

EQUALITY OF SKEW SCHUR FUNCTIONS IN NONCOMMUTING VARIABLES

EMMA YU JIN AND STEPHANIE VAN WILLIGENBURG

ABSTRACT. The question of classifying when two skew Schur functions are equal is a substantial open problem, which remains unsolved for over a century. In 2022, Aliniaiefard, Li and van Willigenburg introduced skew Schur functions in noncommuting variables, $s_{(\delta, \mathcal{D})}$, where \mathcal{D} is a connected skew diagram with n boxes and δ is a permutation in the symmetric group S_n .

In this paper, we combine these two and classify when two skew Schur functions in noncommuting variables are equal: $s_{(\delta, \mathcal{D})} = s_{(\tau, \mathcal{T})}$ such that $\mathcal{D} \neq \mathcal{T}$ if and only if \mathcal{D} is a nonsymmetric ribbon, \mathcal{T} is the antipodal rotation of \mathcal{D} and $\overline{\tau^{-1}\delta}$ is an explicit bijection between two set partitions determined by \mathcal{D} .

1. INTRODUCTION

The problem of classifying when two skew Schur functions are equal has been open since Schur introduced them in 1901 [14]. While partial progress has been made [10, 11, 12, 16], only a few special cases have been classified, for example [6, 8, 9, 15]. The most notable of these classifications was that of ribbon Schur functions [6], where the classification had further impact [2, 3, 4, 7]. Another long-standing problem was to find a basis analogous to that of Schur functions for the algebra of symmetric functions in noncommuting variables. This problem was posed in 2004 by Rosas and Sagan [13], and was eventually resolved in 2022 by Aliniaiefard, Li and van Willigenburg [1]. In this short paper we combine these two problems and classify succinctly when two skew Schur functions in noncommuting variables are equal:

Main Theorem. *Given two connected skew diagrams \mathcal{D} and \mathcal{T} such that $\mathcal{D} \neq \mathcal{T}$, we have that the skew Schur functions in noncommuting variables*

$$s_{(\delta, \mathcal{D})} = s_{(\tau, \mathcal{T})}$$

Date: March 27, 2024.

2020 Mathematics Subject Classification. Primary 05E05; Secondary 05A05, 05A18, 16T30.

Key words and phrases. noncommuting variables, ribbons, skew Schur functions.

The first author was supported by the Austrian Research Fund FWF Elise-Richter Project V 898-N, is supported by the Fundamental Research Funds for the Central Universities, Project No. 20720220039 and the National Nature Science Foundation of China (NSFC), Project No. 12201529. The second author is supported in part by the Natural Sciences Engineering and Research Council of Canada.

if and only if

- (1) \mathcal{D} is a nonsymmetric ribbon and
- (2) $\mathcal{T} = \mathcal{D}^*$, and
- (3) the bijection $\overline{\tau^{-1}\delta} : j \mapsto n + 1 - \tau^{-1}\delta(j)$ preserves each block of the set partition $[\alpha]$ where $n = |\mathcal{D}|$ and α is the row-length composition of \mathcal{D} .

In particular, in Section 2, we give all the background needed to understand the above theorem, and prove some small but valuable lemmas. Then we devote Section 3 to proving the above theorem, which we restate just before the proof as Theorem 3.1.

2. BACKGROUND

We begin by reviewing the combinatorial concepts we need, before introducing our algebras and functions of study.

A *composition* $\alpha = (\alpha_1, \dots, \alpha_l)$ of n is a finite sequence of positive integers $\alpha_1, \dots, \alpha_l$ such that $\sum_{i=1}^l \alpha_i = n$, denoted by $\alpha \vDash n$. We call the α_i the *parts* of α , $\ell(\alpha) = l$ the *length* of α and $|\alpha| = n$ the *size* of α . We denote by 0 the unique composition of length and size 0 . If the parts of α appear in weakly decreasing order, then we call this a *partition* λ , denoted by $\lambda \vdash n$, and if the parts of α are allowed to include 0 then we call this a *weak composition*. Note that every composition α determines a partition $\lambda(\alpha) = (\lambda(\alpha)_1, \dots, \lambda(\alpha)_{\ell(\alpha)})$ obtained by listing the parts of α in weakly decreasing order. Given a composition $\alpha = (\alpha_1, \dots, \alpha_{\ell(\alpha)})$ we define

$$\alpha! = \alpha_1! \cdots \alpha_{\ell(\alpha)}! \text{ and } \alpha^* = (\alpha_{\ell(\alpha)}, \dots, \alpha_1).$$

Meanwhile, given two compositions of n , $\alpha = (\alpha_1, \dots, \alpha_{\ell(\alpha)})$ and $\beta = (\beta_1, \dots, \beta_{\ell(\beta)})$, we say that β is a *coarsening* of α (or α is a *refinement* of β), denoted by $\beta \succcurlyeq \alpha$, if β is obtained from α by adding together adjacent parts of α , and β *dominates* α if $\sum_{j=1}^i \beta_j \geq \sum_{j=1}^i \alpha_j$ for all $1 \leq i \leq \min\{\ell(\alpha), \ell(\beta)\}$.

Example 2.1. If $\alpha = (1, 2, 1, 3, 2) \vDash 9$ then $\ell(\alpha) = 5$, $|\alpha| = 9$ and $\lambda(\alpha) = (3, 2, 2, 1, 1)$. Note that $\alpha! = 1!2!1!3!2! = 24$, $\alpha^* = (2, 3, 1, 2, 1)$, and $(1, 3, 3, 2) \succcurlyeq (1, 2, 1, 3, 2)$.

Given a partition $\lambda = (\lambda_1, \dots, \lambda_{\ell(\lambda)})$, we say that its *diagram*, also denoted by λ , is the array of left-justified boxes with λ_i boxes in row i from the top. Given two partitions $\lambda = (\lambda_1, \dots, \lambda_{\ell(\lambda)}) \vdash n$ and $\mu = (\mu_1, \dots, \mu_{\ell(\mu)}) \vdash m$ such that $\ell(\mu) \leq \ell(\lambda)$ and $\mu_i \leq \lambda_i$ for all $1 \leq i \leq \ell(\mu)$ we say that μ is *contained* in λ , denoted by $\mu \subseteq \lambda$, and moreover the *skew diagram* λ/μ of *size* $(n - m) = |\lambda/\mu|$ is the array of boxes contained in λ but not in μ when the array of boxes of μ is positioned in the top-left corner of the array of boxes of λ . Given a skew diagram λ/μ we define $(\lambda/\mu)^*$ to be the 180° antipodal rotation of λ/μ , and we say that it is *symmetric* if $\lambda/\mu = (\lambda/\mu)^*$.

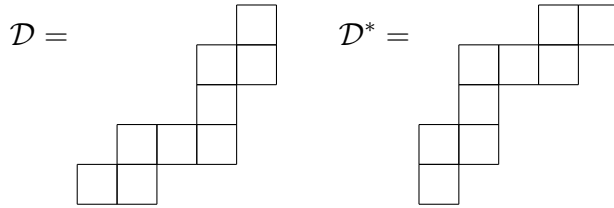
We say that a skew diagram is *connected* if each pair of adjacent rows overlap in at least one column, and say that it is a *ribbon* if each pair of adjacent rows overlap in *exactly* one column. How much certain sets of rows overlap will be important for the rest of our paper and so we define this formally now, in addition to often referring to skew diagrams by letters such as \mathcal{D} in order to streamline notation.

Definition 2.2. Let \mathcal{D} be a skew diagram occupying r rows. For each $k \in \{1, \dots, r\}$, we define the k -row *overlap composition* to be the (weak) composition

$$\alpha^{(k)}(\mathcal{D}) = (r_1^{(k)}, \dots, r_{r-k+1}^{(k)})$$

where $r_i^{(k)}$ is the number of columns occupied in common by the rows $i, i+1, \dots, i+k-1$. Let $\lambda^{(k)}(\mathcal{D})$ be the k -row *overlap partition*, that is, the partition determined by rearranging the parts of $\alpha^{(k)}(\mathcal{D})$ in weakly decreasing order. Furthermore, the *row-length composition* is $\alpha(\mathcal{D}) = \alpha^{(1)}(\mathcal{D})$, and the *row-length partition* is $\lambda(\mathcal{D}) = \lambda(\alpha(\mathcal{D}))$.

Example 2.3. The skew diagram for $\mathcal{D} = (5, 5, 4, 4, 2)/(4, 3, 3, 1)$ is a ribbon where $|\mathcal{D}| = 9$ and is the skew diagram on the left, while \mathcal{D}^* is the skew diagram on the right.



Note that $\alpha(\mathcal{D}) = \alpha^{(1)}(\mathcal{D}) = (1, 2, 1, 3, 2)$, $\alpha^{(2)}(\mathcal{D}) = (1, 1, 1, 1)$, $\alpha^{(3)}(\mathcal{D}) = (0, 0, 0)$, $\alpha^{(4)}(\mathcal{D}) = (0, 0)$, $\alpha^{(5)}(\mathcal{D}) = (0)$ and $\lambda(\mathcal{D}) = (3, 2, 2, 1, 1)$.

Observe that for a skew diagram \mathcal{D} ,

$$\alpha(\mathcal{D}^*) = (\alpha(\mathcal{D}))^*$$

and that if \mathcal{D} is a ribbon, then because we know that every pair of adjacent rows overlap in exactly one column we have that $\alpha(\mathcal{D})$ determines the ribbon exactly, and hence there is a natural bijection between ribbons of size n and compositions of size n .

We now move from diagrams to set partitions. Given $[n] = \{1, \dots, n\}$, a *set partition* $\pi = \pi_1 / \dots / \pi_l$ of $[n]$ is a family of disjoint nonempty subsets of positive integers π_1, \dots, π_l such that $\cup_{i=1}^l \pi_i = [n]$, denoted by $\pi \vdash [n]$. We call the π_i the *blocks* of π , $\ell(\pi) = l$ the *length* of π , and $|\pi| = n$ the *size* of π . We usually list the blocks by increasing least element, omitting set parentheses and commas for ease of legibility, and denote by \emptyset the unique set partition of length and size 0. Note that every set partition π determines a partition $\lambda(\pi) = (\lambda(\pi)_1, \dots, \lambda(\pi)_{\ell(\pi)})$, obtained by listing the cardinalities of the block sizes of π in weakly decreasing order, and we set $\lambda(\pi)! = \lambda(\pi)_1! \dots \lambda(\pi)_{\ell(\pi)}!$.

Example 2.4. Note that the family of disjoint subsets $\{1\}, \{2, 4\}, \{3\}, \{5, 6, 7\}, \{8, 9\}$ is a set partition of $[9]$, and as a set partition π we write

$$\pi = 1/24/3/567/89 \vdash [9]$$

with $\ell(\pi) = 5$, $|\pi| = 9$, $\lambda(\pi) = (3, 2, 2, 1, 1)$ and $\lambda(\pi)! = 3!2!2!1!1! = 24$.

Given two set partitions $\pi \vdash [n]$ and $\sigma = \sigma_1/\cdots/\sigma_{\ell(\sigma)} \vdash [m]$, we say that their *slash product* $\pi \mid \sigma$ is

$$(2.1) \quad \pi \mid \sigma = \pi/(\sigma_1 + n)/\cdots/(\sigma_{\ell(\sigma)} + n) \vdash [n + m],$$

where $\sigma_i + n = \{s + n : s \in \sigma_i\}$ for $1 \leq i \leq \ell(\sigma)$. We also say that two set partitions $\pi, \sigma \vdash [n]$ satisfy $\pi \leq \sigma$ if σ is obtained from π by merging blocks of π . Returning to compositions, given a composition $\alpha = (\alpha_1, \dots, \alpha_{\ell(\alpha)}) \vDash n$, its corresponding set partition $[\alpha] \vdash [n]$ is

$$(2.2) \quad [\alpha] = 1 \cdots \alpha_1 / (\alpha_1 + 1) \cdots (\alpha_1 + \alpha_2) / \cdots / \left(\sum_{i=1}^{\ell(\alpha)-1} \alpha_i \right) + 1 \cdots n \\ = [\alpha_1] \mid [\alpha_2] \mid \cdots \mid [\alpha_{\ell(\alpha)}].$$

We use $[\alpha_i]_\alpha$ to denote each block of $[\alpha]$, in order to avoid confusion with the set $[\alpha_i] = \{1, \dots, \alpha_i\}$.

Example 2.5. If $\pi = 1/24/3 \vdash [4]$ and $\sigma = 123/45 \vdash [5]$ then $\pi \mid \sigma = 1/24/3/567/89 \vdash [9]$. Meanwhile if $\alpha = (1, 2, 1, 3, 2)$ then

$$[\alpha] = 1/23/4/567/89 = [1] \mid [2] \mid [1] \mid [3] \mid [2]$$

with $[\alpha_4]_\alpha = \{5, 6, 7\}$.

We now turn our attention to the Hopf algebras of symmetric functions Sym and symmetric functions in noncommuting variables NCSym , respectively. The *Hopf algebra of symmetric functions* Sym is the graded Hopf algebra

$$\text{Sym} = \text{Sym}^0 \oplus \text{Sym}^1 \oplus \cdots \subset \mathbb{Q}[[x_1, x_2, \dots]]$$

where $[[\cdot]]$ means that the variables commute, $\text{Sym}^0 = \text{span}\{1\}$ and the n th graded piece for $n \geq 1$ has the following bases

$$\text{Sym}^n = \text{span}\{h_\lambda : \lambda \vdash n\} = \text{span}\{s_\lambda : \lambda \vdash n\}$$

where these functions are defined as follows, given a partition $\lambda = (\lambda_1, \dots, \lambda_{\ell(\lambda)}) \vdash n$.

The *complete homogeneous symmetric function*, h_λ , is given by

$$h_\lambda = h_{\lambda_1} \cdots h_{\lambda_{\ell(\lambda)}}$$

where $h_i = \sum_{j_1 \leq \dots \leq j_i} x_{j_1} \cdots x_{j_i}$.

Example 2.6. $h_{(2,1)} = (x_1x_2 + x_1^2 + \cdots)(x_1 + x_2 + \cdots)$

For the next basis we begin with two partitions $\lambda = (\lambda_1, \dots, \lambda_{\ell(\lambda)})$ and $\mu = (\mu_1, \dots, \mu_{\ell(\mu)})$ such that $\mu \subseteq \lambda$, and define the *Jacobi-Trudi matrix* of the skew diagram λ/μ to be

$$(2.3) \quad JT(\lambda/\mu) = (h_{\lambda_i - \mu_j - i + j})_{1 \leq i, j \leq \ell(\lambda)}$$

where we set $\mu_j = 0$ for $\ell(\mu) < j \leq \ell(\lambda)$, $h_0 = 1$ and any function with a negative index equals 0. Then the *skew Schur function*, $s_{\lambda/\mu}$, is given by

$$(2.4) \quad s_{\lambda/\mu} = \det JT(\lambda/\mu)$$

and if $\mu = 0$ then the *Schur function*, s_λ , is given by

$$(2.5) \quad s_\lambda = \det JT(\lambda).$$

Example 2.7. $s_{(2,1)} = \det \begin{pmatrix} h_2 & h_3 \\ h_0 & h_1 \end{pmatrix} = h_2h_1 - h_3h_0 = h_{(2,1)} - h_{(3)}$

In particular, if λ/μ is a ribbon corresponding to the composition α , then the *ribbon Schur function*, r_α , is given by [6, Proposition 2.1]

$$r_\alpha = (-1)^{\ell(\alpha)} \sum_{\beta \succ \alpha} (-1)^{\ell(\beta)} h_\beta.$$

Example 2.8. $r_{(2,1)} = h_{(2,1)} - h_{(3)}$

The Jacobi-Trudi matrix also satisfies some particularly useful properties.

Lemma 2.9. *Given a skew diagram λ/μ , let $\mathcal{A}_{ij} = \lambda_i - \mu_j - i + j$ for any i, j . Then for any $1 \leq i < j \leq \ell(\lambda)$, and $1 \leq k < m \leq \ell(\lambda)$, we have that $\mathcal{A}_{ik} + \mathcal{A}_{jm} = \mathcal{A}_{im} + \mathcal{A}_{jk}$.*

Proof. This follows by the fact that both sides equal $\lambda_i + \lambda_j - \mu_k - \mu_m - i + k - j + m$. \square

Lemma 2.10. [12, Proposition 6.2] *Let λ/μ be a skew diagram with $\ell = \ell(\lambda)$.*

(1) *The largest subscript occurring on any nonzero entry h_L in the Jacobi-Trudi matrix $JT(\lambda/\mu)$ is $L = \lambda_1 + \ell - 1$ and this subscript occurs exactly once, on the $(1, \ell)$ -entry h_L .*

(2) *The subscripts on the diagonal entries in $JT(\lambda/\mu)$ are exactly*

$$\alpha(\lambda/\mu) = (\lambda_1 - \mu_1, \dots, \lambda_\ell - \mu_\ell)$$

and the monomial $h_{\lambda_1 - \mu_1} \cdots h_{\lambda_\ell - \mu_\ell}$ occurs in the determinant $s_{\lambda/\mu}$

(a) with coefficient +1, and

(b) as the monomial whose subscripts rearranged into weakly decreasing order give the smallest partition of $|\lambda/\mu|$ in dominance order among all nonzero monomials.

Meanwhile, the Hopf algebra of symmetric functions in noncommuting variables NCSym is the graded Hopf algebra

$$\text{NCSym} = \text{NCSym}^0 \oplus \text{NCSym}^1 \oplus \cdots \subset \mathbb{Q}\langle\langle x_1, x_2, \dots \rangle\rangle$$

where $\langle\langle \cdot \rangle\rangle$ means that the variables do not commute, $\text{NCSym}^0 = \text{span}\{1\}$ and the n th graded piece for $n \geq 1$ has the following bases

$$\text{NCSym}^n = \text{span}\{h_\pi : \pi \vdash [n]\} = \text{span}\{s_\pi : \pi \vdash [n]\}$$

where these functions are defined following [1], given a set partition $\pi = \pi_1 / \cdots / \pi_{\ell(\pi)} \vdash [n]$.

The complete homogeneous symmetric function in NCSym , h_π , is given by [1, Lemma 2.14]

$$h_\pi = \sum_{\varepsilon} \sum_{(i_1, \dots, i_n)} x_{i_{\varepsilon(1)}} \cdots x_{i_{\varepsilon(n)}}$$

where

- (1) the first sum is over all $\varepsilon \in S_n$ that fixes the blocks of π ,
- (2) the second sum is over all n -tuples of positive integers (i_1, \dots, i_n) such that if j and k are in the same block of π with $j < k$, then $i_j \leq i_k$.

Example 2.11. $h_{13/2} = 2x_1x_1x_1 + x_1x_1x_2 + x_2x_1x_1 + 2x_1x_2x_1 + x_1x_2x_3 + \cdots$

These functions multiply together in a natural way.

Lemma 2.12. [5, Corollary 2.41] *For set partitions π and σ we have that*

$$h_\pi h_\sigma = h_{\pi|\sigma}.$$

Hence, much like their counterparts in Sym , given a composition $\alpha = (\alpha_1, \dots, \alpha_{\ell(\alpha)})$ we have that

$$h_{[\alpha]} = h_{[\alpha_1]} \cdots h_{[\alpha_{\ell(\alpha)}]}.$$

We also have that [13, Section 2] given a permutation $\delta \in S_n$

$$\delta \circ h_\pi = h_{\delta\pi}$$

where δ acts on π by sending every element i to $\delta(i)$.

Example 2.13. $321 \circ h_{12/3} = h_{1/23}$, where the permutation $321 \in S_3$ is written in one-line notation.

For the next basis, much like classical Schur functions, we will again need to compute a determinant, however now it will need to be noncommutative, which we recall as follows [1, Equation 2.6]. We define the *noncommutative analogue of Leibniz' determinantal formula* for any matrix $A = (a_{ij})_{1 \leq i, j \leq n}$ with noncommuting entries a_{ij} to be

$$(2.6) \quad \mathbf{det}(A) = \sum_{\varepsilon \in S_n} \text{sgn}(\varepsilon) a_{1\varepsilon(1)} \cdots a_{n\varepsilon(n)}$$

that takes the product of the entries from the top row to the bottom row, and $\text{sgn}(\varepsilon)$ is the sign of the permutation ε . Given two partitions $\lambda = (\lambda_1, \dots, \lambda_{\ell(\lambda)})$ and $\mu = (\mu_1, \dots, \mu_{\ell(\mu)})$ such that $\mu \subseteq \lambda$, we define the *noncommutative Jacobi-Trudi matrix* of the skew diagram λ/μ to be

$$(2.7) \quad \mathbf{JT}(\lambda/\mu) = \left(\frac{1}{(\lambda_i - \mu_j - i + j)!} h_{[\lambda_i - \mu_j - i + j]} \right)_{1 \leq i, j \leq \ell(\lambda)}$$

where we set $\mu_j = 0$ for $\ell(\mu) < j \leq \ell(\lambda)$, $h_{[0]} = h_\emptyset = 1$ and any function with a negative index equals 0. If $|\lambda/\mu| = n$ and $\delta \in S_n$, then the *skew Schur function in NCSym*, $s_{(\delta, \lambda/\mu)}$, is given by [1, Definition 4.2]

$$(2.8) \quad s_{(\delta, \lambda/\mu)} = \delta \circ \mathbf{det} \mathbf{JT}(\lambda/\mu).$$

If $\delta = \text{id}$, then we call this the *source skew Schur function in NCSym*, $s_{[\lambda/\mu]}$, hence given by [1, Definition 3.1]

$$(2.9) \quad s_{[\lambda/\mu]} = \mathbf{det} \mathbf{JT}(\lambda/\mu) = \mathbf{det} \left(\frac{1}{(\lambda_i - \mu_j - i + j)!} h_{[\lambda_i - \mu_j - i + j]} \right)_{1 \leq i, j \leq \ell(\lambda)}.$$

Source skew Schur functions have the following property.

Lemma 2.14. *For any connected skew diagram \mathcal{D} of size n , the source skew Schur function $s_{[\mathcal{D}]}$ is a linear combination of $h_{[\gamma]}$ with $\gamma \vDash n$.*

Proof. Let $\mathcal{D} = \lambda/\mu$ with $|\lambda/\mu| = n$. For any $w \in S_{\ell(\lambda)}$, let $w(\alpha) = (w(\alpha_1), \dots, w(\alpha_{\ell(\lambda)}))$ where $w(\alpha_i) = \lambda_i - \mu_{w(i)} - i + w(i)$. Then the determinantal formula of $s_{[\lambda/\mu]}$ gives that

$$(2.10) \quad \begin{aligned} s_{[\lambda/\mu]} &= \mathbf{det} \left((\lambda_i - \mu_j - i + j)!^{-1} h_{[\lambda_i - \mu_j - i + j]} \right)_{1 \leq i, j \leq \ell(\lambda)} \\ &= \sum_{w \in S_{\ell(\lambda)}} \text{sgn}(w) \prod_{i=1}^{\ell(\lambda)} w(\alpha)!^{-1} h_{[w(\alpha)]}. \end{aligned}$$

Now by construction $w(\alpha) \vDash w(\alpha_1) + \cdots + w(\alpha_{\ell(\lambda)})$ and

$$\sum_{i=1}^{\ell(\lambda)} w(\alpha_i) = \sum_{i=1}^{\ell(\lambda)} \lambda_i - \mu_{w(i)} - i + w(i) = \sum_{i=1}^{\ell(\lambda)} \lambda_i - \mu_i = n$$

as desired, because w is a permutation. □

Given $\pi \vdash [n]$ let us arrange the blocks such that reading from left to right

- (1) block sizes weakly decrease, and
- (2) smallest elements of blocks of the same size are strictly increasing.

Let δ_π be the permutation in one-line notation obtained by removing the slashes from π with the blocks arranged as just described. Then the (*standard*) Schur function in NCSym, s_π , is given by [1, Definition 4.2]

$$(2.11) \quad s_\pi = \delta_\pi \circ s_{[\lambda(\pi)]} = \delta_\pi \circ \mathbf{det} \left(\frac{1}{(\lambda(\pi)_i - i + j)!} h_{[\lambda(\pi)_i - i + j]} \right)_{1 \leq i, j \leq \ell(\pi)}.$$

Example 2.15. The source Schur function in noncommuting variables $s_{[(2,1)]}$ is

$$\begin{aligned} s_{[(2,1)]} &= \mathbf{det} \begin{pmatrix} \frac{1}{2!} h_{[2]} & \frac{1}{3!} h_{[3]} \\ \frac{1}{0!} h_{[0]} & \frac{1}{1!} h_{[1]} \end{pmatrix} = \mathbf{det} \begin{pmatrix} \frac{1}{2!} h_{12} & \frac{1}{3!} h_{123} \\ \frac{1}{0!} h_\emptyset & \frac{1}{1!} h_1 \end{pmatrix} \\ &= \frac{1}{2!} h_{12} \frac{1}{1!} h_1 - \frac{1}{3!} h_{123} \frac{1}{0!} h_\emptyset = \frac{1}{2} h_{12|1} - \frac{1}{6} h_{123} = \frac{1}{2} h_{12/3} - \frac{1}{6} h_{123}. \end{aligned}$$

If $\pi = 13/2$, then $\delta_\pi = 132 \in S_3$ in one-line notation and so

$$s_{13/2} = 132 \circ s_{[(2,1)]} = 132 \circ \left(\frac{1}{2} h_{12/3} - \frac{1}{6} h_{123} \right) = \frac{1}{2} h_{13/2} - \frac{1}{6} h_{123}.$$

Meanwhile the skew Schur function in noncommuting variables $s_{(321, (2,2)/(1))}$, where $321 \in S_3$ is in one-line notation, is

$$\begin{aligned} s_{(321, (2,2)/(1))} &= 321 \circ \mathbf{det} \begin{pmatrix} \frac{1}{1!} h_{[1]} & \frac{1}{3!} h_{[3]} \\ \frac{1}{0!} h_{[0]} & \frac{1}{2!} h_{[2]} \end{pmatrix} = 321 \circ \mathbf{det} \begin{pmatrix} \frac{1}{1!} h_1 & \frac{1}{3!} h_{123} \\ \frac{1}{0!} h_\emptyset & \frac{1}{2!} h_{12} \end{pmatrix} \\ &= 321 \circ \left(\frac{1}{1!} h_1 \frac{1}{2!} h_{12} - \frac{1}{3!} h_{123} \frac{1}{0!} h_\emptyset \right) = 321 \circ \left(\frac{1}{2} h_{1|12} - \frac{1}{6} h_{123} \right) \\ &= 321 \circ \left(\frac{1}{2} h_{1/23} - \frac{1}{6} h_{123} \right) = \frac{1}{2} h_{12/3} - \frac{1}{6} h_{123}. \end{aligned}$$

Our classification will explain why $s_{[(2,1)]} = s_{(321, (2,2)/(1))}$ later.

In particular, if λ/μ is a ribbon corresponding to a composition α , then the *ribbon Schur function* in NCSym, $r_{[\alpha]}$, is given by [1, Corollary 6.4]

$$(2.12) \quad r_{[\alpha]} = (-1)^{\ell(\alpha)} \sum_{\beta \succ \alpha} (-1)^{\ell(\beta)} \frac{h_{[\beta]}}{\beta!}.$$

Example 2.16. $r_{[(2,1)]} = \frac{1}{2} h_{[(2,1)]} - \frac{1}{6} h_{[(3)]}$

Connecting our two Hopf algebras Sym and NCSym is the homomorphism

$$\rho : \mathbb{Q}\langle\langle x_1, x_2, \dots \rangle\rangle \rightarrow \mathbb{Q}[[x_1, x_2, \dots]]$$

that lets the variables commute, and explicitly relates the functions we have met as follows.

Lemma 2.17.

- (1) [13, Theorem 2.1] $\rho(h_\pi) = \lambda(\pi)!h_{\lambda(\pi)}$
- (2) [1, Lemma 4.4] $\rho(s_{(\delta, \lambda/\mu)}) = s_{\lambda/\mu}$
- (3) [1, Corollary 6.4] $\rho(r_{[\alpha]}) = r_\alpha$

As a direct consequence, it follows that necessary conditions for the equality of two skew Schur functions are also necessary conditions for the equality of two skew Schur functions in NCSym , and so one ingredient for the proof of our classification is the following necessary condition, and it is in fact the final ingredient necessary to prove our classification.

Lemma 2.18. [12, Corollary 8.11] *For two skew diagrams \mathcal{D} and \mathcal{T} , if $s_{\mathcal{D}} = s_{\mathcal{T}}$, then \mathcal{D} and \mathcal{T} must have the same k -row overlap partitions for all k .*

3. PROOF OF OUR MAIN RESULT

We are now ready to prove our main result, in which we classify when two skew Schur functions in NCSym are equal, recalling that a skew diagram \mathcal{D} has its 180° antipodal rotation denoted by \mathcal{D}^* , is nonsymmetric if $\mathcal{D} \neq \mathcal{D}^*$, and is a ribbon if each pair of adjacent rows overlap in exactly one column.

Theorem 3.1. *Given two connected skew diagrams \mathcal{D} and \mathcal{T} such that $\mathcal{D} \neq \mathcal{T}$, we have that the skew Schur functions in NCSym*

$$s_{(\delta, \mathcal{D})} = s_{(\tau, \mathcal{T})}$$

if and only if

- (1) \mathcal{D} is a nonsymmetric ribbon and
- (2) $\mathcal{T} = \mathcal{D}^*$, and
- (3) the bijection $\overline{\tau^{-1}\delta} : j \mapsto n + 1 - \tau^{-1}\delta(j)$ preserves each block of the set partition $[\alpha]$ where $n = |\mathcal{D}|$ and α is the row-length composition of \mathcal{D} .

Proof. In order to make the notation in our proof easier to read, if \mathcal{D}, \mathcal{T} are skew diagrams of size n and $\delta, \tau \in S_n$, then we write $(\delta, \mathcal{D}) \sim (\tau, \mathcal{T})$, if $s_{(\delta, \mathcal{D})} = s_{(\tau, \mathcal{T})}$. Additionally, we denote (id, \mathcal{T}) by (\mathcal{T}) where id is the identity permutation.

Note that $(\delta, \mathcal{D}) \sim (\tau, \mathcal{T})$, that is, $\delta \circ s_{[\mathcal{D}]} = \tau \circ s_{[\mathcal{T}]}$ if and only if $(\tau^{-1}\delta) \circ s_{[\mathcal{D}]} = s_{[\mathcal{T}]}$, namely $(\tau^{-1}\delta, \mathcal{D}) \sim (\tau, \mathcal{T})$. Hence it suffices to only prove the sufficient and necessary conditions in this latter case.

Throughout we also use the notations of our Lemmas in the previous section.

For one direction, if \mathcal{D} is a ribbon corresponding to a composition α , and $\mathcal{T} = \mathcal{D}^* \neq \mathcal{D}$, then by comparing entries on the main diagonal of $JT(\mathcal{D})$ and $JT(\mathcal{T})$ we see that $(\tau^{-1}\delta) \circ h_{[\alpha]} = h_{(\tau^{-1}\delta)[\alpha]} = h_{[\alpha^*]}$ since the bijection $\overline{\tau^{-1}\delta} : j \mapsto n + 1 - \tau^{-1}\delta(j)$ preserves each block of $[\alpha]$. For any composition β satisfying $\alpha \preceq \beta$, we have $[\alpha] \leq [\beta]$ and thus the greatest lower bound of all $[\beta]$ such that $\alpha \preceq \beta$ equals $[\alpha]$. As a result, $(\tau^{-1}\delta) \circ h_{[\beta]} = h_{[\beta^*]}$ for all $\beta \succ \alpha$. In view of (2.12), we find that $(\tau^{-1}\delta) \circ r_{[\alpha]} = r_{[\alpha^]}$, namely $(\delta, \mathcal{D}) \sim (\tau, \mathcal{T})$. This completes the proof in this direction.

For the other direction, suppose that $(\delta, \mathcal{D}) \sim (\tau, \mathcal{T})$, then we have $(\tau^{-1}\delta) \circ s_{[\mathcal{D}]} = s_{[\mathcal{T}]}$, that is, by (2.10) with $\ell(\lambda) = \ell$,

$$s_{[\mathcal{T}]} = \sum_{w \in S_\ell} \text{sgn}(w) \prod_{i=1}^{\ell} w(\alpha)!^{-1} h_{(\tau^{-1}\delta)[w(\alpha)]}.$$

Let

$$\alpha = (\mathcal{A}_{11}, \dots, \mathcal{A}_{ii}, \mathcal{A}_{i+1i+1}, \dots, \mathcal{A}_{\ell\ell}) \text{ and } \gamma = (\mathcal{A}_{11}, \dots, \mathcal{A}_{ii+1}, \mathcal{A}_{i+1i}, \dots, \mathcal{A}_{\ell\ell})$$

be the two (weak) compositions $w(\alpha)$ corresponding to when $w = \text{id}$ and $w = (ii+1)$, respectively. So in particular, α is our usual row-length composition.

By Lemma 2.14 we find that $(\tau^{-1}\delta)[\alpha] = [\mu]$ and $(\tau^{-1}\delta)[\gamma] = [\nu]$ for some $\mu, \nu \vDash n$. Both $[\mu]$ and $[\nu]$ have common blocks $(\tau^{-1}\delta)[\mathcal{A}_{jj}]_\alpha$ for all $j \notin \{i, i+1\}$ since $[\alpha]$ and $[\gamma]$ have common blocks $[\mathcal{A}_{jj}]_\alpha = [\mathcal{A}_{jj}]_\gamma$ for all $j \notin \{i, i+1\}$.

We claim that two blocks $(\tau^{-1}\delta)[\mathcal{A}_{ii}]_\alpha$ and $(\tau^{-1}\delta)[\mathcal{A}_{i+1i+1}]_\alpha$ must be adjacent in the sense that the union of $(\tau^{-1}\delta)[\mathcal{A}_{ii}]_\alpha$ and $(\tau^{-1}\delta)[\mathcal{A}_{i+1i+1}]_\alpha$ is a set of consecutive integers.

To see this, since

$$\mathcal{A}_{ii} + \mathcal{A}_{i+1i+1} = \mathcal{A}_{ii+1} + \mathcal{A}_{i+1i},$$

by Lemma 2.9, and $\mathcal{A}_{ii} < \mathcal{A}_{ii+1}$, by the definition of the Jacob-Trudi matrix, we have that

$$(3.1) \quad [\mathcal{A}_{ii}]_\alpha \subsetneq [\mathcal{A}_{ii+1}]_\gamma \subseteq [\mathcal{A}_{ii+1}]_\gamma \cup [\mathcal{A}_{i+1i}]_\gamma = [\mathcal{A}_{ii}]_\alpha \cup [\mathcal{A}_{i+1i+1}]_\alpha.$$

It follows that

$$(3.2) \quad (\tau^{-1}\delta)[\mathcal{A}_{ii}]_\alpha \subsetneq (\tau^{-1}\delta)[\mathcal{A}_{ii+1}]_\gamma \subseteq (\tau^{-1}\delta)[\mathcal{A}_{ii}]_\alpha \cup (\tau^{-1}\delta)[\mathcal{A}_{i+1i+1}]_\alpha.$$

Note that $(\tau^{-1}\delta)[\mathcal{A}_{ii+1}]_\gamma$ is a block of $[\nu]$, which by definition is a set of consecutive integers, implying that $(\tau^{-1}\delta)[\mathcal{A}_{ii}]_\alpha$ and $(\tau^{-1}\delta)[\mathcal{A}_{i+1i+1}]_\alpha$ must be adjacent as claimed. Consequently, because this is true for all $1 \leq i \leq \ell - 1$, we have that either

- the permutation $\tau^{-1}\delta$ preserves every block of $[\alpha]$, or

- the bijection $\overline{\tau^{-1}\delta} : j \mapsto n + 1 - \tau^{-1}\delta(j)$ preserves every block of $[\alpha]$.

Before we analyze these two cases, we observe that $(\tau^{-1}\delta)[\mathcal{A}_{1\ell}] = [\mathcal{A}_{1\ell}]$. This is true because $[\mathcal{A}_{1\ell}]$ is the unique largest block of any set partition $[w(\alpha)]$ and $(\tau^{-1}\delta)[w(\alpha)]$ as a result of (1) of Lemma 2.10. Now we begin our analysis.

We begin with the simpler case, where $\overline{\tau^{-1}\delta} : j \mapsto n + 1 - \tau^{-1}\delta(j)$ preserves every block of $[\alpha]$ and $\alpha \neq \alpha^*$. In this case we have that

$$\mathcal{A}_{11} + \cdots + \mathcal{A}_{\ell\ell} = \mathcal{A}_{1\ell}$$

because of the following. First note that since $\overline{\tau^{-1}\delta} : j \mapsto n + 1 - \tau^{-1}\delta(j)$ preserves every block of $[\alpha]$, this implies that $1 \in \overline{\tau^{-1}\delta}[\mathcal{A}_{11}]$, and hence that $n \in (\tau^{-1}\delta)[\mathcal{A}_{11}]$. From our observation above we also know that $(\tau^{-1}\delta)[\mathcal{A}_{1\ell}] = [\mathcal{A}_{1\ell}]$. Therefore, since $\mathcal{A}_{11} < \mathcal{A}_{1\ell}$ (for $\ell \neq 1$) we obtain that

$$n \in (\tau^{-1}\delta)[\mathcal{A}_{11}] \subsetneq (\tau^{-1}\delta)[\mathcal{A}_{1\ell}] = [\mathcal{A}_{1\ell}]$$

implying that $\mathcal{A}_{1\ell} \geq n$. Second, note that $\mathcal{A}_{1\ell} \leq n$ because $\mathcal{A}_{1\ell}$ is a part of a composition of n .

Consequently, since $\mathcal{A}_{1\ell} \geq n$ and $\mathcal{A}_{1\ell} \leq n$, we have that

$$\mathcal{A}_{1\ell} = n = \mathcal{A}_{11} + \cdots + \mathcal{A}_{\ell\ell}.$$

By repeatedly applying Lemma 2.9, we are led to

$$\mathcal{A}_{1\ell} = \mathcal{A}_{11} + \cdots + \mathcal{A}_{\ell\ell} = \mathcal{A}_{1\ell} + \mathcal{A}_{21} + \cdots + \mathcal{A}_{\ell\ell-1},$$

which gives that $\mathcal{A}_{j+1j} = 0$ for all $1 \leq j \leq \ell - 1$, because \mathcal{D} is connected and so by definition $\mathcal{A}_{j+1j} \geq 0$. Hence, $s_{[\mathcal{D}]}$ is the determinant of a matrix whose subdiagonal entries are all $h_0 = 1$, which implies that the row overlap of each pair of adjacent rows in \mathcal{D} is 1, that is, \mathcal{D} is a ribbon. Also $\mathcal{T} = \mathcal{D}^* \neq \mathcal{D}$ because $\alpha \neq \alpha^*$ by assumption. This completes the case where $\overline{\tau^{-1}\delta} : j \mapsto n + 1 - \tau^{-1}\delta(j)$ preserves every block of $[\alpha]$ and $\alpha \neq \alpha^*$.

For the other case, where $\tau^{-1}\delta$ preserves every block of $[\alpha]$, we will show that the only possible way to realize $(\tau^{-1}\delta, \mathcal{D}) \sim (\mathcal{T})$ is that $\mathcal{D} = \mathcal{T}$.

Since $(\tau^{-1}\delta)[\alpha] = [\alpha]$ by assumption, and $(\tau^{-1}\delta) \circ h_\pi = h_{(\tau^{-1}\delta)\pi}$ for any set partition π , and we know (3.2), we obtain that

$$[\mathcal{A}_{ii}]_\alpha \subsetneq (\tau^{-1}\delta)[\mathcal{A}_{i+1i}]_\gamma \subseteq [\mathcal{A}_{ii}]_\alpha \cup [\mathcal{A}_{i+1i+1}]_\alpha = [\mathcal{A}_{i+1i}]_\gamma \cup [\mathcal{A}_{i+1i}]_\gamma,$$

by (3.1). Consequently, because we know that $(\tau^{-1}\delta)[\mathcal{A}_{i+1i}]_\gamma$ and $(\tau^{-1}\delta)[\mathcal{A}_{i+1i}]_\gamma$ are blocks of $[\nu]$ they must each be a set of consecutive integers, and so $(\tau^{-1}\delta)[\mathcal{A}_{i+1i}]_\gamma = [\mathcal{A}_{i+1i}]_\gamma$ and $(\tau^{-1}\delta)[\mathcal{A}_{i+1i}]_\gamma = [\mathcal{A}_{i+1i}]_\gamma$. We also have that

$$(\tau^{-1}\delta)[\mathcal{A}_{jj}]_\gamma = (\tau^{-1}\delta)[\mathcal{A}_{jj}]_\alpha = [\mathcal{A}_{jj}]_\alpha = [\mathcal{A}_{jj}]_\gamma$$

for all $j \notin \{i, i+1\}$ since $[\alpha]$ and $[\gamma]$ have common blocks $[\mathcal{A}_{jj}]_\alpha = [\mathcal{A}_{jj}]_\gamma$ for all $j \notin \{i, i+1\}$, and so we have established that $\tau^{-1}\delta$ preserves every block of $[\gamma]$, that is, $(\tau^{-1}\delta)[\gamma] = [\gamma]$.

Now if $(\tau^{-1}\delta, \mathcal{D}) \sim (\mathcal{T})$, then $s_{(\tau^{-1}\delta, \mathcal{D})} = s_{(\text{id}, \mathcal{T})}$ and so

$$(3.3) \quad s_{\mathcal{D}} = \rho(s_{(\tau^{-1}\delta, \mathcal{D})}) = \rho(s_{(\text{id}, \mathcal{T})}) = s_{\mathcal{T}}$$

according to Lemma 2.17. We will now keep (3.3) in mind to employ both the commutative and noncommutative settings at once, in order to obtain our result for noncommuting variables. This we will do by determining the subscripts on the main diagonal and subdiagonal entries of the Jacobi-Trudi matrix $JT(\mathcal{T})$.

First we determine the subscripts on the main diagonal entries of the Jacobi-Trudi matrix $JT(\mathcal{T})$. In view of Lemma 2.18, the 1- and 2-row overlap partitions of \mathcal{D} and \mathcal{T} are the same, that is, the *set* of subscripts on the main diagonal and subdiagonal entries of $JT(\mathcal{D})$ are the same as the ones of $JT(\mathcal{T})$, respectively. By (2) (a)–(b) of Lemma 2.10, only the identity permutation, id , gives rise to the term with subscripts \mathcal{A}_{jj} for $1 \leq j \leq \ell$ in $s_{[\mathcal{D}]}$ and $s_{[\mathcal{T}]}$ respectively. As a result, it follows from $(\tau^{-1}\delta)[\alpha] = [\alpha]$ that the subscript of the (j, j) th entry of $JT(\mathcal{T})$ must be \mathcal{A}_{jj} . Thus, we conclude that the subscripts on the main diagonal entries of the Jacobi-Trudi matrix $JT(\mathcal{T})$ are identical to those of the Jacobi-Trudi matrix $JT(\mathcal{D})$.

Now we turn our attention to the subscripts on the subdiagonal entries of the Jacobi-Trudi matrix $JT(\mathcal{T})$. Because we verified earlier that $(\tau^{-1}\delta)[\gamma] = [\gamma]$, the (weak) composition $\gamma = (\mathcal{A}_{11}, \dots, \mathcal{A}_{i+1, i}, \mathcal{A}_{i+1, i}, \dots, \mathcal{A}_{\ell\ell})$ appears as the subscript of a complete homogeneous symmetric function in the determinantal expansion of $JT(\mathcal{T})$. Since the \mathcal{A}_{jj} are the subscripts of the (j, j) th entries on the main diagonal of $JT(\mathcal{T})$, by above, and

$$\mathcal{A}_{i+1, i} > \mathcal{A}_{ii} > \mathcal{A}_{i+1, i},$$

the subscript of the $(i+1, i)$ th entry of $JT(\mathcal{T})$ must be $\mathcal{A}_{i+1, i}$ by Lemma 2.9.

Because this is true for all $1 \leq i \leq \ell - 1$, we obtain that the subscripts on the subdiagonal entries of the Jacobi-Trudi matrix $JT(\mathcal{T})$ are identical to those of the Jacobi-Trudi matrix $JT(\mathcal{D})$. Thus,

$$JT(\mathcal{T}) = JT(\mathcal{D})$$

because the subscripts on the main diagonal and subdiagonal entries of any Jacobi-Trudi matrix determine the remaining subscripts of the remaining entries by the definition of the Jacobi-Trudi matrix. This leads to $\mathcal{D} = \mathcal{T}$ and $\tau^{-1}\delta$ preserves every block of $w(\alpha)$ for any $w \in S_{\ell}$, by definition. This completes the case where $\tau^{-1}\delta$ preserves every block of $[\alpha]$.

Consequently, putting both these cases together, if $\mathcal{D} \neq \mathcal{T}$, then \mathcal{D} must be a ribbon and $\mathcal{T} = \mathcal{D}^* \neq \mathcal{D}$. This completes the proof of the other direction, and we are done. \square

REFERENCES

- [1] F. Aliniaefard, S. Li and S. van Willigenburg, Schur functions in noncommuting variables, *Adv. Math.* **406** (2022) 37pp.
- [2] F. Aliniaefard, V. Wang and S. van Willigenburg, Extended chromatic symmetric functions and equality of ribbon Schur functions, *Adv. in Appl. Math.* **128** (2021) 29pp.

- [3] J. Aliste-Prieto, L. Crew, S. Spirkl and J. Zamora, A vertex-weighted Tutte symmetric function, and constructing graphs with equal chromatic symmetric function, *Electron. J. Combin.* **28** (2021) 33pp.
- [4] J. Aliste-Prieto and J. Zamora, Proper caterpillars are distinguished by their chromatic symmetric function, *Discrete Math.* **315** (2014) 158–164.
- [5] A. Bergeron-Brleck, Words and noncommutative invariants of finite groups, PhD thesis, York University, 2010.
- [6] L. Billera, H. Thomas and S. van Willigenburg, Decomposable compositions, symmetric quasisymmetric functions and equality of ribbon Schur functions, *Adv. Math.* **204** (2006) 204–240.
- [7] A. Borodin, P. Diaconis and J. Fulman, On adding a list of numbers (and other one-dependent determinantal processes), *Bull. Amer. Math. Soc.* **47** (2010) 639–670.
- [8] C. Gutschwager, Equality of multiplicity free skew characters, *J. Algebraic Combin.* **30** (2009) 215–232.
- [9] E.-Y. Jin and S.-X. Li, Towards equivalent thickened ribbon Schur functions, [arXiv:2403.01843v1](https://arxiv.org/abs/2403.01843v1)
- [10] P. McNamara and R. Ward, Equality of P -partition generating functions, *Ann. Comb.* **18** (2014) 489–514.
- [11] P. McNamara and S. van Willigenburg, Towards a combinatorial classification of skew Schur functions, *Trans. Amer. Math. Soc.* **361** (2009) 4437–4470.
- [12] V. Reiner, K. Shaw and S. van Willigenburg, Coincidences among skew Schur functions, *Adv. Math.* **216** (2007) 118–152.
- [13] M. Rosas and B. Sagan, Symmetric functions in noncommuting variables, *Trans. Amer. Math. Soc.* **358** (2004) 215–232.
- [14] I. Schur, *Über eine Klasse von Matrizen, die sich einer gegebenen Matrix zuordnen lassen*. Dissertation, Berlin, 1901. In: I. Schur, *Gesammelte Abhandlungen I*, Springer, Berlin, (1973) 1–70.
- [15] S. van Willigenburg, Equality of Schur and skew Schur functions, *Ann. Comb.* **9** (2005) 355–362.
- [16] K. Yeats, A Hopf algebraic approach to Schur function identities, *Electron. J. Combin.* **24** (2017) 16pp.

SCHOOL OF MATHEMATICAL SCIENCES, XIAMEN UNIVERSITY, XIAMEN 361005, CHINA

Email address: yjin@xmu.edu.cn

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRITISH COLUMBIA, VANCOUVER, BRITISH COLUMBIA V6T 1Z2, CANADA

Email address: steph@math.ubc.ca