Lewis

APPENDIX A

COMPACTLY GENERATED SPACES

The first half of this thesis developed from the discovery that the obvious procedures for constructing smash products and colimits for the category of spectra of May [77] fail to work for purely point-set topological reasons. The successful construction of these objects in this thesis depends heavily on the special properties of compactly generated spaces. As no source in the literature contains a treatment of these spaces adequate for this thesis, a summary of their properties is included here. The focus of this appendix will be on providing a thorough catalog of results and counterexamples. Proofs will be sketched or omitted except for those which illustrate the techniques uniquely applicable to compactly generated spaces and those for new results. Other discussions of compactly generated spaces appear in Steenrod [67], Vogt [71], Wyler [73], and McCord [69].

1. Definitions, Examples, and Basic Properties

We begin with a sequence of basic definitions. Throughout this appendix compact means compact Hausdorff; the term quasicompact is used for the finite subcovering property. A space X is weak Hausdorff if for every continuous map $g:K \to X$ with K compact, g(K) is closed in X. The weak Hausdorff property is strictly between T_1 (points are closed) and the Hausdorff property. The category of weak Hausdorff spaces and continuous maps is denoted wH. A subset A of a space X is compactly closed (open) if for every continuous map $g:K \to X$ with K compact, $g^{-1}(A)$ is closed (open). Note that $A \subset X$ is compactly closed if and only if its complement is compactly open. A space X is a k-space if every compactly closed subset is closed. $\mathcal{K}$ denotes the full subcategory of Top, the category of topological spaces, consisting of k-spaces. A space is compactly generated if it is both weak Hausdorff and a k-space. $\mathcal{U}$ denotes the full subcategory of Top consisting of compactly generated spaces.

Compactly generated spaces are the spaces of actual interest in this thesis. We have introduced k-spaces because many colimit-related results on compactly generated spaces must first be proved for k-spaces and then transferred to compactly generated spaces. The following examples should provide some feeling for the generality of k-spaces and compactly generated spaces. A space is locally (quasi) compact if each neighborhood of every point contains a (quasi) compact neighborhood.

k-spaces include

a) locally compact spaces

b) first countable spaces (including metric spaces, discrete spaces, and indiscrete spaces) and, more generally, sequential spaces (see Section 11)

c) any quotient space of a k-space

d) arbitrary disjoint unions of k-spaces

e) arbitrary wedges of based k-spaces

f) any intersection of an open and a closed subset of a k-space.

Compactly generated spaces include:

a) locally compact Hausdorff spaces

b) weak Hausdorff first countable spaces (including metric spaces and discrete spaces) and, more generally, sequential spaces with unique sequential limits (see Section 11)

c) arbitrary disjoint unions of compactly generated spaces

d) arbitrary wedges of compactly generated based spaces

e) any intersection of an open and a closed subset of a compactly generated space.

The following two results on k-spaces and weak Hausdorff spaces are the basis for most of their nice properties.

LEMMA 1.1. *If* $X \in wH$, K *is compact, and* $g:K \to X$ *is continuous, then* $g(K)$ *is compact.*

Proof. Clearly $g(K)$ is quasicompact. Let $y_1, y_2 \in g(K)$ with $y_1 \neq y_2$. Since K is normal, there are disjoint open sets U_1 and U_2 about $g^{-1}(y_1)$ and $g^{-1}(y_2)$; $g(K)-g(K-U_1)$ and $g(K)-g(K-U_2)$ are disjoint open sets about y_1 and y_2 in $g(K)$. ///

LEMMA 1.2. a) *If* $X \in \mathcal{K}$ *and* $Y \in \text{Top}$, *then a set function* $f:X \to Y$ *is continuous if and only if* $f \circ g$ *is continuous for every continuous* $g:K \to X$ *with* K *compact*.

b) *If, further,* X *is weak Hausdorff and thus in* $\mathcal{U}$, *then* f *is continuous if and only if it is continuous on every compact subset of* X.

2. The Functor k; Products and Other Limits in $\mathcal{K}$ and $\mathcal{U}$

In order to construct products and other small limits in $\mathcal{K}$ and $\mathcal{U}$ we construct a functor $k:\text{Top} \to \mathcal{K}$ which is right adjoint to the inclusion functor $\mathcal{K} \to \text{Top}$. For any space X, the set of compactly open subsets of X satisfies the conditions necessary to be a topology on the underlying set of X; kX is defined to be this set with this topology. Since open subsets are compactly open, $1:kX \to X$ is continuous. It is easy to see that k is a functor and that $1:kX \to X$ is a natural transformation with the following properties:

LEMMA 2.1. a) *If* $f:X \to Y$ *is a set function such that* fg *is continuous for every continuous* $g:K \to X$ *with* K *compact, then* $kf:kX \to kY$ *is continuous*.

b) *If* $X \in \mathcal{K}$, *then* $kX = X$.

c) *If* $X \in \mathcal{K}$ *then* $f:X \to Y$ *is continuous if and only if* $kf:X \to kY$ *is continuous*.

d) *If* $X \in wH$ *then* $kX \in \mathcal{U}$.

e) *If* $K \subset X$ *is compact then* K *is compact in* kX.

f) *If* $X \in wH$ *then the compact subsets of* X *and* kX *are the same.*

g) $1{:}kX \to X$ *induces isomorphisms on singular homology and homotopy groups.*

Properties c) and d) are summarized categorically as follows.

PROPOSITION 2.2. $k{:}\mathrm{Top} \to \mathcal{K}$ *and* $k{:}wH \to \mathcal{U}$ *are right adjoints to the inclusions* $\mathcal{K} \to \mathrm{Top}$ *and* $\mathcal{U} \to wH$ *respectively; that is,*

$$\mathcal{K}(X,kY) \cong \mathrm{Top}(X,Y) \ \text{for } X \in \mathcal{K},\ Y \in \mathrm{Top}$$

$$\mathcal{U}(X,kY) \cong wH(X,Y) \ \text{for } X \in \mathcal{U},\ Y \in wH.$$

Top has all (small) limits. Also, since the category wH is closed under arbitrary products with the standard product topology and under subspaces, wH is closed under limits in Top. Therefore we have

PROPOSITION 2.3. $\mathcal{K}$ *and* $\mathcal{U}$ *have all limits. Limits in* $\mathcal{K}$ *and* $\mathcal{U}$ *are obtained by applying* k *to the corresponding limits in* Top.

Thus limits in $\mathcal{K}$ and $\mathcal{U}$ are generally different from the corresponding limits in Top. In particular, if X and Y are k-spaces, then $X \times Y$ with the cartesian product topology needn't be a k-space. For this reason we henceforth denote products with the cartesian product topology by $X \times_c Y$ and $\Pi_c X_i$ and products with the k-space topology by $X \times Y$ and ΠX_i. The spaces

$X \times Y$ and ΠX_i are of course the categorical products for K and U. The two most useful properties of the cartesian product topology--that neighborhoods of the form $U \times V$ form a basis and that a product net $\{(x_\alpha, y_\alpha)\}$ in $X \times_c Y$ converges if and only if the nets $\{x_\alpha\}$ and $\{y_\alpha\}$ converge--fail in the k-space product topology. The following results provide the tools with which one can manipulate k-space products.

LEMMA 2.4. *If* X *and* Y *are k-spaces and* X *is locally compact, then* $X \times Y = X \times_c Y$.

Proof. Let A be a compactly closed subset of $X \times_c Y$ and let $(x_0, y_0) \in (X \times_c Y) - A$. Let N be a compact neighborhood of x_0. Then $A \cap (N \times_c \{y_0\})$ is closed. Therefore there is an open neighborhood U of x_0 with $\overline{U} \subset N$ and $A \cap (\overline{U} \times \{y_0\}) = \phi$. Let B denote the projection in Y of $A \cap (\overline{U} \times_c Y)$. B is easily seen to be compactly closed and therefore closed. But $y_0 \notin B$. Hence $U \times_c (Y-B)$ is an open neighborhood of (x_0, y_0) in $(X \times_c Y) - A$. Therefore A is closed. ///

LEMMA 2.5. *If* X *and* Y *are first countable (and so in* K*), then* $X \times Y = X \times_c Y$.

The point is that $X \times_c Y$ is also first countable.

LEMMA 2.6. *If* $\{X_i\}_{i \in I}$ *is a set of spaces in* K*, then the projection* $\pi_j : \Pi X_i \to X_j$ *is an open map for each* $j \in I$.

It is enough to show that $\pi_j(U)$ is compactly open if U is open in ΠX_j. This follows easily from Lemma 2.4.

LEMMA 2.7. *If* $Y \in K$ *and* X *is compact then the projection* $X\times Y \to Y$ *is a closed map.*

For the proof use Lemma 2.4 to show that the image of a closed set is compactly closed.

3. More on Weak Hausdorff Spaces; Quotient Maps and Inclusions

This section contains three key results for manipulating k-spaces and compactly generated spaces. First, the weak Hausdorff condition for k-spaces is re-expressed in terms of the closure of the diagonal in the k-space product. Second, the first result is used to give conditions under which the quotient of a k-space is weak Hausdorff. Finally, the most useful description of a k-space inclusion is given.

We begin by establishing that the relation between the k-space product and the weak Hausdorff condition is identical to the relation between the cartesian product and the Hausdorff property.

PROPOSITION 3.1. *If* $X \in K$, *then* X *is weak Hausdorff if and only if the diagonal is closed in* $X\times X$.

Proof. Let Δ_X denote the diagonal in $X\times X$. Suppose that $X \in wH$

so that X and X×X are in $\mathcal{U}$. To show that Δ_X is closed in X×X it is enough to show that Δ_X is compactly closed. Let π_1, π_2: X×X → X be the projections, let g:K → X×X be a continuous map from a compact space and let $L = \pi_1(g(K)) \cap \pi_2(g(K))$. By Lemma 1.1, L is a compact subset of X. Thus Δ_L, the diagonal of L×L regarded as a subset of X×X, is compact and therefore closed in X×X. But $g^{-1}(\Delta_X) = g^{-1}(\Delta_L)$ so $g^{-1}(\Delta_X)$ is closed. Conversely, suppose that $\Delta_X \subset X \times X$ is closed and let h:K → X be a continuous map of a compact space into X. To show that h(K) is closed in X it is enough to show that it is compactly closed. Let j:L → X be a second continuous map of a compact space into X and let π:K×L → L be the projection. Then $j^{-1}(h(k)) = \pi((h \times \ell)^{-1}(\Delta_X))$ which is closed since Δ_X is closed and K×L is compact. Therefore h(K) is compactly closed. ///

It is now possible to clarify the relation between Hausdorff and weak Hausdorff. From Lemmas 2.4, 2.5 and 3.1 it is clear that the two properties are equivalent for first countable spaces and for locally compact spaces. These two conditions for equivalence are about as much as can be expected since Franklin [67] describes a countable, quasicompact, sequential, weak Hausdorff space which is not Hausdorff.

Another easy consequence of Lemma 3.1 is that a map f:X → Y in $\mathcal{U}$ is a categorical epi (that is, it does not equalize any unequal pair g_1, g_2:Y → Z of maps in $\mathcal{U}$) if and only if it has dense image. In more traditional language, two maps $g_1, g_2 : Y \to Z$ in $\mathcal{U}$ which agree on a dense subset of Y are equal.

Equipped with Proposition 3.1 we can now attack the question of when a quotient space of a k-space is weak Hausdorff. It is not enough for the total space to be weak Hausdorff. For a counterexample, consider the map from [0,1] to {0,1} which is zero on the rationals and one on the irrationals. With the quotient topology from this map, {0,1} is an indiscrete space. The most useful necessary and sufficient condition for a weak Hausdorff quotient is:

PROPOSITION 3.2. _If_ $X \in \mathcal{K}$ _and_ $\rho : X \to Y$ _is a quotient map, then_ Y _is weak Hausdorff, and thus in_ $\mathcal{U}$, _if and only if_ $(\rho \times \rho)^{-1}(\Delta_Y)$ _is closed in_ $X \times X$.

The proof is an easy consequence of Proposition 5.8. Applications of this criterion are given in Sections 7 and 9.

We close this section with a discussion of subspaces and the definition of an inclusion, which is, as Proposition 3.4 indicates, the dual notion to a quotient map. Since a subspace of a k-space with the relative topology needn't be a k-space, we always take the "k-ification" of the relative topology as the subspace topology. Note that for a subspace which is the intersection of a closed and an open subset, this is just the relative topology. The "k-ified" subspace topology gives us the appropriate universal property for subspaces.

LEMMA 3.3. <u>If</u> X,Y ∈ K, A⊂Y <u>and</u> f:X → Y <u>is a set function with</u> f(X)⊂A <u>then</u> f:X → Y <u>is continuous if and only if</u> f:X → A <u>is continuous</u>.

Just as we alter the notion of a subspace, we also alter the notion of an inclusion. A map f:A → X in K is an inclusion if it is a homeomorphism onto its image with the k-subspace topology. Since this topology needn't be the relative topology, a k-space inclusion needn't be a topological inclusion. For this reason closed inclusions (that is, inclusions with closed images) are generally better behaved than arbitrary inclusions.

The following result gives the standard techniques for proving that a map of k-spaces is an inclusion. It is a clear generalization of the lemma above and an obvious dualization of the usual characterization of quotient maps.

PROPOSITION 3.4. a) <u>If</u> f:X → Y <u>is a map in</u> K <u>such that either</u> f <u>is injective or</u> X <u>is</u> T_0, <u>then</u> f <u>is an inclusion if and only if condition</u> (*) <u>below holds for all</u> Z ∈ K.

b) <u>If</u> f:X → Y <u>is a map in</u> U <u>then</u> f <u>is an inclusion if and only if condition</u> (*) <u>holds for all</u> Z ∈ U.

(*) <u>If</u> g:Z → X <u>is a set function with fg continuous, then</u> g <u>is continuous</u>.

If X is T_0 then (*) implies the injectivity of f. Otherwise it must be assumed (indeed, (*) holds trivially when X is indiscrete). Note that by Lemma 1.2, attention can always be restricted to maps into X from compact spaces Z.

4. Colimits in $\mathcal{K}$ and $\mathcal{U}$

Since colimits are constructed from coproducts and quotients and since $\mathcal{K}$ is closed under these, $\mathcal{K}$ inherits (small) colimits from Top. Colimits for $\mathcal{U}$ cannot be obtained in this fashion because the weak Hausdorff property is not in general preserved by quotients. This section is devoted to showing that $\mathcal{U}$ nevertheless does have colimits. Unfortunately, very little can be said in general about their structure.

Colimits for $\mathcal{U}$ are obtained via a left adjoint $q:\mathcal{K} \to \mathcal{U}$ to the inclusion functor $\mathcal{U} \to \mathcal{K}$. The following proof of the existence of q is non-constructive. We also give an explicit construction for use in Section 8; but it is not much more enlightening.

PROPOSITION 4.1. <u>The inclusion functor</u> $\mathcal{U} \to \mathcal{K}$ <u>has a left adjoint</u> $q:\mathcal{K} \to \mathcal{U}$. <u>Therefore</u> $\mathcal{U}$ <u>has all (small) colimits. Colimits in</u> $\mathcal{U}$ <u>are obtained by applying</u> q <u>to the corresponding colimits in</u> $\mathcal{K}$.

Proof. By Proposition 2.3, $\mathcal{U}$ has all small limits and the inclusion functor $\mathcal{U} \to \mathcal{K}$ preserves limits. Also, for any space X there is clearly only a set of isomorphism classes of continuous surjections $X \to Y$ with $Y \in \mathcal{U}$. Therefore Freyd's adjoint functor theorem (Mac Lane [71], p. 117) gives the conclusion. ///

The following corollary, which can be proved formally from Proposition 4.1, makes it clear that colimits in $\mathcal{U}$ are quotients of the corresponding colimits in $\mathcal{K}$.

COROLLARY 4.2. The unit $\eta : X \to qX$ of the adjunction between q and the inclusion functor $\mathcal{U} \to \mathcal{K}$ is a quotient map.

The corollary is also an immediate consequence of the following explicit construction of q.

CONSTRUCTION 4.3. $q : \mathcal{K} \to \mathcal{U}$.

The intuition for the construction is simple. A function $J : \mathcal{K} \to \mathcal{K}$ and a natural transformation $\lambda : 1 \to J$ are constructed so that JX is a first approximation to making a space X weak Hausdorff and $\lambda : X \to JX$ is a quotient map which is a first approximation to the unit $\eta : X \to qX$ of the adjunction. Then J is iterated until a weak Hausdorff space is obtained. In detail, let $X \in \mathcal{K}$ and define an equivalence relation $\approx$ on X by taking the transitive closure of the relation $x \sim y$ if

$(x,y) \in \bar{\Delta} \subset X \times X$ where $\bar{\Delta}$ is the closure of the diagonal. Let $JX = X/\approx$ and $\lambda: X \to JX$ be the quotient map. Clearly λ is an isomorphism if and only if $X \in \mathcal{U}$. Also if $f: X \to Y$ with $Y \in \mathcal{U}$, then there is a unique map $\tilde{f}: JX \to Y$ with $f = \tilde{f}\lambda$. By transfinite induction we define functors $J^\alpha: \mathcal{K} \to \mathcal{K}$ and natural transformations $\lambda^\alpha_\beta: J^\alpha \to J^\beta$ for α, β ordinals with $\alpha < \beta$ starting with $J^0 = 1$, $J^1 = J$, and $\lambda^0_1 = \lambda$ such that

1) For all $X \in \mathcal{K}$ and $\alpha < \beta$, $\lambda^\alpha_\beta: J^\alpha X \to J^\beta X$ is a quotient map

2) $\lambda^\alpha_\gamma = \lambda^\beta_\gamma \lambda^\alpha_\beta$ for $\alpha < \beta < \gamma$

3) If $f: X \to Y$ with $Y \in \mathcal{U}$ then for each α there is a unique map $\tilde{f}_\alpha: J^\alpha X \to Y$ with $f = \tilde{f}_\alpha \lambda^0_\alpha$.

The definitions of $J^\beta X$ and the λ^α_β are by use of J and λ for successor ordinals and of colimits for the limit ordinals. For each $X \in \mathcal{K}$, there is an ordinal α with $J^\alpha X \in \mathcal{U}$. Indeed, if $J^\alpha X \notin \mathcal{U}$ for an ordinal α, then $\lambda^0_{\alpha+1}: X \to J^{\alpha+1}X$ must collapse a pair of points in X which were not collapsed by $\lambda^0_\alpha: X \to J^\alpha X$. X has only a set of pairs of points to be collapsed. If α is an ordinal of cardinality greater than the cardinality of $X \times X$ then $J^\alpha X \in \mathcal{U}$. Now take qX to be $J^\alpha X$ for the least ordinal with $J^\alpha X \in \mathcal{U}$ and let $\eta: X \to qX$ be λ^0_α. By property (3) above, η is a universal map from X to spaces in $\mathcal{U}$. Therefore q can be made into a functor $q: \mathcal{K} \to \mathcal{U}$ left adjoint to the inclusion $\mathcal{U} \to \mathcal{K}$ and η is the unit of the adjunction. ///

5. Function Spaces and Products

The principal advantage of K and U over Top is that, with their natural function space topologies, they are cartesian closed categories (see Mac Lane [71] for a discussion of this notion). This has useful consequences for the behavior of limits and colimits in K and U. In this section we describe the function space topology and prove the closed structure. We demonstrate its utility by showing that the product of two quotient maps in K or U is again a quotient map.

We begin by introducing a modified compact open topology on the set of maps from a space X to a space Y. If K is compact, $h:K \to X$ is continuous and $U \subset Y$ is open, let

$$N(h,U) = \{f:X \to Y \mid f \text{ continuous and } f(h(K)) \subset U\}.$$

Let C(X,Y) be the set Top(X,Y) with the topology generated by the subbasis $\{N(h,U)\}$. Clearly C defines a functor $Top^{op} \times Top \to Top$. Evaluation gives a set function $\varepsilon:C(X,Y)\times_c X \to Y$.

LEMMA 5.1. *If* K *is compact and* $g:K \to X$ *is continuous, then the map* $\varepsilon(1\times_c g):C(X,Y)\times_c K \to Y$ *is continuous. In particular, evaluation* $\varepsilon_x:C(X,Y) \to Y$ *at a point* $x \in X$ *is continuous.*

Proof. Suppose $U \subset Y$ is open and $(f,z) \in (\varepsilon(1\times_c g))^{-1}(U)$. Then $z \in (fg)^{-1}(U)$. Since $(fg)^{-1}(U)$ is open there is an open neighborhood V of z in K with $\overline{V} \subset (fg)^{-1}(U)$. $N(g|_{\overline{V}},U)\times_c V$ is an open neighborhood of (f,z) in $(\varepsilon(1\times_c g))^{-1}(U)$. Therefore $\varepsilon(1\times_c g)$ is continuous. ///

LEMMA 5.2. <u>If</u> $X \in \text{Top}$ <u>and</u> $Y \in wH$, <u>then</u> $C(X,Y) \in wH$.

<u>Proof</u>. Let $Z = \prod_{x\in X} Y$ be the cartesian product of one copy of Y for each point in X. Define $j:C(X,Y) \to Z$ to be evaluation at $x \in X$ on the factor corresponding to x. Since j is a continuous injection and since $Z \in wH$ (because wH is closed under arbitrary products), $C(X,Y) \in wH$. ///

Now define $Y^X = kC(X,Y)$. This gives functors $K^{op}\times K \to K$ and $U^{op}\times U \to U$ which provide the function spaces for K and U. Note that this function space topology is generally strictly finer than the compact-open topology. However, if both X and Y are second countable Hausdorff spaces and X is locally compact, then C(X,Y) is second countable (Dugundji [66], p. 265) and therefore in U. Also, if X is compact and Y is a metric space, then C(X,Y) is a metric space (Dugundji [66], p. 270) and therefore in U. Unfortunately, these two results apparently represent all that is known about when the compact-open topology agrees with its "k-ification."

Evaluation gives a set function $\varepsilon:Y^X\times X \to Y$ and we have the following result.

PROPOSITION 5.3. <u>If</u> $X, Y \in K$, <u>then</u> $\varepsilon:Y^X\times X \to Y$ <u>is continuous</u>.

<u>Proof</u>. Since $Y^X \times X = k(C(X,Y)\times_c X)$ it is enough to show that for any $g:K \to C(X,Y)\times_c X$, $\varepsilon g:K \to Y$ is continuous. This is immediate from Lemma 5.1. ///

If $f:Y\times X \to Z$ is continuous for $X, Y, Z \in \mathcal{K}$ then there is an adjoint set function $\tilde{f}:Y \to Z^X$ given by $\tilde{f}(y)(x) = f(y,x)$.

PROPOSITION 5.4. *If $X, Y, Z \in \mathcal{K}$ and $f:Y\times X \to Z$ is continuous, then so is $\tilde{f}:Y \to Z^X$. In particular for $X, Y \in \mathcal{K}$ the map $\eta:Y \to (Y\times X)^X$ given by $\eta(y)(x) = (y,x)$ is continuous.*

<u>Proof</u>. Since $Y \in \mathcal{K}$ it is enough to show that $\tilde{f}:Y \to C(X,Z)$ is continuous. Therefore it is enough to check that $\tilde{f}^{-1}(N(h,U))$ is open for $h:K \to X$ continuous with K compact and U open in Z. If $y \in \tilde{f}^{-1}(N(h,U))$ then $\{y\}\times K \subset (f(1\times_c h))^{-1}(U)$ and, by the compactness of K, there is an open neighborhood V of y with $V\times K \subset (f(1\times_c h))^{-1}(U)$. But then $V \subset \tilde{f}^{-1}(N(h,U))$ and $\tilde{f}^{-1}(N(h,U))$ is open. ///

For fixed X, the maps $\eta:Y \to (Y\times X)^X$ and $\varepsilon:Y^X\times X \to Y$ satisfy the identities necessary to specify the following adjunctions.

THEOREM 5.5. *There are natural isomorphisms*

$$\mathcal{K}(Y\times X,Z) \cong \mathcal{K}(Y,Z^X) \quad \text{for } X,Y,Z \in \mathcal{K}$$

and

$$\mathcal{U}(Y\times X,Z) \cong \mathcal{U}(Y,Z^X) \quad \text{for } X,Y,Z \in \mathcal{U}.$$

Therefore $\mathcal{K}$ and $\mathcal{U}$ are cartesian closed categories.

The following results are formal consequences.

COROLLARY 5.6. *For* $X,Y,Z \in K$ (*or* U)

a) $(Z^Y)^X$ *is homeomorphic to* $Z^{Y\times X}$.

b) *Composition* $C:Z^Y\times Y^X \to Z^X$ *is continuous*.

c) *The functor* $?\times X$ *preserves colimits*.

d) *The functor* $?^X$ *preserves limits*.

e) *The functor* $Y^?$ *takes colimits to limits*.

To give some idea of the power of this formal structure we turn to the often tricky question of the preservation of quotient maps by products. In K and U this preservation follows immediately from the preservation of colimits by products via the following lemma.

LEMMA 5.7. *If* $X, Y \in K$ *then* $f:X \to Y$ *is a quotient map if and only if it is a coequalizer*.

Proof. Coequalizers are quotient maps by the definition of colimits in K. Conversely if $f:X \to Y$ is a quotient map then for each $y \in Y$ select a point $x_y \in f^{-1}(y) \subset X$. Define two maps from $\coprod_{y\in Y} f^{-1}(y)$ to X by taking one to be the inclusion of $f^{-1}(y)$ on each factor and the other to be $f^{-1}(y) \to x_y$ on each factor. Then f is easily seen to be the coequalizer of these two maps. ///

PROPOSITION 5.8. *If* $X, X', Y, Y' \in K$ *and* $p:X \to Y$, $q:X' \to Y'$ *are quotient maps, then* $p\times q:X\times X' \to Y\times Y'$ *is a quotient map*.

6. Based Spaces

$\mathcal{T}$ and $\mathcal{K}_*$ denote the categories of compactly generated based spaces and based k-spaces respectively. Since $\mathcal{U}$ and $\mathcal{K}$ are cartesian closed categories with all limits and colimits it follows formally that $\mathcal{T}$ and $\mathcal{K}_*$ are closed categories with all limits and colimits. We record these completeness properties.

PROPOSITION 6.1. The categories $\mathcal{T}$ and $\mathcal{K}_*$ have all (small) limits and colimits. Limits in $\mathcal{T}$ and $\mathcal{K}_*$ are obtained by applying k to the corresponding limits in Top_*. Colimits in $\mathcal{K}_*$ are just colimits in Top_*. Colimits in $\mathcal{T}$ are obtained by applying q to the corresponding colimits in $\mathcal{K}_*$.

The closed structures on $\mathcal{K}_*$ and $\mathcal{T}$ merit a bit more discussion. The smash product pairing with respect to which $\mathcal{K}_*$ becomes a symmetric monoidal category is defined by $X \wedge Y = X \times Y / X \vee Y$ for $X, Y \in \mathcal{K}_*$. The associativity of the smash product follows from Proposition 5.8 and it is easy to check that S^0 is the unit. The coherence diagrams necessary for $\mathcal{K}_*$ to be a symmetric monoidal category follow from the diagrams for $\mathcal{K}$. The following lemma indicates that the smash product restricts to give $\mathcal{T}$ a symmetric monoidal structure.

LEMMA 6.2. If $X, Y \in \mathcal{T}$ then $X \wedge Y \in \mathcal{T}$ and the quotient map $\pi : X \times Y \to X \wedge Y$ is a closed map.

This holds because $X \vee Y$ is a closed subset of $X \times Y$.

The function space $F(X,Y)$ for K_* and T is just the subspace of Y^X consisting of based maps with the k-subspace topology. If $Y \in T$ then $F(X,Y)$ is a closed subset of Y^X. For $X,Y,Z \in K_*$, $F(Y \wedge X, Z) \to Z^{Y \times X}$ and $F(X,F(Y,Z)) \to (Z^Y)^X$ are inclusions by Proposition 7.7. This yields the following result.

THEOREM 6.3. *The following conclusions hold for* X,Y,Z *in* K_* *or* T.

a) $? \wedge X$ *and* $F(X,?)$ *are adjoint functors*.

b) K_* *and* T *are closed categories*.

c) $F(Y \wedge X, Z)$ *and* $F(X,F(Y,Z))$ *are naturally homeomorphic*.

d) *Composition* $C: F(Y,Z) \wedge F(X,Y) \to F(X,Z)$ *is continuous*.

e) *The functor* $? \wedge X$ *preserves colimits*.

f) *The functor* $F(X,?)$ *preserves limits*.

g) *The functor* $F(?,Y)$ *takes colimits to limits*.

Since coequalizers in K_* are the same as coequalizers in K, the analog of Lemma 5.7 holds for pointed spaces. Therefore we have

PROPOSITION 6.4. *If* $X, X', Y, Y' \in K_*$ *and* $p: X \to Y$ *and* $q: X' \to Y'$ *are quotient maps then* $p \wedge q: X \wedge X' \to Y \wedge Y'$ *is a quotient map*.

7. Preservation of Inclusions under Various Constructions

This section is concerned with results on the preservation of inclusions and closed inclusions under such constructions as products, smash products, wedges, pushouts, and function spaces. The relation between inclusions and directed colimits and between inclusions and pullbacks is discussed in Sections 9 and 10.

The following results are direct consequences of Proposition 3.4 which characterizes inclusions.

PROPOSITION 7.1. a) *An arbitrary product of (closed) inclusions in* K *is a (closed) inclusion.*

b) *An arbitrary disjoint union of (closed) inclusions in* K *is a (closed) inclusion.*

c) *An arbitrary wedge of (closed) inclusions in* T *is a (closed) inclusion.*

For part (c) the fact that condition (*) of Proposition 3.4 needs to hold only on compact spaces allows a reduction of the problem to the case of a finite wedge where it is elementary.

The following technical lemma on closed inclusions has several applications.

LEMMA 7.2. *If* f *is a closed inclusion,* g *is injective,* p *is surjective and either* q *is closed or* q *is a quotient map with* $q^{-1}(g(Z)) \subset f(X)$ *in the following commuting diagram in* K, *then* g *is a closed inclusion.*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ Z & \xrightarrow{g} & W \end{array}$$

The preservation of closed inclusions by smash products in $\mathcal{T}$ now follows easily.

PROPOSITION 7.3. *If $f:X \to Y$ and $g:X' \to Y'$ are closed inclusions in $\mathcal{T}$, then $f \wedge g: X \wedge X' \to Y \wedge Y'$ is a closed inclusion.*

One might expect the analog of this proposition to hold for non-closed inclusions. This is not the case. Let $I = [0,1]$ and $[0,1)$ have basepoint 0. Then the map $I \wedge [0,1) \to I \wedge I$ is not an inclusion because the sequence $\frac{1}{n} \wedge (1-\frac{1}{n})$ is discrete in $I \wedge [0,1)$ and converges to the basepoint in $I \wedge I$. This misbehavior of inclusions is one of the problems with the construction of smash products of spectra. The following is the best available result on the preservation of inclusions by smash products.

PROPOSITION 7.4. *If $f:X \to Y$ is an inclusion in $\mathcal{K}_*$, K is compact and K^+ denotes the union of K with a disjoint basepoint, then $f \wedge 1: X \wedge K^+ \to Y \wedge K^+$ is an inclusion.*

This follows from the existence of a map $X \wedge K^+ \to X$ and the compactness of K.

For pushouts we have

PROPOSITION 7.5. *If the diagram below is a pushout in $\mathcal{K}$ and i is a closed inclusion then j is a closed inclusion.*

$$\begin{array}{ccc} A & \xrightarrow{\;i\;} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{\;j\;} & Y \cup_f X \end{array}$$

If A, X, $Y \in \mathcal{U}$, *then* $Y \cup_f X$ *is weak Hausdorff and is thus also the pushout in* $\mathcal{U}$.

Proof. The injectivity of j follows because pushouts preserve injectivity in the category of sets. For the rest, note that the diagram below is also a pushout and that (j,g) is a quotient map.

$$\begin{array}{ccc} Y \amalg A & \xrightarrow{\;1 \amalg i\;} & Y \amalg X \\ {\scriptstyle (1,f)}\downarrow & & \downarrow{\scriptstyle (j,g)} \\ Y & \xrightarrow{\;j\;} & Y \cup_f X \end{array}$$

Now apply Lemma 7.2 and Proposition 3.2. ///

In $\mathcal{U}$, pushouts of non-closed inclusions need not even be injective. For example, if $f:[-1,0) \cup (0,1] \to S^0$ is the collapse of each component to a point and $i:[-1,0) \cup (0,1] \to [-1,1]$ is the inclusion, then the pushout P of f and i in $\mathcal{K}$ is a three point space in which only one point is closed. The pushout in $\mathcal{U}$ is a single point because q collapses P.

We now relate equalizers to inclusions and closed inclusions and thereby derive the preservation properties for function spaces. The following result is dual to Lemma 5.7 and admits a dual proof, with $Y/f(X)$ replacing the coproduct used there.

LEMMA 7.6. a) <u>A map</u> $f:X \to Y$ <u>in</u> K <u>is an inclusion if and only if it is an equalizer</u>.

b) <u>A map</u> $f:X \to Y$ <u>in</u> U <u>is a closed inclusion if and only if it is an equalizer in</u> U.

PROPOSITION 7.7. a) <u>If</u> $X \in K$ <u>and</u> $Y \in K_*$, <u>then the functors</u> $?^X:K \to K$ <u>and</u> $F(Y,?):K_* \to K_*$ <u>preserve inclusions and the functors</u> $X^?:K^{op} \to K$ <u>and</u> $F(?,Y):K_*^{op} \to K_*$ <u>take quotient maps to inclusions</u>.

b) <u>If</u> $X \in U$ <u>and</u> $Y \in T$, <u>then the functors</u> $?^X:U \to U$ <u>and</u> $F(Y,?):T \to T$ <u>preserve inclusions and closed inclusions and the functors</u> $X^?:U^{op} \to U$ <u>and</u> $F(?,Y):T^{op} \to T$ <u>take quotient maps to closed inclusions</u>.

8. <u>Identification of Various Maps as Inclusions</u>

In this section we show that such special maps as cofibrations and the unit of the product-function space adjunction are inclusions or closed inclusions. First we have a technical result on retractions.

LEMMA 8.1. <u>If</u> $j:X \to Y$ <u>and</u> $r:Y \to X$ <u>are maps in</u> K <u>with</u> $rj = 1$ <u>then</u> j <u>is an inclusion and</u> r <u>is a quotient map. Further, if</u> $Y \in U$, <u>then</u> j <u>is a closed inclusion</u>.

The point is that j is the equalizer and r is the coequalizer of the map $jr:Y \to Y$ and the identity. An easy consequence of this is

PROPOSITION 8.2. *If* $i:A \to X$ *is a cofibration in* $\mathcal{U}$ *or* $\mathcal{T}$, *then* i *is a closed inclusion.*

Proof. Assume i is a cofibration in $\mathcal{U}$. Then there is a left inverse $r:X\times I \to Mi$ to the natural map $j:Mi \to X\times I$ where Mi is the mapping cylinder. Therefore j is a closed inclusion. If $\theta:A \to Mi$ is the map into the free end of the mapping cylinder and $i_1:X \to X\times I$ is the map of X onto the top end of the cylinder, then θ and i_1 are closed inclusions and the diagram

$$\begin{array}{ccc} A & \xrightarrow{\theta} & Mi \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ X & \xrightarrow{i_1} & X\times I \end{array}$$

commutes. Hence, since $j\theta$ is a closed inclusion and i_1 is injective, i is a closed inclusion. The proof for i a cofibration in $\mathcal{T}$ is similar. ///

The following proposition provides a useful relation between inclusions and the (smash) product-function space adjunction. Note that none of the converses of the statements below are true.

PROPOSITION 8.3. a) *If* $f:Y\times X \to Z$ *is an inclusion in* $\mathcal{K}$ *and* $X \neq \phi$, *then the adjoint* $\tilde{f}:Y \to Z^X$ *is an inclusion. If, further,* $X, Y, Z \in \mathcal{U}$ *and* f *is a closed inclusion, then so is* $\tilde{f}$. *In particular the map* $\eta:Y \to (Y\times X)^X$ *specified by* $\eta(y)(x) = (y,x)$

is a (closed) inclusion if $X, Y \in \mathcal{K}$ $(X,Y \in \mathcal{U})$ *and* X *is nonempty*.

b) *If* $f: Y \wedge X \to Z$ *is an inclusion in* $\mathcal{K}_*$ *and* $X \neq \{*\}$, *then so is the adjoint* $\tilde{f}: Y \to F(X,Z)$. *If, further,* $X,Y,Z \in \mathcal{T}$ *and* f *is a closed inclusion, then so is* $\tilde{f}$. *In particular, the map* $\eta: Y \to F(X, Y \wedge X)$ *specified by* $\eta(y)(x) = y \wedge x$ *is a (closed) inclusion if* $X,Y \in \mathcal{K}$ $(X,Y \in \mathcal{U})$ *and* X *is not a point*.

The proof that $\tilde{f}$ is an inclusion is immediate from Proposition 3.4 and an application of the relevant adjunction. To prove that the image is closed under the specified conditions, first show that η is a closed inclusion using the closed diagonal condition. Then express $\tilde{f}$ in terms of η and f and apply Proposition 7.7.

The result below describes the natural map relating function spaces and wedges.

PROPOSITION 8.4. *If* K *is a compact based space and* $\{Y_i\}_{i \in I} \subset \mathcal{T}$, *then the natural map* $j: \bigvee_{i \in I} F(K, Y_i) \to F(K, \bigvee_{i \in I} Y_i)$ *is a closed inclusion*.

Proof. Since K is compact the proof that j is an inclusion can be reduced to consideration of $F(K,X) \vee F(K,Y) \to F(K, X \vee Y)$ which is an inclusion by comparison with the homeomorphism $X^K \times Y^K \to (X \times Y)^K$. To see that the image of j is closed, note that its complement is the set of functions $f:K \to \bigvee_{i \in I} Y_i$ such that for some $x,y \in K$ and $i \neq j \in I$, $f(x) \in Y_i-\{*\}$ and $f(y) \in Y_j-\{*\}$. ///

If the indexing set I is finite, then the compactness assumption on K is not needed. If the indexing set is infinite, then the result cannot be proved using the homeomorphism $\prod_{i \in I} (Y_i^K) \to (\Pi Y_i)^K$ since the natural map $\bigvee_{i \in I} Y_i \to \prod_{i \in I} Y_i$ needn't be an inclusion (the map of a countable wedge of copies of S^1 into the product is a counterexample). Thus some condition like the compactness of K seems to be needed.

The following result is useful but not as general as one might hope.

PROPOSITION 8.5. *If* $X,Y \in \mathcal{T}$, K *is nonempty and compact, and* K^+ *denotes the union of* K *and a disjoint basepoint, then the natural map* $j:F(X,Y)\wedge K^+ \to F(X,Y\wedge K^+)$ *is a closed inclusion.*

Proof. j is easily seen to be a continuous injection. Let $\pi:Y\wedge K^+ \to Y$ be the projection and let $\rho:F(X,Y)\times K \to F(X,Y)\wedge K^+$ be the quotient map. In order to show that j is an inclusion, it is enough by Proposition 3.4 to show that if $g:Z \to F(X,Y)\wedge K^+$ is a set function with jg continuous, then g is continuous. Let $h = \pi_* jg:Z \to F(X,Y)$; h is continuous. If $z \in Z-h^{-1}(*)$ then it is easy to see that g is continuous at z by using the evaluation map $\varepsilon_x: F(X, Y\wedge K^+) \to Y\wedge K^+$ for some $x \in X$ such that $h(z)(x) \neq *$. Therefore it is enough to show that g is continuous at any point in $h^{-1}(*)$. For such a point z_0, let U be an open neighborhood of $g(z_0) = *$ in $F(X,Y)\wedge K^+$. Then $\{*\}\times K \quad \rho^{-1}(U)$. And, since K is compact, there is an open neighborhood V of $*$ in $F(X,Y)$ with $V\times K \subset \rho^{-1}(U)$. But then $h^{-1}(V)$ is an open neighborhood of z_0 in Z and $g(h^{-1}(V)) \subset U$. Therefore g is continuous at z_0 and j is an inclusion. In order to see that the image of j is closed in $F(X, Y\wedge K^+)$, note that its complement is the set of functions $f: X \to Y\wedge K^+$ such that for some $x_1, x_2 \in X$, $h(x_1) = y_1\wedge z_1 \neq *$, $h(x_2) = y_2\wedge z_2 \neq *$, and $z_1 \neq z_2$. With this description it is easy to see that the complement is open. ///

The adjunction of the disjoint basepoint is essential to the above result as the following counterexample shows.

COUNTEREXAMPLE 8.6. If $I = [0,1]$ with basepoint zero, then $F(I,I)\wedge I \to F(I, I\wedge I)$ is not an inclusion. To see this let $f_n \in F(I,I)$ for $n \geq 1$ be the function which is zero on $[0, 1 - \frac{1}{n+1}]$ and a spike of height one and width $\frac{1}{n+1}$ on $[1 - \frac{1}{n+1}, 1]$. Then $\{f_n\}_{n\geq 1}$ is discrete in $F(I,I)$ and is the standard example of a non-uniformly convergent sequence of functions which converges pointwise. The sequence $\{f_n \wedge \frac{1}{n}\}_{n\geq 1}$ is discrete in $F(I,I)\wedge I$ but $\{j(f_n\wedge\frac{1}{n})\}_{n\geq 1}$ converges to the zero function in $F(I, I\wedge I)$. ///

9. Directed Systems and Their Colimits

Three aspects of directed colimits are considered here--their preservation by products and function spaces, the relation of certain properties of the maps in the directed system such as being injections or inclusions to the behavior of the natural maps into the colimit, and conditions under which the colimit in K of a directed system in U is weak Hausdorff and thus also the colimit in U. Note that a directed colimit in K_* (or T) is obtained by assigning the obvious basepoint to the colimit in K (or U) of the same directed system. Thus everything said about K and U applies equally well to K_* and T. The relation between directed systems and pullbacks is discussed in the next section.

We begin with the relation of products and colimits.

PROPOSITION 9.1. If $\{X_\alpha, \lambda^\alpha_\beta : X_\alpha \to X_\beta\}$ and $\{Y_\alpha, \gamma^\alpha_\beta : Y_\alpha \to Y_\beta\}$ are directed systems in K (or U) indexed on the same directed set $\{\alpha\}$ and if $X = \text{colim } X_\alpha$ and $Y = \text{colim } Y_\alpha$ in K (or U) then $X \times Y = \text{colim } X_\alpha \times Y_\alpha$.

The proof follows from the colimit preserving property of products and a cofinality argument.

The next two propositions relate properties of the maps in the directed system to the natural maps into the colimit. A condition for a colimit in K to be weak Hausdorff is also given.

PROPOSITION 9.2. *Let* $\{X_\alpha, \lambda^\alpha_\beta : X_\alpha \to X_\beta\}$ *be a directed system in* $\mathcal{K}$ *(or* $\mathcal{U}$*) such that all the* λ^α_β *are surjective. Set* $X =$ colim X_α *in* $\mathcal{K}$ *(or* $\mathcal{U}$*) and let* $\{\lambda_\alpha : X_\alpha \to X\}$ *be the natural maps into the colimit. Then the maps* λ_α *are surjective.*

Proof. In $\mathcal{K}$ a map is a surjection if and only if it is an epi and the analogous assertion for epis holds in any category. The result for $\mathcal{U}$ follows from the result for $\mathcal{K}$, Proposition 4.1 and Corollary 4.2. ///

PROPOSITION 9.3. *Let* $\{X_\alpha, \lambda^\alpha_\beta : X_\alpha \to X_\beta\}$ *be a directed system in* $\mathcal{K}$ *such that all the* λ^α_β *are injective. Let* $X =$ colim X_α *in* $\mathcal{K}$ *and let* $\{\lambda_\alpha : X_\alpha \to X\}$ *be the natural maps into the colimit. Then*

a) *The maps* λ_α *are injective.*

b) *If* $\{X_\alpha\} \subset \mathcal{U}$ *then* X *is weak Hausdorff and therefore also the colimit in* $\mathcal{U}$.

c) *If* $\{X_\alpha\} \subset \mathcal{U}$ *and the* λ^α_β *are (closed) inclusions then the* λ_α *are (closed) inclusions.*

Proof. (a) follows from the construction of colimits in sets. (b) follows from Proposition 9.1 and the closed diagonal condition for weak Hausdorff. The result in (c) that the image of any λ_α is closed is an easy consequence of directedness. For the inclusion part of (c), fix $\beta \in \{\alpha\}$. To show that $\lambda_\beta : X_\beta \to X$ is an inclusion it is enough to show that for any closed $C \subset X_\beta$, $D = \lambda_\beta(C)$ is compactly closed in $\lambda_\beta(X_\beta)$. Let

$K \subset \lambda_\beta(X_\beta)$ be compact. Since K is closed in X, $K_\alpha = \lambda_\alpha^{-1}(K)$ is closed in X_α for all α. It is enough to show that for each α, $\lambda_\alpha^{-1}(K \cap D)$ is closed in X_α. Select $\alpha' \in \{\alpha\}$ with $\alpha \leq \alpha'$ and $\beta \leq \alpha'$. Then $\lambda_\alpha^{-1}(K \cap D) = (\lambda_{\alpha'}^{\alpha})^{-1}(\lambda_{\alpha'}^{\beta}(K_\beta \cap C))$. Since $\lambda_{\alpha'}^{\beta}(K_\beta) = K_{\alpha'}$, $\lambda_{\alpha'}^{\beta}|_{K_\beta}$ is a closed inclusion. Therefore $\lambda_{\alpha'}^{\beta}(K_\beta \cap C)$ is closed, so $\lambda_\alpha^{-1}(K \cap D)$ is closed. ///

We now consider the behavior of maps of compact spaces into directed colimits.

LEMMA 9.4. <u>Let</u> $\{x_n, \lambda_n : X_n \to X_{n+1}\}$ <u>be a directed sequence in</u> Top. If X=colim X_n is a T_1 space (points are closed), $\{\varphi_n : X_n \to X\}$ <u>are the maps into the colimit and</u> K <u>is any quasicompact subset of</u> X, <u>then there is an</u> n <u>with</u> $K \subset \varphi_n(X_n)$.

If $K \not\subset \varphi_n(X_n)$ for every n then a discrete sequence of points in K can be constructed contrary to the quasicompactness of K.

This lemma does not extend to colimits over arbitrary directed systems. I am indebted to Myles Tierney for the following counterexample. The unit interval I, being a sequential space (see § 11), is the colimit of its closed countable subsets and these subsets form a directed system ordered by inclusion. However, I is clearly not contained in any one of its countable closed subsets.

For compactly generated spaces, this lemma yields an important preservation property for directed colimits and function spaces.

PROPOSITION 9.5. <u>Let</u> $\{X_n, \lambda_n : X_n \to X_{n+1}\}$ <u>be a directed sequence in</u> $\mathcal{U}$ <u>such that each</u> λ_n <u>is an inclusion</u>. <u>If</u> $X = \operatorname{colim} X_n$ <u>and</u> K <u>is a compact space, then</u> $X^K = \operatorname{colim} X_n^K$ <u>and similarly in</u> $\mathcal{T}$.

The proof is an easy consequence of Lemma 9.4 and Proposition 9.3(c).

We close this section with a result on maps given by the universality of a colimit.

PROPOSITION 9.6. <u>Let</u> $\{X_\alpha, \lambda_\beta^\alpha : X_\alpha \to X_\beta\}$ <u>be a directed system in</u> $\mathcal{K}$ (<u>or</u> $\mathcal{U}$). <u>If</u> $X = \operatorname{colim} X_\alpha$ <u>in</u> $\mathcal{K}$ (<u>or</u> $\mathcal{U}$), $Y \in \mathcal{K}$ (<u>or</u> $\mathcal{U}$) <u>and</u> $\{\varphi_\alpha : X_\alpha \to Y\}$ <u>is a collection of injective maps which commute with the</u> λ_β^α, <u>then the induced map</u> $\varphi : X \to Y$ <u>is injective</u>.

The point is that any pair of points in X must come from a common X_α since the system is directed.

One might hope for an analog to the effect that if the φ_α are inclusions or closed inclusions then so is φ. No such result is true. Let Q be the rationals with the subspace topology from the reals and let $\{r_m\}_{m\geq 0}$ be an enumeration of Q. If $X_n = \{r_m\}_{0\leq m\leq n}$, then the maps $X_n \to X_{n+1}$ are closed inclusions and so are the maps $X_n \to Q$. However $X = \operatorname{colim} X_n$ is a countable discrete space so the induced map $X \to Q$ is not an inclusion.

10. Pullbacks, Inclusions, and Colimits

We begin by relating pullbacks to injections, inclusions, closed inclusions and quotient maps in $\mathcal{K}$ and $\mathcal{U}$. The preservation results on pullbacks and quotient maps are used to show that under very general circumstances pulling back preserves colimits in $\mathcal{K}$. We then show that under conditions applicable in bundle theory pulling back in $\mathcal{U}$ preserves colimits in $\mathcal{U}$. Note that pullbacks in $\mathcal{K}_*$ and $\mathcal{T}$ are pullbacks in $\mathcal{K}$ and $\mathcal{U}$ respectively, with appropriate basepoints, so that all of the results of this section apply equally well in the based context.

PROPOSITION 10.1. *Let*

$$\begin{array}{ccc} P & \xrightarrow{j} & A \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ C & \xrightarrow[i]{} & B \end{array}$$

be a pullback diagram in $\mathcal{K}$. *Then*

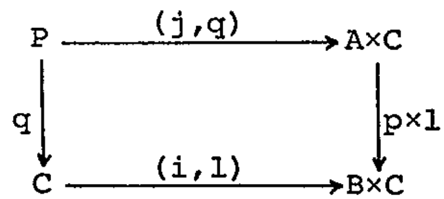
a) *if* i *is injective so is* j.

b) *if* i *is an inclusion so is* j.

c) *if* i *is a closed inclusion and* $B \in \mathcal{U}$ *then* j *is a closed inclusion.*

d) *if* p *is a quotient map and* $B \in \mathcal{U}$ *then* q *is a quotient map.*

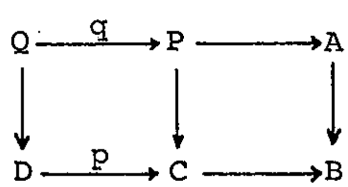
Proof. (a) and (b) are formal. For (c) it suffices to prove that j(P) is compactly closed and this is straightforward. For (d) note that the following diagram is also a pullback:



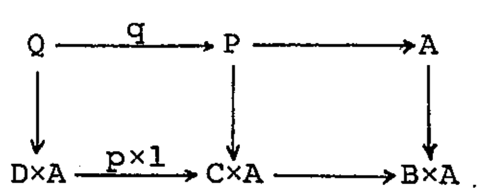
Here p×1 is a quotient map; (i,1), (j,q) are easily seen to be inclusions and the closed diagonal condition on B can be used to show that their images are closed. It then follows easily from the diagram that q is a quotient map. ///

The following extension of the result above is needed for the study of colimits and pullbacks.

LEMMA 10.2. *Suppose that, in the diagram below, both squares are pullbacks*, A, C, D $\in \mathcal{K}$ *and* B $\in \mathcal{U}$. *Then if* p *is a quotient map, so is* q.



Proof. In the diagram



both squares are pullbacks and from Proposition 10.1 and the closed diagonal condition on B, all the vertical maps are closed inclusions. Also $p \times 1$ is a quotient map. It follows easily that q is a quotient map. ///

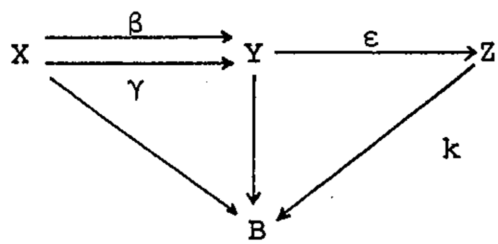
For $A \in \mathcal{K}$, let $\mathcal{K} \downarrow A$ denote the category whose objects are maps $f:X \to A$ in $\mathcal{K}$ and whose morphisms $\alpha:(f:X \to A) \to (g:Y \to A)$ are maps $\alpha:X \to Y$ in $\mathcal{K}$ with $f = g\alpha$. For $A \in \mathcal{U}$, the notation $\mathcal{U} \downarrow A$ has the corresponding meaning. The categories $\mathcal{K} \downarrow A$ and $\mathcal{U} \downarrow A$ have all small colimits. Colimits in $\mathcal{K} \downarrow A$ are constructed by taking the appropriate colimit of the domain spaces in $\mathcal{K}$ together with the induced map into A; for example, if $\lambda_n:(f_n:X_n \to A) \to (f_{n+1}:X_{n+1} \to A)$ is a sequence in $\mathcal{K} \downarrow A$, then colim f_n is the object $f:X \to A$ where $X = \operatorname{colim} X_n$ and f is induced by the f_n. Colimits in $\mathcal{U} \downarrow A$ are constructed similarly. Suppose $g:A \to B$ is a map in $\mathcal{K}$. Then there is a functor $P_g:\mathcal{K} \downarrow B \to \mathcal{K} \downarrow A$ defined by pulling back along $g:A \to B$; that is, $f:X \to B$ goes to $\tilde{f}:P \to A$ where the following diagram is a pullback.

$$\begin{array}{ccc} P & \xrightarrow{\tilde{f}} & A \\ \downarrow & & \downarrow g \\ X & \xrightarrow[f]{} & B \end{array}$$

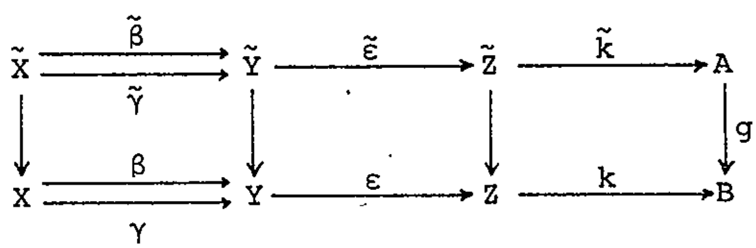
For $g:A \to B$ in $\mathcal{U}$, $P_g:\mathcal{U} \downarrow B \to \mathcal{U} \downarrow A$ is defined similarly. Our purpose is to give conditions under which these functors preserve colimits.

PROPOSITION 10.3. *Let* $g\colon A \to B$ *be a map in* K *and let* $B \in U$. *Then* $P_g\colon K \downarrow B \to K \downarrow A$ *preserves all (small) colimits.*

Proof. Since colimits are constructed from coproducts (disjoint unions) and coequalizers, it is enough to show that P_g preserves these. The preservation of coproducts is trivial. Suppose the following is a coequalizer diagram in $K \downarrow B$.



Taking pullbacks we have



Note that $\tilde{X}$ is well defined since $k\circ\varepsilon\circ\beta = k\circ\varepsilon\circ\gamma$ because ε is a coequalizer. $\tilde{\varepsilon}$ coequalizes $\tilde{\gamma}$ and $\tilde{\beta}$ and $\tilde{\varepsilon}$ is a quotient map since $B \in U$ and ε, being a coequalizer, is a quotient map. The question, then, is whether or not $\tilde{\varepsilon}$ is the coequalizer of $\tilde{\gamma}$ and $\tilde{\beta}$. There are two ways to see that it is. One can directly check the equivalence relation from which $\tilde{Z}$ is constructed as a quotient space of $\tilde{Y}$ or, more easily, one can note that in Sets pulling back preserves colimits and the forgetful functor from $\mathcal{K}$ to Sets preserves colimits. ///

Since $\mathcal{U}$ is the category of actual interest, it would be nice to modify this proposition to say that for $g:A \to B$ in $\mathcal{U}$, P_g preserves all colimits. This is false. Suppose $f:X \to B$ is a non-surjective map in $\mathcal{U}$ with dense image. Then, thought of as a map $f:(f:X \to B) \to (1:B \to B)$ in $\mathcal{U} \downarrow B$, f is epi. Let $g:\{*\} \to B$ be a map of $*$ to a point not in $f(X)$. Then $P_g(f)$ is the map $\phi \to *$ which is certainly not epi. Therefore P_g can't preserve all colimits since a functor which preserves all colimits must preserve epis.

There are two ways in which a version of Proposition 10.3 may be obtained for colimits in $\mathcal{U} \downarrow B$. The first is the following observation, part (b) of which is a special case of (a).

COROLLARY 10.4. a) *Suppose* $g:A \to B$ *is a map in* $\mathcal{U}$. *Then* $P_g:\mathcal{U} \downarrow B \to \mathcal{U} \downarrow A$ *preserves all colimits in* $\mathcal{U} \downarrow B$ *which are also colimits in* $\mathcal{K} \downarrow B$. *In particular,* P_g *preserves directed colimits over systems of injections.*

b) *Suppose* $Y \subset X \in \mathcal{U}$. *If* $\{F_\alpha X\}$ *is a filtration of* X *indexed over a directed set--that is, for each* α, $F_\alpha X \subset X$ *and* X *has the colimit topology from the* $F_\alpha X$*--then* $F_\alpha Y = Y \cap F_\alpha X$ *is a filtration of* Y.

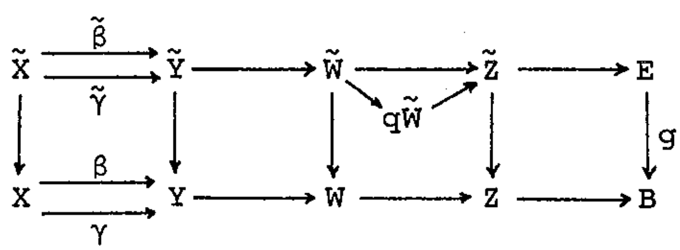
One immediate application of this result is that if B is a C.W. complex and $\xi:E \to B$ is a bundle (or fibration) then

ξ is the colimit of the bundles (or fibrations) over the individual cells of B.

The second approach is to place special conditions on g such that P_g preserves all colimits. The required conditions on g are exactly those applicable in bundle theory.

PROPOSITION 10.5. *Suppose* $g:E \to B$ *in* $\mathcal{U}$ *is locally trivial--that is,* B *has an open cover such that for each* U *in the cover,* $g^{-1}(U) \cong U \times F$ *over* U *for some fixed* F. *Then* $P_g : \mathcal{U} \downarrow B \to \mathcal{U} \downarrow E$ *preserves all colimits in* $\mathcal{U} \downarrow B$.

Proof. The preservation of coproducts follows from Corollary 8.4. In order to show that coequalizers are preserved it is necessary to use the explicit construction of the functor $q:\mathcal{K} \to \mathcal{U}$ given in Section 4. Suppose that in the diagram below $Y \to W$ is the coequalizer of β and γ in $\mathcal{K}$, $Y \to Z = qW$ is the coequalizer in $\mathcal{U}$ and all of the squares are pullbacks.



Then $\tilde{Y} \to \tilde{W}$ is the coequalizer in $\mathcal{K}$ of $\tilde{\beta}$ and $\tilde{\gamma}$ by Proposition 10.3 and $\tilde{Y} \to q\tilde{W}$ is the coequalizer of $\tilde{\beta}$ and $\tilde{\gamma}$ in $\mathcal{U}$. Therefore it suffices to show that $q\tilde{W} \cong \tilde{Z}$. By the construction of q it suffices to show that

$$\begin{array}{ccc} J^{\alpha}\tilde{W} & \to & E \\ \downarrow & & \downarrow g \\ J^{\alpha}W & \to & B \end{array}$$

is a pullback for all ordinals α. But then, by Proposition 10.3 and the definition of the J^{α}'s, it suffices to show that, for any $f:W \to B$ in K, when the left square below is a pullback so is the right:

$$\begin{array}{ccc} \tilde{W} & \to & E \\ \pi\downarrow & & \downarrow g \\ W & \xrightarrow{f} & B \end{array} \qquad\qquad \begin{array}{ccc} J\tilde{W} & \to & E \\ \downarrow & & \downarrow g \\ JW & \to & B \end{array}$$

For this it suffices to show that, considering $\tilde{W}$ as a subspace of $W\times E$,

$$\overline{\Delta}_{\tilde{W}} = \tau(\overline{\Delta}_W \times \Delta_E) \cap (\tilde{W}\times\tilde{W})$$

where $\tau:W\times W\times E\times E \to W\times E\times W\times E$ is the twist map. $\overline{\Delta}_{\tilde{W}} \subset \tau(\overline{\Delta}_W\times\Delta_E) \cap (\tilde{W}\times\tilde{W})$ is immediate. Suppose $(w_1,e,w_2,e) \in \tau(\overline{\Delta}_W\times\Delta_E) \cap (\tilde{W}\times\tilde{W})$. Then $(w_1,w_2) \in \overline{\Delta}_W$. Therefore $f(w_1) = f(w_2) \in B$ since $B \in \mathcal{U}$. Let U be an open neighborhood of $f(w_1)$ such that g is trivial over U. Since π is trivial over $f^{-1}(U)$ and $w_1,w_2 \in f^{-1}(U)$, $(w_1,e,w_2,e) \in \overline{\Delta}_{\tilde{W}}$. Therefore $\overline{\Delta}_{\tilde{W}} = \tau(\overline{\Delta}_W\times\Delta_E) \cap (\tilde{W}\times\tilde{W})$ and the right square above is a pullback. ///

11. Sequentially Generated Spaces

Any discussion of convenient categories of topological spaces would be incomplete without the mention of sequentially generated spaces. If, in the definitions of compactly open and closed, one replaced arbitrary compact spaces by arbitrary compact metric spaces, one would obtain notions of compact metrically open and closed. One could then study spaces in which compact metrically open sets were open. It is fairly easy to see that a set $A \subset X$ is compact metrically open if and only if it is sequentially open--that is, if $\{x_n\} \subset X$ is a sequence coverging to $x \in A$ then for some N, $\{x_n\}_{n>N} \subset A$. Spaces in which sequentially open sets are open are called sequential spaces and have been studied by Franklin [65, 66, 67], Wyler [73], Meyer [72], and Johnstone [78].

Sequential spaces have several important advantages, including:

1) All point-set questions can be settled with sequential arguments.

2) All C.W. complexes are sequential.

3) In sequential spaces the notions of sequentially quasicompact and countably quasicompact coincide. As a result, a sequential space is weak Hausdorff if and only if it has unique sequential limits; such spaces are called sequentially generated. The closed diagonal condition in the sequential product topology is equivalent to these two separation properties.

4) All the results of this appendix carry over with

sequential spaces replacing k-spaces and sequentially generated spaces replacing compactly generated spaces.

5) If X and Y are sequential spaces and Y^X is the function space with the appropriate sequential topology then a sequence $\{f_n\}_{n\geq 0} \subset Y^X$ converges to $f \in Y^X$ if and only if for every sequence $\{x_m\}_{m\geq 0}$ in X converging to a point x in X, the doubly indexed sequence $f_n(x_m)$ converges to $f(x)$.

Property (5) is more important than it might initially seem. The function space topology for compactly generated spaces is in general strictly finer than the compact open topology and is virtually untouchable by standard techniques. Therefore it is far easier to study function space related questions in sequentially generated spaces than in compactly generated spaces. The only real disadvantage of the category of sequentially generated spaces is that it is too small, in that it does not contain all compact Hausdorff spaces, and this is of negligible significance in algebraic topology.