HOMEWORK 4

Due on 31 March 2020, v2.

(1) In this question, you may assume that the formation of the nerve $N\mathbf{C}$ of a small category is functorial, in that a functor $h : \mathbf{C} \to \mathbf{D}$ induces a map of simplicial sets $Nh : N\mathbf{C} \to N\mathbf{D}$. Moreover, a natural transformation $\Phi : h \to h'$ induces a (left) homotopy $Nh \simeq Nh'$.

  (a) Suppose $\iota : \mathbf{C} \to \mathbf{D}$ is an equivalence of small categories. Show that $N\iota$ is a weak equivalence of simplicial sets. (You may use any well known definition of "equivalence of categories")

  (b) Suppose $h : M \to S$ is a surjective function of sets. Form a category $\mathbf{C}$ where the objects are the elements of M and there is a unique morphism from m to m' if and only if $h(m) = h(m')$, and no morphisms otherwise. Prove that $N\mathbf{C}$ is homotopy equivalent to the discrete set S.

  (c) Let $\mathbf{C}, \tau$ be a site with enough points. Let $X \in \mathbf{C}$ be an object. Let $\mathcal{U} = \{f_i : U_i \to X\}_{i \in I}$ be a covering family. Form the *nerve of the covering*, i.e., the simplicial presheaf $N\mathcal{U}$ having as m-th level a disjoint union of all $(m+1)$-fold products $\eta_{U_{i_0}} \times_{\eta_X} \cdots \times_{\eta_X} \eta_{U_{i_m}}$. Prove that the evident map $N\mathcal{U} \to X$ is a local weak equivalence.

(2) Prove that a trivial fibration $f : X \to Y$ of simplicial sets is surjective on 0-simplices (this should be easy).

(3) Suppose $X \to Y$ is the inclusion of a CW subcomplex in the category of k-spaces. Consider the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \\ * & \end{tikzcd}$$

Prove that the natural map from the homotopy colimit to the colimit of this diagram is a weak equivalence. Hint: collapsing a contractible subcomplex of a CW complex Z results in a space equivalent to Z. Either prove this or find a reference.

(4) Let $p \geq 2$ be a prime number. Consider the self map $f_n : S^1 \to S^1$ given by $z \mapsto z^p$. This is a map in the pointed category of CW complexes. Suppose (X, x_0) is a pointed simply connected CW complex that is $\{f_p\}$-local. Prove that $\pi_i(X, x_0)$ is uniquely p-divisible for all $i \geq 0$. Suppose Y is an $\{f_p\}$-local replacement for S^2. Describe the cohomology $\mathrm{H}^*(Y, \mathbb{Z})$.

You may assume without proof that $\tilde{\mathrm{H}}^i(Y, A) = [Y, K(A, i)]$ in the pointed Quillen model structure.