$

Proposition 11.1.5. *There is an isomorphism*

$$E^i\Big(\bigvee_{\alpha\in A} X_\alpha\Big) = \Big[\bigvee_{\alpha\in A} X_\alpha, E\Big] \cong \prod_{\alpha\in A} E^i(X_\alpha).$$

Remark 11.1.6. That is, the Ω-spectrum E defines a generalized cohomology theory on $\mathbf{M}$.

Example 11.1.7. In the Quillen model structure on Kelly spaces, we may set $E_i = K(A, i)$. Then $\Omega E_{i+1} \simeq E_i$, so we have an infinite loop space. The associated cohomology theory satisfies

$$E^i(S^n) = \begin{cases} A & \text{if } i = n \\ 0 & \text{otherwise.} \end{cases}$$

That is, this is ordinary reduced cohomology with coefficients in A.

11.2 Motivic Cohomology

The construction of Motivic Eilenberg–Mac Lane objects in this manner really belongs to [Voe10]. This is not easy to read, however. The general idea is sketched in [VRØ07, p162–164], and we attempt to explain that a little more.

Let k be a perfect field, i.e., one without finite inseparable extensions. Let c denote the exponential characteristic of the field throughout. This is 1 if the field has characteristic 0 and p if it has characteristic p.

To define motivic cohomology, we define an Ω-spectrum in the $\mathbb{A}^1$-homotopy theory. In fact, we define a family of Ω-spectra.

Definition 11.2.1. Suppose $X, Y \in \mathbf{Sm}_k$. A *finite correspondence* from X to Y is a finite formal sum $\sum_{i=1}^n a_i Z_i$ where each $a_i \in \mathbb{Z}$ and each Z_i is a closed subvariety of $X \times Y$ such that the maps induced by the projection $Z_i \to X$ are finite and surjective onto a connected component of X.

Definition 11.2.2. Suppose Y is an object of $\mathbf{Sm}_k$, and A is an abelian group. Define $L(Y)$ to be the functor

$$L(Y) : \mathbf{Sm}_k^{\mathrm{op}} \to \mathbf{Set} \subseteq \mathbf{sSet}$$

given by

$$L(Y)(X) = \text{finite correspondences } X \to Y.$$

In fact, this takes values in abelian groups. We define $L(Y; A) = L(Y) \otimes_{\mathbb{Z}} A$.

Remark 11.2.3. This really is functorial in X. If $f : X_1 \to X_2$ is a map of varieties, the functoriality in f is given by pulling finite surjective maps $Z \to X_2$ back along f.

Proposition 11.2.4. *Fix an abelian group A. Then the construction $Y \mapsto L(Y; A)$ is a functor*

$$\mathbf{Sm}_k \to \mathbf{sPre}(\mathbf{Sm}_k)$$

Remark 11.2.5. It suffices to prove this when $A = \mathbb{Z}$.

In fact, more is true. It is possible to define a category $\mathbf{Cor}_{k,A}$ having the same objects as $\mathbf{Sm}_k$ but in which $\mathbf{Cor}_{k,A}(X, Y)$ is the group of A-linear finite correspondences $X \to Y$. The fact that there is actually a composition of correspondences is proved in [MVW06, Chapter 1]. The composition is defined as follows:

Suppose $Z_0 \subset X_0 \times X_1$ and $Z_1 \subset X_1 \times X_2$ are finite and surjective over a component of X_0 and X_1 respectively. One can pull Z_0 and Z_1 back to $p_3^*(Z_0), p_1^*(Z_1)$, both closed in $X_0 \times X_1 \times X_2$, then intersect $p_3^*(Z_0) \cap p_1^*(Z_1)$, then take the image under the projection $X_0 \times X_1 \times X_2$. The fact that this all actually works out to give a well-defined operation is done in [MVW06, Chapter 1].

Now suppose $f : Y \to W$ is a map of varieties. Then the graph $\Gamma_f \subset X \times Y$ is a closed subvariety, consisting of pairs $(x, f(x))$. The graph provides a finite correspondence $X \to Y$. In this way, the category $\mathbf{Sm}_k$ embeds in $\mathbf{Cor}_k$. From this point of view $L(Y)$, as defined previously, is essentially the Yoneda functor of $Y \in \mathbf{Cor}_k$.

Remark 11.2.6. The presheaf $L(Y; A)$ is actually a Nisnevich sheaf. This is a kind of descent argument, not given further here.

Notation 11.2.7. Let A be an abelian group.

If X, x_0 is a pointed scheme, then we define $L(X, x_0; A)$ as the cokernel $L(x_0; A) \to L(X; A)$.

If $\mathcal{X}, x_0$ is a pointed presheaf on $\mathbf{Sm}_k$, then we may define $L(\mathcal{X}, x_0; A)$ by writing $\mathcal{X}$ as a filtered colimit of pointed representable presheaves $(\mathcal{X}, x_0) = \operatorname{colim}_i(U_i, u_{i,0})$, then setting

$$L(\mathcal{X}, x_0; A) = \operatorname*{colim}_i L(U_i, u_{i,0}; A).$$

Remark 11.2.8. The $\mathbb{A}^1$-homotopy type of $L(\mathcal{X}, x_0; A)$ depends only on the $\mathbb{A}^1$-homotopy type of $(\mathcal{X}, x_0)$ and on A. This is difficult, and we refer to [Voe10].

Remark 11.2.9. We have not extended $L(\cdot; A)$ to simplicial presheaves, but this isn't necessary. Given any simplicial presheaf, there is some presheaf concentrated in degree 0 that is $\mathbb{A}^1$-equivalent. Most importantly, we can find a version of the mapping cone construction: given $f : \mathcal{X} \to \mathcal{Y}$, a map of presheaves, produce $\mathcal{X} \times \mathbb{A}^1$, then replace $\mathcal{X} \times \{1\}$ by $\mathcal{Y} \times \{1\}$ using f, and collapse $\mathcal{X} \times \{0\}$ to a point.

Since S^0 is a presheaf, we can find a presheaf model for S^n for all $n \geq 0$.

Definition 11.2.10. Let A be an abelian group in which the exponential characteristic c of k is invertible. Then define

$$K(A, p, q) := R_{\mathbb{A}^1} L(S^p \wedge \mathbb{G}_m^{\wedge q}, *; A).$$

Proposition 11.2.11. *There are weak equivalences*

$$\operatorname{Map}_+(S^1, K(A, p, q)) \to K(A, p-1, q)$$
$$\operatorname{Map}_+(\mathbb{G}_m, K(A, p, q)) \to K(A, p, q-1).$$

Corollary 11.2.12. *Let A be an abelian group in which c is invertible. For any object X of $\mathbf{sPre}(\mathbf{Sm}_k)$, we define the motivic cohomology*

$$\tilde{\mathrm{H}}^p(X, A(q)) = [X, K(A, p, q)]_\bullet$$

Remark 11.2.13. We have not defined an S^1-$\mathbb{G}_m$-bispectrum, but this is what $K(A, \cdot, \cdot)$ is. This allows us to define $\mathrm{H}^p(X, A(q))$ even for negative values of p and q.

Example 11.2.14. Why is this remotely plausible?

consider the following construction in classical topology. Start with a space, X, equipped with a basepoint $x_0 \in X$. Take $\mathrm{Sing}\, X$, a simplicial set pointed at x_0. Then form the free simplicial abelian group $\tilde{\mathbb{Z}}[X] := \mathbb{Z}\,\mathrm{Sing}\, X/\mathbb{Z}x_0$ object. The Dold–Kan correspondence says that $\pi_i(\tilde{\mathbb{Z}}[X])$ calculates the singular homology $\mathrm{H}_i(X;\mathbb{Z})$, and therefore $|\tilde{\mathbb{Z}}[X]|$ is a space for which the homotopy groups are the $\mathbb{Z}$-homology of X. Similar constructions apply also for groups other than $\mathbb{Z}$.

The space $|\tilde{\mathbb{Z}}[X]|$ is a kind of topological construction of the "free abelian group on (X, x_0)", and this construction is a version of the Dold–Thom theorem.

If X is a sphere, S^n, then $|\tilde{\mathbb{Z}}[S^n]|$ gives a construction of the Eilenberg–Mac Lane space $K(\mathbb{Z}, n)$.

Suppose k for now is algebraically closed of characteristic 0. Suppose U, u_0 is a pointed scheme that is also a motivic sphere. Let's consider $L(U)(k)$. This consists of formal sums of closed subschemes of $U \times k = U$ that are finite (and surjective) over k. That is, $L(U)(k)$ consists of formal sums of the elements in $U(k)$.

There is a difference between $L(U)$ and $\mathbb{Z}U$—the presheaf that sends an arbitrary X to formal sums of elements of $U(X)$—in that $L(U)$ takes more account of the geometry of U.

The presheaf $L(U, u_0)(k)$ is the quotient of $L(U)(k)$ obtained by setting $u_0 = 0$. Therefore, $L(U, u_0)$ is a version of the same construction used in the Dold–Thom theorem.

Here are some more results about motivic cohomology, which can't be deduced immediately from what we've already proved.

They are all proved in [MVW06] (although the definition of motivic cohomology there is superficially different from here, it turns out to be equivalent).

Proposition 11.2.15. *Suppose $Z \to X$ is a smooth pair of codimension c. Then there is a "Thom Isomorphism"*

$$\tilde{\mathrm{H}}^p(Th_Z(X); A(q)) \cong \mathrm{H}^{p-2c}(Z; A(q-x))$$

Proposition 11.2.16. *There are vanishing results:*

1. *For all X, the groups $\mathrm{H}^p(X, A(q)) = 0$ when $q < 0$.*
2. *When X is a smooth k-variety, $\mathrm{H}^p(X, \mathbb{Z}(q)) = 0$ whenever $p > 2q$ and $\mathrm{H}^{2p}(X, \mathbb{Z}(p)) \cong \widetilde{CH}^p(X)$.*
3. *If X is a smooth k-variety of dimension d, then $\mathrm{H}^p(X, \mathbb{Z}(q)) = 0$ whenever $p > q + d$, and if $X = \mathrm{Spec}\, k$, then $\mathrm{H}^p(\mathrm{Spec}\, k, \mathbb{Z}(p)) \cong K_p^M(k)$, the Milnor K-theory.*

The following is the Beilinson–Lichtenbaum conjecture, now proved as a part of the Norm–Residue Isomorphism Theorem ([HW19]).

Proposition 11.2.17. *Let X be a smooth variety over a field k (of characteristic different from ℓ). Then there is an isomorphism*

$$\mathrm{H}^p(X, \mathbb{Z}/(\ell)(q)) \cong \mathrm{H}^p_{\acute{e}t}(X, \mu_\ell^{\otimes q})$$

for all $p \leq q$.

(here μ_ℓ denotes the étale sheaf of ℓ-th roots of unity).

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