HOMEWORK 1

Due on 10 February 2020.

(1) Give an example, with proof, of a map of topological spaces $f : X \to Y$ that is a weak equivalence but not a homotopy equivalence. In order to prove a map is not a homotopy equivalence, it may be helpful to know about the Strøm model structure on compactly generated weak Hausdorff spaces ([BR13, Section 2]).

(2) Suppose $\mathbf{C}$ is a model category. A *pointed object* of $\mathbf{C}$ is an object X and a map $x_0 : * \to X$ (a basepoint). A map of pointed objects is a map $f : (X, x_0) \to (Y, y_0)$ such that $f(x_0) = y_0$. There is a category $\mathbf{C}_+$ of pointed objects in $\mathbf{C}$. There is a forgetful functor $U : \mathbf{C}_+ \to \mathbf{C}$, and a disjoint-basepoint functor $P : X \mapsto X \coprod *$. Put a model structure on $\mathbf{C}_+$ where the weak equivalences, fibrations and cofibrations are weak equivalences, fibrations and cofibrations in $\mathbf{C}$. You do not have to verify that this is a model structure (it is not a particularly difficult proof, [Hov99, Proposition 1.1.8]) Prove that P, U form a Quillen adjunction.

(3) Let $f : R \to S$ be a map of (unital) rings, and let $\mathbf{Ch}(R)$ and $\mathbf{Ch}(S)$ denote the categories of nonnegatively grded chain complexes of left R- and S-modules. There exists a model structure on $\mathbf{Ch}(R)$ where the weak equivalences are the quasi-isomorphisms, the fibrations are the levelwise surjective maps and the cofibrant objects are the complexes that are levelwise projective. This is a variant of the structure in [Hov99, Section 2.3]

There is an adjoint pair of functors $\epsilon : R\mathbf{Mod} \rightleftarrows S\mathbf{Mod} : \rho$ given by extension $\epsilon(M) = S \otimes_R M$ and restriction $\rho(M)$ of scalars. The R-module $\rho(M)$ agrees with M as an abelian group and the R-structure is given by $r \cdot m = f(r)m$. Prove that ϵ and ρ induce an adjoint pair of functors between $\mathbf{Ch}(R)$ and $\mathbf{Ch}(S)$. Prove that ρ is a right Quillen functor, and describe what the total derived functor of ϵ does to objects (it's enough to give the isomorphism class of $L\epsilon(X)$ when X is a chain complex—don't worry about functoriality of the construction).

Note: a version of this construction also makes sense for unbounded complexes, but it is harder to describe the cofibrant objects in that case, so I have not assigned this as a problem.

(4) The simplicial set $\Delta[n]$ has $\binom{n+1}{k+1}$ nondegenerate k simplices, each corresponding to a subset of $\{0, 1, \dots, n\}$. Consider the 0-simplices of $\Delta[n] \times \Delta[1]$. These can be identified with $\{0, 1, \dots, n\} \times \{0, 1\}$. Write $(i, 0) = i_0$ and $(i, 1) = i_1$, so that the 0-simplices of $\Delta[n] \times \Delta[1]$ are $\{0_0, \dots, n_0, 0_1, \dots, n_1\}$. Describe the nondegenerate $n + 1$ simplices of $\Delta[n] \times \Delta[1]$, specifically, how many are there and what are their 0-simplicies?

References

[BR13] Tobias Barthel and Emily Riehl. On the construction of functorial factorizations for model categories. *Algebraic & Geometric Topology*, 13(2):1089–1124, 2013.

[Hov99] Mark Hovey. *Model Categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999.