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Resolving Stanley's e-positivity of claw-contractible-free graphs

Abstract. In Stanley's seminal 1995 paper on the chromatic symmetric function, he stated that there was no known graph that was not contractible to the claw and whose chromatic symmetric function was not e-positive, that is, not a positive linear combination of elementary symmetric functions. We resolve this by giving infinite families of graphs that are not contractible to the claw and whose chromatic symmetric functions are not e-positive. Moreover, one such family is additionally claw-free, thus establishing that the e-positivity of chromatic symmetric functions is in general not dependent on the existence of an induced claw or of a contraction to a claw.

Keywords. Chromatic symmetric function, claw-free, claw-contractible, elementary symmetric function

1. Introduction

The chromatic polynomial is a classical graph invariant dating back to Birkhoff [4], while symmetric functions date back even further to Cauchy [9]. Both the chromatic polynomial and symmetric functions continue to see wide application in diverse areas such as algebraic geometry [17, 30] and quantum physics [10, 20]. In 1995 Stanley [31] introduced for a finite simple graph G a symmetric function generalization of the chromatic polynomial of G, known as the chromatic symmetric function, X_G. Stanley [32] further defined symmetric function generalizations of the Tutte polynomial of a graph (in the guise of the bad colouring polynomial) and the chromatic polynomial of a hypergraph. Noble and Welsh [24] introduced the U-polynomial as a symmetric function generalization of the Tutte polynomial different from Stanley's, but showed them to be equivalent. Some time before Stanley's chromatic symmetric function was conceived, Brylawski had introduced the polychromate [8] as a multivariate generalization of the Tutte polynomial;

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this was later shown by Sarmiento [29] to be equivalent to the U-polynomial and hence to Stanley's symmetric function generalization of the Tutte polynomial too.

Regarding applications, the chromatic symmetric function has been applied to scheduling problems via a generalization of Breuer and Klivans [5], while the symmetric bad colouring polynomial arose in the study of the Potts model by Klazar et al. [20]. The chromatic symmetric function has also been shown to distinguish various non-isomorphic trees, for example in [1, 23], which explore Stanley's fundamental question of whether X_G distinguishes non-isomorphic trees [31, p. 170]. Although X_G does not satisfy a deletion-contraction recurrence, Orellana and Scott [25] established a three-term deletion recurrence for X_G when G has a triangle. This was generalized by the first and third authors [13] to a k-term deletion recurrence for X_G when G has a k-cycle.

Other quasisymmetric function generalizations of Stanley's chromatic symmetric function have been defined by Humpert [18] and Shareshian and Wachs [30], the latter having been further studied, for example in [2, 6, 11]. The generalization defined by Shareshian and Wachs [30] was in part motivated by Stanley's conjecture [31, Conjecture 5.1] that if a poset is $(\mathbf{3} + \mathbf{1})$-free, then the chromatic symmetric function of its incomparability graph is e-positive, that is, a non-negative linear combination of elementary symmetric functions. Stanley observes that this conjecture is equivalent to the Stanley–Stembridge poset chain conjecture [34, Conjecture 5.5]. The incomparability graph of a $(\mathbf{3}+\mathbf{1})$-free poset is claw-free, that is, does not contain $K_{1,3}$ as an induced subgraph. After formulating his conjecture, Stanley asks whether it could not be widened from incomparability graphs to a larger class of graphs, and is led to venture that it may hold for graphs not contractible to a claw (allowing removal of any parallel edges). Stanley establishes the truth of this wider conjecture for paths and cycles [31, Propositions 5.3 and 5.4].

Gebhard and Sagan [16] proved a number of special cases of the conjecture via generalizing the chromatic symmetric function from commuting variables to non-commuting variables. Gasharov [15] moved closer to the conjecture by proving the weaker statement that the chromatic symmetric function of the incomparability graph of a $(\mathbf{3}+\mathbf{1})$-free poset is s-positive, that is, a non-negative linear combination of Schur functions. Gasharov's work has led to its own avenue of research, for example in [19, 35].

In this paper we produce infinitely many graphs not contractible to a claw (even regarding multiple edges of a contraction as a single edge) but which are not e-positive, thereby resolving Stanley's problem,

> We don't know of a graph which is not contractible to K_{13} (even regarding multiple edges of a contraction as a single edge) which is not e-positive

in the negative. The original conjecture for incomparability graphs of $(\mathbf{3} + \mathbf{1})$-free posets remains open, as does Stanley's question as to precisely which graphs have e-positive chromatic symmetric function.

Our paper is structured as follows. In Section 2 we recall relevant concepts that we will require later, and give the four graphs with the fewest number of vertices that are not contractible to the claw and whose chromatic symmetric functions are not e-positive in Figure 2. Section 3 exhibits eight infinite families of graphs that are distinguished by whether or not they are claw-free, are not contractible to the claw, or have a chromatic

Table 1. Infinite graph families that satisfy every combination of claw-free, claw-contractible-free and *e*-positive. Definitions for specific graphs can be found in Sections 3, 4 and 5.

Claw-free	Claw-contractible-free	*e*-Positive	Example graph family
yes	yes	yes	path, cycle, complete
yes	yes	no	triangular tower $TT_{n,n,n}, n \geq 3$
yes	no	yes	generalized bull
yes	no	no	generalized net
no	yes	yes	tadpole $T_{a,b}, a \geq 4$
no	yes	no	saltire $SA_{n,n}$, augmented
			saltire $AS_{n,n}, AS_{n,n+1}, n \geq 3$
no	no	yes	spider $S(n, n-1, 1), n \geq 2$
no	no	no	star

symmetric function that is *e*-positive. This result is Theorem 3.2 and thus establishes that the *e*-positivity of the chromatic symmetric function of a graph is independent of whether it is claw-free or claw-contractible-free. A summary of this theorem can be found in Table 1. Then in Section 4 we generalize two of the graphs from Section 2 into two infinite families of graphs. First is the family of saltire graphs, $SA_{a,b}$, where $a, b \geq 2$, which generalizes the graph from Section 2 with the fewest edges. We prove that these graphs are not contractible to the claw in Lemma 4.1, and additionally for $n \geq 3$ we prove that $X_{SA_{n,n}}$ is not *e*-positive in Lemma 4.4.

Having discovered a family of graphs with an even number of vertices that resolves Stanley's problem we then extend our results to graphs with any number of vertices via the second family of augmented saltire graphs, $AS_{a,b}$, where $a \geq 2, b \geq 3$, which we also prove are not contractible to the claw in Lemma 4.6. For $n \geq 3$ we further prove that $X_{AS_{n,n}}$ and $X_{AS_{n,n+1}}$ are not *e*-positive in Lemma 4.9.

Finally, in Section 5 we introduce the family of triangular tower graphs, $TT_{a,b,c}$, where $a, b, c \geq 2$, and prove in Lemma 5.1 that they are claw-free and do not contract to the claw, and for $n \geq 3$ prove in Lemma 5.4 that $X_{TT_{n,n,n}}$ is not *e*-positive. This last example shows that the chromatic symmetric function of a graph need not be *e*-positive for graphs that have neither an induced claw nor a claw obtained by contraction of edges (allowing removal of parallel edges). This final family allows us to establish that there exists an infinite family of graphs that satisfies every combination of claw-free, not being contractible to the claw and whose chromatic symmetric function is *e*-positive. In all cases we see that classical techniques suffice to yield our proofs, though a number of technical lemmas such as Lemma 5.2 are required to yield the final results.

2. Background

We begin by recalling some necessary combinatorial, algebraic and graph-theoretic results that will be useful later. A *partition* $\lambda = (\lambda_1, \ldots, \lambda_{\ell(\lambda)})$ of N, denoted by $\lambda \vdash N$, is a list of positive integers whose *parts* λ_i satisfy $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\ell(\lambda)} > 0$ and $\sum_{i=1}^{\ell(\lambda)} \lambda_i = N$. If λ has exactly m_i parts equal to i for $1 \leq i \leq N$ we often denote λ by $\lambda = (1^{m_1}, 2^{m_2}, \ldots, N^{m_N})$.

The algebra of symmetric functions is a subalgebra of $\mathbb{Q}[[x_1, x_2, \ldots]]$ that can be defined as follows. The *i-th elementary symmetric function* e_i for $i \geq 1$ is given by

$$e_i = \sum_{j_1 < \cdots < j_i} x_{j_1} \cdots x_{j_i},$$

and given a partition $\lambda = (\lambda_1, \ldots, \lambda_{\ell(\lambda)})$ the *elementary symmetric function* e_λ is given by

$$e_\lambda = e_{\lambda_1} \cdots e_{\lambda_{\ell(\lambda)}}.$$

The *algebra of symmetric functions*, Λ, is then the graded algebra

$$\Lambda = \Lambda^0 \oplus \Lambda^1 \oplus \cdots$$

where $\Lambda^0 = \text{span}\{1\} = \mathbb{Q}$ and for $N \geq 1$,

$$\Lambda^N = \text{span}\{e_\lambda \mid \lambda \vdash N\}.$$

Moreover, the elementary symmetric functions form a basis for Λ and if a symmetric function can be written as a non-negative linear combination of elementary symmetric functions, then we say it is *e-positive*.

However, while the basis of elementary symmetric functions is central to the problem we wish to resolve, it is another basis, the basis of power sum symmetric functions, which will be central to our proofs. In terms of elementary symmetric functions the *i-th power sum symmetric function* p_i for $i \geq 1$ is given by

$$p_i = \sum_{\mu = (1^{m_1}, 2^{m_2}, \ldots, i^{m_i}) \vdash i} (-1)^{i - \ell(\mu)} \frac{i(\ell(\mu) - 1)!}{\prod_{j=1}^{i} m_j!} e_\mu, \tag{2.1}$$

and given a partition $\lambda = (\lambda_1, \ldots, \lambda_{\ell(\lambda)})$ the *power sum symmetric function* p_λ is given by

$$p_\lambda = p_{\lambda_1} \cdots p_{\lambda_{\ell(\lambda)}}.$$

This particular basis, of power sum symmetric functions, will be useful later, as will the following. Given two partitions $\lambda, \mu \vdash N$ we write $\lambda \succeq_p \mu$ if the parts of λ are obtained by summing (not necessarily adjacent) parts of μ. For example, $(5, 4, 2) \succeq_p (3, 3, 2, 2, 1)$ since $5 = 3 + 2$, $4 = 3 + 1$ and $2 = 2$. Therefore by (2.1) we get the following key observation.

Observation 2.1. When calculating the coefficient of e_μ in a symmetric function written in the basis of power sum symmetric functions, we need only focus on those p_λ where $\lambda \succeq_p \mu$.

Since we will often want to compute the coefficient of a symmetric function $f \in \Lambda$ when written in the basis $\{b_\lambda\}_{\lambda \vdash N \geq 1}$, we will denote this by $[b_\lambda]f$. More details on these classical symmetric functions can be found in [22, 27, 33], but for now we turn our attention to a more recent symmetric function, the chromatic symmetric function.

The chromatic symmetric function is reliant on a graph that is *finite* and *simple* and from now on we will assume that all our graphs satisfy these properties. This function is also reliant on all proper colourings of a graph. More precisely, given a graph G with vertex set V, a *proper colouring* κ of G is a function

$$\kappa : V \to \{1, 2, \ldots\}$$

such that if $u, v \in V$ are adjacent, then $\kappa(u) \neq \kappa(v)$. With this in mind we can now define the chromatic symmetric function, which we do in two ways before giving an example in Example 2.4.

Definition 2.2 ([31, Definition 2.1]). For a graph G with vertex set $V = \{v_1, \ldots, v_N\}$ and edge set E, the *chromatic symmetric function* is defined to be

$$X_G = \sum_{\kappa} x_{\kappa(v_1)} \cdots x_{\kappa(v_N)}$$

where the sum is over all proper colourings κ of G.

For brevity, we say a graph G is *e-positive* when X_G is *e*-positive, and *not e-positive* otherwise.

A more useful realisation of the chromatic symmetric function for us will be the following lemma, also due to Stanley, which requires some more notation. Given a graph G with vertex set $V = \{v_1, \ldots, v_N\}$, edge set E, and a subset $S \subseteq E$, let $\lambda(S)$ be the partition of N whose parts are equal to the number of vertices in the connected components of the spanning subgraph of G with vertex set V and edge set S. If the number of vertices in a connected component is λ_i, then for succinctness we may refer to the connected component as a *piece of size λ_i*.

Lemma 2.3 ([31, Theorem 2.5]). *For a graph G with vertex set V and edge set E we have*

$$X_G = \sum_{S \subseteq E} (-1)^{|S|} p_{\lambda(S)}.$$

Example 2.4. The *claw* (also known as $K_{1,3}$ or, as in Stanley's quote, K_{13}) shown in Figure 1 has chromatic symmetric function

$$p_{(1^4)} - 3p_{(2,1^2)} + 3p_{(3,1)} - p_{(4)} = e_{(2,1^2)} - 2e_{(2,2)} + 5e_{(3,1)} + 4e_{(4)}.$$
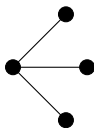
Thus, the claw is not *e*-positive.

Fig. 1. The claw.

While the focus of our paper will be on the claw, a number of well known graphs will also play a role: the *complete graph*, K_N, $N \geq 1$; the *N-path*, P_N, $N \geq 1$; and the *N-cycle*, C_N, $N \geq 3$, and we set $C_i = K_i$ for $i = 1, 2$. Two particular claw-related properties of a graph will also be much in demand, the property of being claw-free and of being claw-contractible-free.

For the former recall that an *induced* subgraph of a graph G is a subgraph consisting of a subset of its vertices together with all edges whose endpoints both lie in the subset, and an *edge induced* subgraph of G is a subgraph consisting of a subset of its edges together with all vertices at the endpoints of some edge in the subset.

We say a graph is *claw-free* if it does not have the claw as an induced subgraph. Claw-free graphs have the following characterization due to Beineke [3, Theorem] who combined the results of Krausz [21, Theorem 1] and van Rooij and Wilf [26, Theorem 4].

Lemma 2.5 ([3, Theorem]). *A graph G is claw-free if there exists a partition of the edges of G into disjoint sets such that every set edge-induces a complete subgraph of G and no vertex of G belongs to more than two of these complete subgraphs.*

For the latter, let G be a graph with vertex set V and edge set E. Recall a subset of V is *independent* if no two vertices in the subset are adjacent, plus when we delete a vertex v from G we delete v and all edges incident to v. Furthermore, if $S \subseteq E$, then we denote by G/S the graph G with the edges in S contracted and the vertices at either end identified. We say that G *contracts to the claw* if there exists $S \subseteq E$ such that G/S yields the claw once multiple edges are replaced by single edges, and G is *claw-contractible-free* if G does not contract to the claw.

As with being claw-free, an elegant characterization exists for a graph to be claw-contractible-free. It depends on deleting independent sets of vertices, and is a special case of a theorem by Brouwer and Veldman.

Lemma 2.6 ([7, Theorem 3]). *A graph G is claw-contractible-free if and only if the deletion of every set of three independent vertices from G results in a disconnected graph.*

We now turn our attention to the graphs with the fewest number of vertices that are claw-contractible-free and not *e*-positive. Since a disconnected graph cannot contract to the claw, we restrict ourselves to *connected* graphs in order to produce a meaningful resolution to Stanley's problem.

Otherwise, by [31, Proposition 2.3], which says that for disjoint graphs G, H we have

$$X_{G \cup H} = X_G X_H, \tag{2.2}$$

calculating $X_{K_1} = e_1$, and using Example 2.4, we can conclude that the disjoint union of the claw and K_1 is a trivial resolution to Stanley's problem.

Using an exhaustive computational search, one can find that the connected graphs with the fewest number of vertices, N, that are claw-contractible-free and not *e*-positive occur at $N = 6$ and are given in Figure 2.
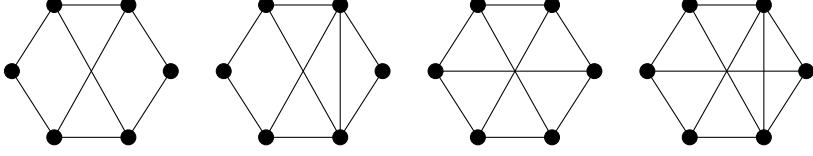
Fig. 2. From left to right: the graphs $SA_{3,3}$, $AS_{3,3}$, $K_{3,3}$, $AK_{3,3}$.

In particular, their chromatic symmetric functions are

$$
\begin{aligned}
X_{SA_{3,3}} &= 2e_{(2,2,2)} - 6e_{(3,3)} + 26e_{(4,2)} + 28e_{(5,1)} + 102e_{(6)} \\
X_{AS_{3,3}} &= 2e_{(3,2,1)} - 6e_{(3,3)} + 24e_{(4,2)} + 40e_{(5,1)} + 120e_{(6)} \\
X_{K_{3,3}} &= 2e_{(2,2,2)} - 12e_{(3,3)} + 30e_{(4,2)} + 24e_{(5,1)} + 186e_{(6)} \\
X_{AK_{3,3}} &= 2e_{(3,2,1)} - 6e_{(3,3)} + 20e_{(4,2)} + 32e_{(5,1)} + 228e_{(6)}.
\end{aligned}
$$

As we will see in Section 4, the first two graphs each naturally give rise to infinite families of graphs that are claw-contractible-free, are not *e*-positive, and moreover we can explicitly identify and calculate a negative coefficient. However, we will first establish that *e*-positivity does not depend on the graph being either claw-free or claw-contractible-free.

3. Independence of *e*-positivity and the claw

In order to prove that the *e*-positivity of a graph does not, in general, depend on the graph being claw-free or claw-contractible-free, we need to establish that there exists an infinite family of graphs that satisfies every combination of claw-free, claw-contractible-free, and *e*-positive. We achieve this in Theorem 3.2, in which we see that some of our families are also those that will resolve Stanley's problem later, and some are well-known families that we have already defined. The remaining families we define now, with an accompanying figure immediately following in Example 3.1.

The *generalized bull graph* $B_{a,b}$, where $a, b \geq 2$, is the graph on $a + b + 1$ vertices created as follows. Consider K_3 with vertices u, v, w, and two paths P_a and P_b. Then $B_{a,b}$ is created by identifying a leaf of P_a with u and identifying a leaf of P_b with v. Similarly the *generalized net graph* GN_a, where $a \geq 3$, is the graph on $a + 3$ vertices created as follows. Consider K_a and three copies of P_2. Then GN_a is created by identifying a leaf of each P_2 with a distinct vertex of K_a. The *tadpole graph* $T_{a,b}$, where $a \geq 3$, $b \geq 2$, is the graph on $a + b - 1$ vertices created as follows. Consider C_a and P_b. Then $T_{a,b}$ is created by identifying a leaf of P_b with a vertex, denoted by v, of C_a.

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_{\ell(\lambda)}) \vdash (N - 1)$, where $\ell(\lambda) \geq 3$, the *spider graph* $S(\lambda_1, \ldots, \lambda_{\ell(\lambda)})$ is the graph on N vertices consisting of $P_{\lambda_1}, \ldots, P_{\lambda_{\ell(\lambda)}}$ and a single vertex v that is joined to a leaf in each P_{λ_i} for $1 \leq i \leq \ell(\lambda)$. Lastly, the *star graph* S_N, where $N \geq 4$, is the spider graph $S(1^{N-1})$.

Example 3.1. Here are small examples for each of the types of graphs defined above. Note that S_4 is the claw.

$B_{2,2}$ GN_4 $T_{4,2}$ $S(2, 1, 1)$
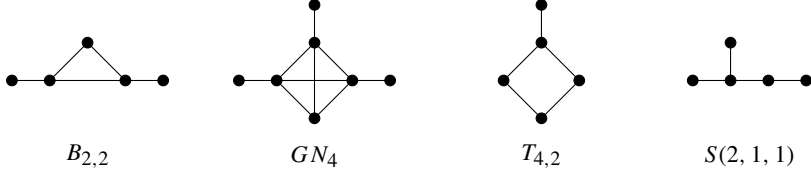
Theorem 3.2. *There exists an infinite family of graphs that satisfies every combination of claw-free, claw-contractible-free, and e-positive.*

Proof. We will provide infinite families of graphs that satisfy all eight combinations of claw-free, claw-contractible-free, and *e*-positive. A summary is provided in Table 1.

Claw-free, claw-contractible-free, e-positive: The complete graphs K_N for $N \geq 1$, the N-paths P_N for $N \geq 1$, and the N-cycles C_N for $N \geq 1$ are straightforwardly claw-free and claw-contractible-free. The graphs P_N and C_N were proved explicitly to be *e*-positive by Stanley [31, Propositions 5.3 and 5.4], and K_N is *e*-positive since $X_{K_N} = N! e_N$.

Claw-free, claw-contractible-free, not e-positive: The triangular tower graphs $TT_{n,n,n}$ for $n \geq 3$ defined in Section 5 and illustrated in Figure 5 are proved to satisfy these three properties in Theorem 5.5.

Claw-free, not claw-contractible-free, e-positive: The generalized bull graphs $B_{a,b}$ for $a, b \geq 2$ are claw-free by Lemma 2.5 because of the following edge partition: The edges in the K_3 form one set, and every remaining edge forms its own set. These graphs contract to the claw by Lemma 2.6 because we can delete three independent vertices (the vertex w, and the two degree 1 vertices) resulting in a connected graph. The generalized bull graphs are *e*-positive as they are a special case of a corollary by Gebhard and Sagan [16, Corollary 7.7].

Claw-free, not claw-contractible-free, not e-positive: The generalized net graphs GN_a for $a \geq 3$ are claw-free by Lemma 2.5 because of the following edge partition: The edges in the K_a form one set, and each of the three remaining edges forms its own set. These graphs contract to the claw by Lemma 2.6 because we can delete the three independent degree 1 vertices resulting in a connected graph, and were proved to be not *e*-positive by Foley et al. [14, Theorem 1].

Not claw-free, claw-contractible-free, e-positive: Consider the tadpole graphs $T_{a,b}$ for $a \geq 4$, $b \geq 2$, that is, all tadpole graphs where $a \neq 3$. These graphs are not claw-free due to the induced claw consisting of v and the three vertices adjacent to it, since $a \neq 3$. They are also claw-contractible-free by Lemma 2.6, because if we delete three independent vertices then at least two vertices belong to the copy of C_a or P_b, resulting in a disconnected graph. All tadpole graphs are *e*-positive by combining the results of Gebhard and Sagan [16, Proposition 6.8] and [16, Theorem 7.6] repeatedly.

Not claw-free, claw-contractible-free, not e-positive: The saltire graphs $SA_{n,n}$ for $n \geq 3$ and augmented saltire graphs $AS_{n,n}$ and $AS_{n,n+1}$ for $n \geq 3$ defined in Section 4 and illustrated in Figures 3 and 4 are proved to be claw-contractible-free and not *e*-positive in Theorems 4.5 and 4.10. These graphs are not claw-free due to the induced claw consisting of v_2 and the three vertices adjacent to it.

Not claw-free, not claw-contractible-free, e-positive: The spider graphs $S(n, n-1, 1)$ for $n \geq 2$ satisfy these properties. Firstly, they are not claw-free due to the induced claw consisting of v and the three vertices adjacent to it. Secondly, they contract to the claw by Lemma 2.6 since deleting the three vertices of degree 1, which are independent, results in a connected graph. Now we will prove these graphs are *e*-positive, and for this we need to recall the following result of Orellana and Scott [25, Theorem 3.1]: If G is a graph with edge set E such that $\epsilon_1, \epsilon_2, \epsilon_3 \in E$ form an edge induced K_3, then

$$X_G = X_{G-\{\epsilon_1\}} + X_{G-\{\epsilon_2\}} - X_{G-\{\epsilon_1, \epsilon_2\}} \tag{3.1}$$

where $G - S$ for some $S \subseteq E$ is the graph with vertex set V and edge set $E - S$.

Consider the generalized bull graph $B_{n,n}$ for $n \geq 2$. Let $\epsilon_1, \epsilon_2, \epsilon_3$ be the three edges in its K_3 such that ϵ_1, ϵ_2 are incident to w and ϵ_3 is the edge uv. By (2.2) and (3.1) we have

$$X_{B_{n,n}} = 2X_{S(n,n-1,1)} - X_{P_{2n}} X_{K_1},$$

so $X_{S(n,n-1,1)} = \frac{1}{2}(X_{B_{n,n}} + X_{P_{2n}} X_{K_1})$. We know from the above that $B_{n,n}$, P_{2n} and K_1 are all *e*-positive, and hence so is $S(n, n-1, 1)$.

Not claw-free, not claw-contractible-free, not e-positive: The star graphs S_N for $N \geq 4$ are not claw-free due to the induced claw consisting of v and any three vertices adjacent to it. These graphs contract to the claw by Lemma 2.6 since deleting any three vertices of degree 1, which are independent, results in a connected graph. Lastly, these graphs were shown to be not *e*-positive by Dahlberg, She and van Willigenburg [12, Example 11]. □

4. Saltire and augmented saltire graphs

In this section we begin by introducing our first infinite family of graphs to resolve Stanley's problem. In particular, this family includes the graph with the fewest number of vertices and edges, as verified by computer, which is claw-contractible-free and yet whose chromatic symmetric function is not *e*-positive.

The *saltire graph* $SA_{a,b}$, where $a, b \geq 2$, is the graph on $a+b$ vertices $\{v_1, \ldots, v_{a+b}\}$ with edges $v_i v_{i+1}$ for $1 \leq i \leq a+b-1$, $v_{a+b}v_1$, $v_1 v_{a+1}$ and $v_2 v_{a+2}$. For example, $SA_{2,2} = K_4$, and $SA_{3,3}$ and a graphical representation of a generic $SA_{a,b}$ are given in Figure 3.
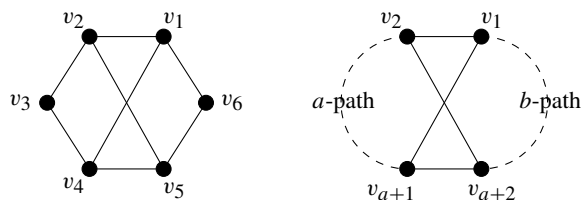

Fig. 3. Left: $SA_{3,3}$. Right: a generic $SA_{a,b}$.

From the graphical representation we see that the edges of $SA_{a,b}$ can be naturally partitioned into three parts as follows. Given $SA_{a,b}$ we refer to the subgraph induced by the edges

$$\{v_i v_{i+1} \mid 2 \leq i \leq a\}$$

as the *a-path*, to the subgraph induced by the edges

$$\{v_i v_{i+1} \mid a+2 \leq i \leq a+b-1\} \cup \{v_{a+b}v_1\}$$

as the *b-path*, and to the subgraph induced by the edges

$$\{v_1 v_2, \, v_1 v_{a+1}, \, v_2 v_{a+2}, \, v_{a+1} v_{a+2}\}$$

as the *middle*. Furthermore, when considering $SA_{n,n}$ we refer to the *a*-path where $a = n$ as the *left n-path*, and to the *b*-path where $b = n$ as the *right n-path*, to distinguish them. With these definitions in hand, we come to our first result on saltire graphs.

Lemma 4.1. *For all $a, b \geq 2$ the graph $SA_{a,b}$ is claw-contractible-free. In particular, for $n \geq 3$ the graph $SA_{n,n}$ is claw-contractible-free.*

Proof. By Lemma 2.6 it suffices to show that the deletion of any three independent vertices from $SA_{a,b}$ results in a disconnected graph. The pigeonhole principle guarantees that at least two of these independent vertices will belong to either the *a*-path or the *b*-path, and we can see from Figure 3 that the removal of any two non-adjacent vertices from the *a*-path or the *b*-path results in a disconnected graph. □
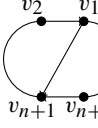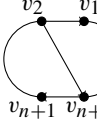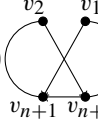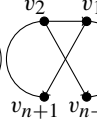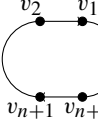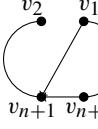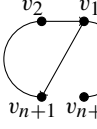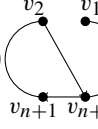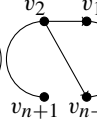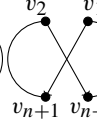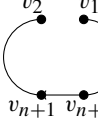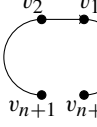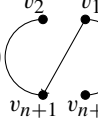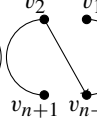
Now that we have proved that $SA_{a,b}$ is claw-contractible-free, we will restrict our attention to $SA_{n,n}$ where $n \geq 3$, and prove that its chromatic symmetric function is not *e*-positive by calculating the coefficient $[e_{(n^2)}]X_{SA_{n,n}}$. Note that since $(2n)$ and (n^2) are the only partitions $\lambda \vdash 2n$ satisfying $\lambda \succeq_p (n^2)$ by Observation 2.1 and Lemma 2.3, in order to calculate $[e_{(n^2)}]X_{SA_{n,n}}$ we need to calculate $[p_{(2n)}]X_{SA_{n,n}}$ and $[p_{(n^2)}]X_{SA_{n,n}}$.

Lemma 4.2. *For $n \geq 3$ we have*

(1) $[p_{(2n)}]X_{SA_{n,n}} = -3n^2 + 4n - 2$ *and*
(2) $[p_{(n^2)}]X_{SA_{n,n}} = 2n - 1.$

Proof. To prove this we will use Lemma 2.3 that involves all subsets S of the edge set E. We are only interested in the subsets S that yield $\lambda(S) = (2n)$ or (n^2), both of which have parts that are at least n. This means we will ignore any S where $\lambda(S)$ has a part smaller than n as these subsets will not affect the coefficient of $p_{(2n)}$ or $p_{(n^2)}$ in $X_{SA_{n,n}}$. Note that if S has two or more edges removed from the left *n*-path (or the right *n*-path), then $\lambda(S)$ certainly has a part smaller than n. Thus, we will only consider subsets S that have *at most one edge* removed from the left *n*-path and *at most one edge* removed from the right *n*-path. In Table 2 we illustrate all possible graphs with one through three middle edges removed from $SA_{n,n}$ since these will be central to our case analysis, consisting of five cases corresponding to the exclusion of zero to four edges from E.

Table 2. All possible $SA_{n,n}$ with one through three middle edges removed.

i	$SA_{n,n}$ with i middle edges removed
1	
2	
3	

First, consider $|S| = |E|$, so S contains all the edges in E. This gives us the term

$$(-1)^{2n+2} p_{(2n)} = p_{(2n)}.$$

Second, consider $|S| = |E| - 1$, so S has one fewer edge than E. Note that if we remove any one of the $2n + 2$ edges from $SA_{n,n}$, then our graph is still connected. This gives us the term

$$(-1)^{2n+1}(2n + 2)p_{(2n)} = -(2n + 2)p_{(2n)}.$$

Third, consider $|S| = |E| - 2$, so S has two fewer edges than E. If we exclude two edges from the middle, then we can see from Table 2 that all six possibilities result in connected graphs, so we get the term $(-1)^{2n} 6 p_{(2n)}$.

Instead we can exclude one edge from the left n-path or right n-path and the other edge from the middle. In all four ways to remove one edge from the middle, as illustrated in Table 2, we can also remove any one of the $2(n - 1)$ edges on the left n-path or right n-path and still have a connected graph. This gives us the term $(-1)^{2n} 8(n - 1)p_{(2n)}$.

Finally, we could exclude no edges from the middle, one of the $n - 1$ edges from the left n-path, and one of the $n - 1$ edges from the right n-path. Any of the $(n - 1)^2$ choices results in a connected graph. This gives us the term $(-1)^{2n}(n - 1)^2 p_{(2n)}$.

Altogether from this case we have the term

$$(n^2 + 6n - 1)p_{(2n)}.$$

Fourth, consider $|S| = |E| - 3$, where we exclude three edges from E. We can see from Table 2 that excluding any three edges from the middle leaves the graph connected. This gives the term $(-1)^{2n-1} 4 p_{(2n)}$.

If instead we remove two edges from the middle and one of the $n - 1$ edges from the left n-path, we can see from Table 2 that only four out of six possibilities do not yield

$\lambda(S)$ having a part smaller than n. In these four possibilities the graph is connected, which gives us the term $(-1)^{2n-1}4(n-1)p_{(2n)}$. We similarly get the term $(-1)^{2n-1}4(n-1)p_{(2n)}$ if we remove two edges from the middle, one from the right n-path, and have all parts being at least n.

Lastly, if we remove one edge from the middle, any one of the $n-1$ edges from the right n-path, and any one of the $n-1$ edges from the left n-path, then we can see from Table 2 that the graph is still connected. This gives us the term $(-1)^{2n-1}4(n-1)^2 p_{(2n)}$.

Altogether from this case we get the term

$$-4n^2 p_{(2n)}.$$

Fifth and finally, consider $|S| = |E| - 4$. If we exclude all four of the edges from the middle, then this disconnects our graph into two pieces of size n. The associated term is $(-1)^{2n-2}p_{(n^2)}$.

Say we exclude three edges from the middle and one from the left n-path or right n-path. In any of these situations $\lambda(S)$ has a part smaller than n.

Instead, say that we remove two edges from the middle, one from the left n-path, and one from the right n-path. From Table 2 we can see that in only the leftmost and rightmost pictures, $\lambda(S)$ is not forced to have a part smaller than n. In the leftmost and rightmost pictures we will disconnect the graph into two pieces. For any of the $n-1$ edges on the left n-path there is exactly one choice of an edge on the right n-path so that we break the graph into two pieces of size n. This gives the term $(-1)^{2n-2}2(n-1)p_{(n^2)}$.

Altogether from this case we get the term

$$(2n-1)p_{(n^2)}.$$

Once we exclude more than four edges from E we are guaranteed that $\lambda(S)$ will have a part smaller than n. Combining everything we see that the coefficient of $p_{(2n)}$ is

$$[p_{(2n)}]X_{SA_{n,n}} = 1 - (2n+2) + (n^2 + 6n - 1) - 4n^2 = -3n^2 + 4n - 2,$$

and

$$[p_{(n^2)}]X_{SA_{n,n}} = 2n - 1$$

is the coefficient for $p_{(n^2)}$.   □

Lemma 4.3. *For $n \geq 1$ we have*

(1) $[e_{(n^2)}]p_{(2n)} = n$ *and*
(2) $[e_{(n^2)}]p_{(n^2)} = n^2$.

Proof. Using (2.1) we see that $[e_{(n^2)}]p_{(2n)} = (-1)^{2n-2}\frac{2n(2-1)!}{2!} = n$.

To prove the other coefficient note by (2.1) that $[e_n]p_n = (-1)^{n-1}\frac{n(1-1)!}{1!} = (-1)^{n-1}n$. In $p_{(n^2)} = p_n p_n$ the coefficient of $e_{(n^2)}$ is purely determined by the multiplication of the coefficients of e_n in p_n, which gives $[e_{(n^2)}]p_{(n^2)} = [e_n]p_n[e_n]p_n = (-1)^{n-1}n(-1)^{n-1}n = n^2$.   □

We now apply these lemmas to determine the e-positivity of $X_{SA_{n,n}}$ for $n \geq 3$ in one final lemma.

Lemma 4.4. *The chromatic symmetric function of $SA_{n,n}$ for $n \geq 3$ is not e-positive. In particular,*

$$[e_{(n^2)}]X_{SA_{n,n}} = -n(n-1)(n-2).$$

Proof. By Observation 2.1 we can see that $e_{(n^2)}$ has non-zero coefficient in p_λ only for $\lambda \vdash 2n$ with $\lambda \succeq_p (n^2)$. There are only two partitions $\lambda \vdash 2n$ with $\lambda \succeq_p (n^2)$, namely $(2n)$ and (n^2).

Since

$$X_{SA_{n,n}} = \sum_{\lambda \vdash 2n} [p_\lambda]X_{SA_{n,n}} p_\lambda,$$

the coefficient of $e_{(n^2)}$ in $X_{SA_{n,n}}$ only arises from the $p_{(2n)}$ and $p_{(n^2)}$ terms. In particular,

$$[e_{(n^2)}]X_{SA_{n,n}} = [e_{(n^2)}]p_{(2n)} \cdot [p_{(2n)}]X_{SA_{n,n}} + [e_{(n^2)}]p_{(n^2)} \cdot [p_{(n^2)}]X_{SA_{n,n}}.$$

Using Lemmas 4.2 and 4.3 we therefore have

$$\begin{aligned}
[e_{(n^2)}]X_{SA_{n,n}} &= [e_{(n^2)}]p_{(2n)} \cdot [p_{(2n)}]X_{SA_{n,n}} + [e_{(n^2)}]p_{(n^2)} \cdot [p_{(n^2)}]X_{SA_{n,n}} \\
&= n \cdot (-3n^2 + 4n - 2) + n^2 \cdot (2n - 1) = -n(n-1)(n-2). \qquad \square
\end{aligned}$$

We can now identify our first family of graphs that are claw-contractible-free and whose chromatic symmetric functions are not *e*-positive.

Theorem 4.5. *The graphs $SA_{n,n}$ for all $n \geq 3$ are claw-contractible-free and not e-positive.*

Proof. This follows immediately from Lemmas 4.1 and 4.4. $\qquad \square$

Since Theorem 4.5 yields an infinite family of graphs with an even number of vertices that are claw-contractible-free and not *e*-positive, a natural question to ask is whether an infinite family of graphs exists with N vertices, for all $N \geq 6$, which are claw-contractible-free and not *e*-positive. Such a family exists and is the family of *augmented* saltire graphs, which we introduce now.

The *augmented saltire graph* $AS_{a,b}$, where $a \geq 2$, $b \geq 3$, is the saltire graph $SA_{a,b}$ with the additional edge $v_1 v_{a+2}$. More precisely, $AS_{a,b}$, where $a \geq 2$, $b \geq 3$, is the graph on $a + b$ vertices $\{v_1, \ldots, v_{a+b}\}$ with edges $v_i v_{i+1}$ for $1 \leq i \leq a + b - 1$, $v_{a+b}v_1$, $v_1 v_{a+1}$, $v_2 v_{a+2}$ and $v_1 v_{a+2}$. For example, $AS_{3,3}$, $AS_{3,4}$ and a graphical representation of a generic $AS_{a,b}$ are given in Figure 4.

Using proofs very similar to those for saltire graphs, but with some additional cases generated by the edge $v_1 v_{a+2}$, we can obtain the following two lemmas.
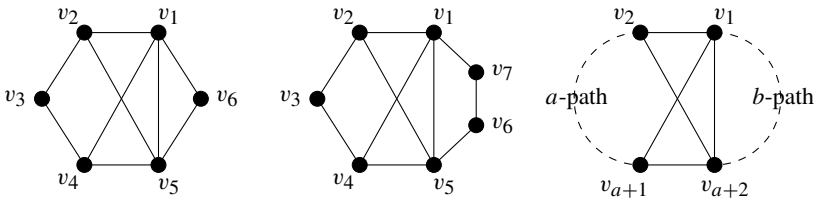

Fig. 4. From left to right: $AS_{3,3}$, $AS_{3,4}$ and a generic $AS_{a,b}$.

Lemma 4.6. *For all $a \geq 2$, $b \geq 3$ the graph $AS_{a,b}$ is claw-contractible-free. In particular, for $n \geq 3$ the graphs $AS_{n,n}$ and $AS_{n,n+1}$ are claw-contractible-free.*

Lemma 4.7. *For $n \geq 3$ we have*

(1) $[p_{(2n)}]X_{AS_{n,n}} = -4n^2 + 6n - 2$,
(2) $[p_{(n^2)}]X_{AS_{n,n}} = 3n - 3$,
(3) $[p_{(2n+1)}]X_{AS_{n,n+1}} = 4n^2 - 2n$ *and*
(4) $[p_{(n+1,n)}]X_{AS_{n,n+1}} = -7n + 4$.

We also obtain the following lemma whose proof is analogous to that of Lemma 4.3.

Lemma 4.8. *For $n \geq 1$ we have*

(1) $[e_{(n+1,n)}]p_{(2n+1)} = -(2n + 1)$ *and*
(2) $[e_{(n+1,n)}]p_{(n+1,n)} = -n(n + 1)$.

Now we are able to determine the e-positivity of $X_{AS_{n,n}}$ and $X_{AS_{n,n+1}}$ for $n \geq 3$.

Lemma 4.9. *The chromatic symmetric functions of $AS_{n,n}$ and $AS_{n,n+1}$ for $n \geq 3$ are not e-positive. In particular,*

$$[e_{(n^2)}]X_{AS_{n,n}} = [e_{(n+1,n)}]X_{AS_{n,n+1}} = -n(n - 1)(n - 2).$$

Proof. By Observation 2.1 we can see that $e_{(n^2)}$ has non-zero coefficient in p_λ only for $\lambda \vdash 2n$ with $\lambda \succeq_p (n^2)$. There are only two partitions $\lambda \vdash 2n$ with $\lambda \succeq_p (n^2)$, namely $(2n)$ and (n^2).

Since

$$X_{AS_{n,n}} = \sum_{\lambda \vdash 2n} [p_\lambda]X_{AS_{n,n}} p_\lambda,$$

the coefficient of $e_{(n^2)}$ in $X_{AS_{n,n}}$ only arises from the $p_{(2n)}$ and $p_{(n^2)}$ terms. In particular,

$$[e_{(n^2)}]X_{AS_{n,n}} = [e_{(n^2)}]p_{(2n)} \cdot [p_{(2n)}]X_{AS_{n,n}} + [e_{(n^2)}]p_{(n^2)} \cdot [p_{(n^2)}]X_{AS_{n,n}}.$$

Using Lemmas 4.7 and 4.3 we therefore have

$$\begin{aligned}
[e_{(n^2)}]X_{AS_{n,n}} &= [e_{(n^2)}]p_{(2n)} \cdot [p_{(2n)}]X_{AS_{n,n}} + [e_{(n^2)}]p_{(n^2)} \cdot [p_{(n^2)}]X_{AS_{n,n}} \\
&= n \cdot (-4n^2 + 6n - 2) + n^2 \cdot (3n - 3) = -n(n - 1)(n - 2).
\end{aligned}$$

Again by Observation 2.1 we can see that $e_{(n+1,n)}$ has non-zero coefficient in p_λ only for $\lambda \vdash 2n + 1$ with $\lambda \succeq_p (n + 1, n)$. There are only two partitions $\lambda \vdash 2n + 1$ with $\lambda \succeq_p (n + 1, n)$, namely $(2n + 1)$ and $(n + 1, n)$.

Since

$$X_{AS_{n,n+1}} = \sum_{\lambda \vdash 2n+1} [p_\lambda]X_{AS_{n,n+1}} p_\lambda,$$

the coefficient of $e_{(n+1,n)}$ in $X_{AS_{n,n+1}}$ only arises from the $p_{(2n+1)}$ and $p_{(n+1,n)}$ terms. In particular,

$[e_{(n+1,n)}]X_{AS_{n,n+1}}$
$$= [e_{(n+1,n)}]p_{(2n+1)} \cdot [p_{(2n+1)}]X_{AS_{n,n+1}} + [e_{(n+1,n)}]p_{(n+1,n)} \cdot [p_{(n+1,n)}]X_{AS_{n,n+1}}.$$

Using Lemmas 4.7 and 4.8 we therefore have

$[e_{(n+1,n)}]X_{AS_{n,n+1}}$
$$= [e_{(n+1,n)}]p_{(2n+1)} \cdot [p_{(2n+1)}]X_{AS_{n,n+1}} + [e_{(n+1,n)}]p_{(n+1,n)} \cdot [p_{(n+1,n)}]X_{AS_{n,n+1}}$$
$$= -(2n+1) \cdot (4n^2 - 2n) - n(n+1) \cdot (-7n+4)$$
$$= -n(n-1)(n-2). \qquad \square$$

We can now identify our second family of graphs that are claw-contractible-free and whose chromatic symmetric functions are not *e*-positive, and, moreover, one such graph exists with N vertices for all $N \geq 6$.

Theorem 4.10. *The graphs* $AS_{n,n}$ *and* $AS_{n,n+1}$ *for all* $n \geq 3$ *are claw-contractible-free and not e-positive.*

Proof. This follows immediately from Lemmas 4.6 and 4.9. $\qquad \square$

5. Triangular tower graphs

Our final section is devoted to answering the following question: does there exist a graph that is claw-contractible-free *and* claw-free whose chromatic symmetric function is not *e*-positive? By exhaustive computational search the smallest such example is
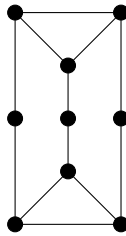
with chromatic symmetric function

$$12e_{(3,3,2,1)} - 12e_{(3,3,3)} + 102e_{(4,3,2)} + 90e_{(4,4,1)} + 18e_{(5,2,2)} + 96e_{(5,3,1)}$$
$$+ 294e_{(5,4)} + 30e_{(6,2,1)} + 180e_{(6,3)} + 342e_{(7,2)} + 294e_{(8,1)} + 666e_{(9)},$$

and moreover it yields an infinite family of graphs that are claw-contractible-free, claw-free and not *e*-positive, the triangular tower graphs.

The *triangular tower graph* $TT_{a,b,c}$, where $a, b, c \geq 2$, is the graph on $a + b + c$ vertices

$$\{v_1, \ldots, v_a\} \cup \{v_{a+1}, \ldots, v_{a+b}\} \cup \{v_{a+b+1}, \ldots, v_{a+b+c}\}$$

with edges $v_i v_{i+1}$ for

$$i \in \{1, \ldots, a - 1\} \cup \{a + 1, \ldots, a + b - 1\} \cup \{a + b + 1, \ldots, a + b + c - 1\}$$

plus $\{v_1 v_{a+1}, v_{a+1} v_{a+b+1}, v_{a+b+1} v_1\}$ and $\{v_a v_{a+b}, v_{a+b} v_{a+b+c}, v_{a+b+c} v_a\}$. Informally we can visualize $TT_{a,b,c}$ as consisting of three disjoint paths with, respectively, a, b, c vertices where we take one leaf from each path and connect them in a triangle to form an induced K_3, and do the same with the remaining three leaves. For example, $TT_{3,2,4}$, $TT_{3,3,3}$ and a graphical representation of a generic $TT_{a,b,c}$ are given in Figure 5.
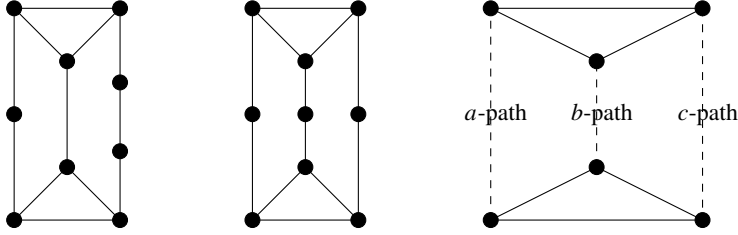

Fig. 5. From left to right: $TT_{3,2,4}$, $TT_{3,3,3}$ and a generic $TT_{a,b,c}$.

Given $TT_{a,b,c}$ we refer to the subgraph induced by the edges

$$\{v_i v_{i+1} \mid 1 \leq i \leq a - 1\}$$

as the *a-path*, to

$$\{v_i v_{i+1} \mid a + 1 \leq i \leq a + b - 1\}$$

as the *b-path*, and to

$$\{v_i v_{i+1} \mid a + b + 1 \leq i \leq a + b + c - 1\}$$

as the *c-path*. Plus we refer to $\{v_1 v_{a+1}, v_{a+1} v_{a+b+1}, v_{a+b+1} v_1\}$ as the *top triangle*, and to $\{v_a v_{a+b}, v_{a+b} v_{a+b+c}, v_{a+b+c} v_a\}$ as the *bottom triangle*.

As in the previous section, we will focus on a subset of this family, namely $TT_{n,n,n}$. In this case we refer to the *a*-path where $a = n$ as the *left n-path*, to the *b*-path where $b = n$ as the *middle n-path*, and to the *c*-path where $c = n$ as the *right n-path*. We are now ready to ascertain the containment of the claw for this new family of graphs.

Lemma 5.1. *For all $a, b, c \geq 2$ the graph $TT_{a,b,c}$ is claw-contractible-free and claw-free. In particular, for $n \geq 3$ the graph $TT_{n,n,n}$ is claw-contractible-free and claw-free.*

Proof. We first show that $TT_{a,b,c}$ is claw-free by demonstrating a partition of the edges into disjoint sets such that every set edge-induces a complete subgraph and no vertex belongs to more than two of the subgraphs. The result will then follow by Lemma 2.5. Note that such a partition is given by the edges in the top triangle, and the bottom triangle, edge-inducing a K_3 subgraph each, and each remaining edge likewise edge-inducing a K_2 subgraph.

Now we show that $TT_{a,b,c}$ is claw-contractible-free by showing that the deletion of any three independent vertices from $TT_{a,b,c}$ results in a disconnected graph. The result will then follow by Lemma 2.6. Note that the removal of at least two non-adjacent vertices from either the *a*-path, *b*-path, or *c*-path results in a disconnected graph. Similarly the removal of one vertex from each of the *a*-path, the *b*-path, and the *c*-path of $TT_{a,b,c}$ results in a disconnected graph unless all three vertices belong to the top triangle, or to the bottom triangle, but neither of these sets of three vertices is itself independent. □

Having proved that $TT_{a,b,c}$ is both claw-contractible-free and claw-free we restrict our attention to $TT_{n,n,n}$ where $n \geq 3$ and prove that its chromatic symmetric function is not *e*-positive by calculating the coefficient $[e_{(n^3)}]X_{TT_{n,n,n}}$. Note that $(3n)$, $(2n, n)$ and (n^3) are the only partitions that satisfy $\lambda \vdash 3n$ and $\lambda \succeq_p (n^3)$ and hence by Observation 2.1 and Lemma 2.3 in order to calculate $[e_{(n^3)}]X_{TT_{n,n,n}}$, we need to calculate $[p_{(3n)}]X_{TT_{n,n,n}}$, $[p_{(2n,n)}]X_{TT_{n,n,n}}$ and $[p_{(n^3)}]X_{TT_{n,n,n}}$, which we do in the following lemma using a case analysis that is similar to, but more substantial and delicate than, Lemma 4.2.

Lemma 5.2. *For $n \geq 3$ we have*

(1) $[p_{(3n)}]X_{TT_{n,n,n}} = (-1)^{3n+3}(12n^2 - 12n + 2)$,
(2) $[p_{(2n,n)}]X_{TT_{n,n,n}} = (-1)^{3n}(4n^2 + 6n - 7)$ *and*
(3) $[p_{(n^3)}]X_{TT_{n,n,n}} = (-1)^{3n-3}(3n - 2)$.

Proof. To prove this we will use Lemma 2.3 that concerns all subsets of the edge set E. We are only interested in subsets $S \subseteq E$ that yield $\lambda(S) = (3n)$, $(2n, n)$ or (n^3). Note that all of these have parts at least n so we will disregard any set S where $\lambda(S)$ has a part smaller than n. If S has two or more edges removed from any of the n-paths, then $\lambda(S)$ will certainly have a part smaller than n. Thus we will only consider subsets $S \subseteq E$ that have *at most one edge* removed from any of the n-paths. In Table 3 we have considered all cases of one through five edges removed from the two triangles and have enumerated and collected all isomorphic graphs. This will be especially useful in our delicate case analysis, consisting of seven cases corresponding to the removal of zero to six edges from E.
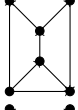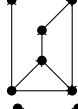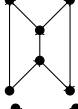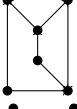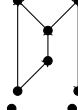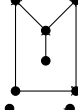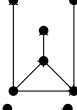
First, consider $|S| = |E|$. This gives us the term

$$(-1)^{3n+3}p_{(3n)}.$$

Second, consider $|S| = |E| - 1$, and note that removing any one of the $3n + 3$ edges yields a connected graph, and hence the term
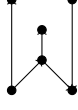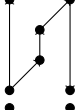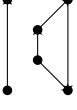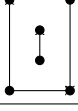
$$(-1)^{3n+2}(3n + 3)p_{(3n)}.$$

Table 3. All possible $TT_{n,n,n}$ with one through five edges removed from the top and bottom triangle.

i	$TT_{n,n,n}$ with i edges removed from the triangles
1	6 of
2	6 of , 3 of , 6 of
3	12 of , 6 of , 2 of
4	6 of , 6 of , 3 of
5	6 of

Third, consider $|S| = |E| - 2$. If the two removed edges come from the triangles, there are 15 possibilities and we can see from Table 3 that all these possibilities are connected, so this contributes the term $(-1)^{3n+1} 15 p_{(3n)}$.

Say we remove one edge from the triangles and one from the paths. In all six identical possibilities of removing one edge from a triangle we can remove any one of the $3(n-1)$ edges from the n-paths and maintain a connected graph, so we get the term $(-1)^{3n+1} 18(n-1) p_{(3n)}$.

Next consider the situation where we remove two edges from the n-paths. We noted earlier that these two edges cannot be from the same n-path. There are $\binom{3}{2}$ ways to choose the two n-paths and $n-1$ edge choices in each n-path. Since the resulting graph is always connected, we have the term $(-1)^{3n+1} 3(n-1)^2 p_{(3n)}$.

Altogether this case contributes the term

$$(-1)^{3n+1}(3n^2 + 12n) p_{(3n)}.$$

Fourth, consider $|S| = |E| - 3$. Now consider the situation of removing those three edges from the triangles. We can see from Table 3 that all 20 possibilities are connected, so contribute the term $(-1)^{3n} 20 p_{(3n)}$.

Say instead we remove two edges from the triangles and one from the n-paths. In the six possibilities on the left in Table 3, if we remove an edge from the left n-path we disconnect the graph, obtaining a part smaller than n. If instead we remove any one of

the $2(n-1)$ edges from the middle or right n-paths, then we have a connected graph, which contributes the term $(-1)^{3n}12(n-1)p_{(3n)}$. In the remaining nine middle and right possibilities of removing two edges from the triangles we can remove any one of the $3(n-1)$ edges from the three n-paths and still have a connected graph, which contributes the term $(-1)^{3n}27(n-1)p_{(3n)}$.

Now say that we remove one edge from the triangle and two edges from the n-paths. Again, these two edges must be on different n-paths and any choice of edges on the two n-paths will leave the graph connected. With six ways to remove an edge from the triangle, $\binom{3}{2}$ ways to choose the two n-paths, and $(n-1)^2$ ways to choose the edges on the n-paths, we get the term $(-1)^{3n}18(n-1)^2p_{(3n)}$.

Finally, consider the situation where we remove all three edges from the n-paths. No two of these removed edges are on the same n-path so we are removing one edge from each n-path. This will certainly disconnect the graph into two pieces. The only two-part partition we are interested in is $(2n, n)$ so we will count the edge removal choices that split the graph yielding a partition of this type. We will first count the number of possibilities that the piece connected to the top triangle has n vertices. Say we remove an edge on the left n-path that results in i vertices from this n-path contributing to this top connected piece. Also, say we remove an edge from the middle n-path so that the middle n-path contributes j vertices to the top connected piece. As long as $1 \leq i, j \leq n-1$ and $2 \leq i + j \leq n-1$, there exists exactly one edge in the right n-path that contributes $n - i - j$ vertices to the top connected piece, which yields our piece with n vertices. The number of choices for i and j is $(n-1)(n-2)/2$. Since there are equally many choices to instead make the bottom connected piece have n vertices, this contributes the term $(-1)^{3n}(n-1)(n-2)p_{(2n,n)}$.

Altogether this case contributes the terms

$$(-1)^{3n}(18n^2 + 3n - 1)p_{(3n)} \quad \text{and} \quad (-1)^{3n}(n^2 - 3n + 2)p_{(2n,n)}.$$

Fifth, consider $|S| = |E| - 4$. There are 15 possibilities for removing all four of the edges from the triangles. We can see in Table 3 that in 12 of the possibilities the graph remains connected so contributes the term $(-1)^{3n-1}12p_{(3n)}$. In the remaining three possibilities the graph becomes an n-path and a $2n$-cycle so contributes the term $(-1)^{3n-1}3p_{(2n,n)}$.

Next consider removing only three edges from the triangles and one edge from the n-paths. In the left 12 possibilities listed in Table 3 we can remove any of the $2(n-1)$ edges from the left and middle n-paths and maintain a connected graph. The removal of any edge from the right n-path will yield a part smaller than n. In the middle six possibilities listed in Table 3 we can again remove any one of the $2(n-1)$ edges from the left or right n-path and maintain a connected graph, but choosing an edge from the middle n-path yields a part smaller than n. In the right two possibilities in Table 3 any edge removed from any n-path would yield a part smaller than n so altogether this contributes the term $(-1)^{3n-1}36(n-1)p_{(3n)}$.

Next say we remove two edges from the triangles and two edges from the n-paths. In Table 3 we can see for the left six possibilities that we can only remove the two edges

from the right and middle n-paths and this will split the graph into two pieces. Any choice of one of the $n-1$ edges from the middle n-path gives us precisely one choice for an edge in the right n-path so that we disconnect the graph to get $(2n, n)$. This contributes the term $(-1)^{3n-1}6(n-1)p_{(2n,n)}$. In the remaining nine middle and right possibilities in Table 3 we can choose any two n-paths in $\binom{3}{2}$ ways and choose any edge in $(n-1)^2$ ways and still have a connected graph, which contributes the term $(-1)^{3n-1}27(n-1)^2p_{(3n)}$.

Finally, suppose we remove only one edge from the triangles and three edges from the n-paths. Very similar to earlier, this breaks the graph into two pieces and there are $(n-1)(n-2)$ ways to choose the edges so that the graph is separated into one piece of size $2n$ and another of size n, which contributes the term $(-1)^{3n-1}6(n-1)(n-2)p_{(2n,n)}$. We cannot remove four edges from the n-paths else we get a part smaller than n.

Altogether this case contributes the terms

$$(-1)^{3n-1}(27n^2 - 18n + 3)p_{(3n)} \quad \text{and} \quad (-1)^{3n-1}(6n^2 - 12n + 9)p_{(2n,n)}.$$

Sixth, consider $|S| = |E| - 5$. If all five edges are removed from the triangles, then we have six possibilities all of which give us a disconnected n-path and $2n$-path that contributes the term $(-1)^{3n-2}6p_{(2n,n)}$.

Say we remove four edges from the triangles and one from the n-paths. The left and middle 12 possibilities in Table 3 will split the graph into two pieces, but not yielding the partition $(2n, n)$. In the right three possibilities in Table 3 we do not want to remove an edge from the left n-path since we would get a part smaller than n, but we can remove any of the $2(n-1)$ other edges from the n-paths and have the graph yield $(2n, n)$, which contributes the term $(-1)^{3n-2}6(n-1)p_{(2n,n)}$.

Say we remove three edges from the triangles and two from the n-paths. In Table 3 we can see with the right two possibilities that there is no choice of edges on the n-paths that disconnects the graph yielding parts we are interested in. In the left and middle 18 possibilities there are two n-paths we can remove the two edges from without automatically getting a part smaller than n. Also, any choice of edges on the n-paths splits the graph into two pieces so we need to count the possibilities that result in the partition $(2n, n)$. For any of the $n-1$ choices for an edge on one n-path there is exactly one choice of an edge on the other n-path such that the graph is partitioned to obtain $(2n, n)$. Together this contributes the term $(-1)^{3n-2}18(n-1)p_{(2n,n)}$.

Finally, consider removing two edges from the triangles and three edges from the n-paths. For the right and middle nine possibilities in Table 3 this splits the graph into two pieces and there are $(n-1)(n-2)$ ways to choose the edges so that the graph is separated into a piece of size $2n$ and another of size n as discussed earlier. In the left six possibilities in Table 3 we would split the graph so that it yields a part smaller than n. This contributes the term $(-1)^{3n-2}9(n-1)(n-2)p_{(2n,n)}$. Again, we cannot remove four or more edges from the n-paths else we get a part smaller than n.

Altogether this case gives us the term

$$(-1)^{3n-2}(9n^2 - 3n)p_{(2n,n)}.$$

Seventh, consider $|S| = |E| - 6$. If we remove all six edges from the triangles, we obtain three disconnected n-paths, which contributes the term $(-1)^{3n-3}p_{(n^3)}$.

If we remove five edges from the triangles and one from the n-paths, then we can see that in all six possibilities in Table 3 we will split our graph into three pieces not yielding (n, n, n).

Say that we remove four edges from the triangles and two from the n-paths. In the left and middle 12 possibilities in Table 3 we will split the graph into pieces we are not interested in. In the right three possibilities in Table 3 we split the graph into three pieces and any one choice of the $n - 1$ edges in the middle n-path will leave us with one choice of an edge in the right n-path so that we split our graph into three pieces of size n. This gives us the term $(-1)^{3n-3}3(n-1)p_{(n^3)}$.

Say that we remove three edges from the triangles and three from the n-paths. In all 20 possibilities in Table 3 we split the graph into pieces of sizes we are not interested in.

Since we cannot remove four or more edges from the n-paths and get pieces of size at least n, altogether this case gives us the term

$$(-1)^{3n-3}(3n - 2)p_{(n^3)}.$$

No matter how we remove seven or more edges in total we will obtain a part smaller than n, so we have considered all sets S that contribute to the partitions $(3n)$, $(2n, n)$ and (n^3). Adding everything we get

$$
\begin{aligned}
[p_{(3n)}]X_{TT_{n,n,n}} \\
&= (-1)^{3n+3}\big(1 - (3n + 3) + (3n^2 + 12n) - (18n^2 + 3n - 1) + (27n^2 - 18n + 3)\big) \\
&= (-1)^{3n+3}(12n^2 - 12n + 2)
\end{aligned}
$$

and

$$
\begin{aligned}
[p_{(2n,n)}]X_{TT_{n,n,n}} &= (-1)^{3n}\big((n^2 - 3n + 2) - (6n^2 - 12n + 9) + (9n^2 - 3n)\big) \\
&= (-1)^{3n}(4n^2 + 6n - 7)
\end{aligned}
$$

and

$$[p_{(n^3)}]X_{TT_{n,n,n}} = (-1)^{3n-3}(3n - 2). \qquad \square$$

Similar to Lemma 4.3 we can prove the following.

Lemma 5.3. *For $n \geq 1$ we have*

(1) $[e_{(n^3)}]p_{(3n)} = (-1)^{3n-3}n,$
(2) $[e_{(n^3)}]p_{(2n,n)} = (-1)^{3n-3}n^2$ *and*
(3) $[e_{(n^3)}]p_{(n^3)} = (-1)^{3n-3}n^3.$

Proof. Using (2.1) we see that $[e_{(n^3)}]p_{(3n)} = (-1)^{3n-3}\frac{3n(3-1)!}{3!} = (-1)^{3n-3}n.$

Recall from Lemma 4.3 and its proof that $[e_{(n^2)}]p_{(2n)} = (-1)^{2n-2}n$ and $[e_n]p_n = (-1)^{n-1}n$. In $p_{(n^3)} = p_n p_n p_n$ the coefficient of $e_{(n^3)}$ is purely determined by the multiplication of the coefficients of e_n in p_n, which gives $[e_{(n^3)}]p_{(n^3)} = ([e_n]p_n)^3 = (-1)^{3n-3}n^3.$

In $p_{(2n,n)} = p_{(2n)}p_n$ the coefficient of $e_{(n^3)}$ is purely determined by the multiplication of the coefficient of $e_{(n^2)}$ in $p_{(2n)}$ and e_n in p_n, which gives $[e_{(n^3)}]p_{(2n,n)} = [e_{(n^2)}]p_{(2n)}[e_n]p_n = (-1)^{3n-3}n^2$. $\qquad\square$

Lemma 5.4. *The chromatic symmetric function of* $TT_{n,n,n}$ *for* $n \geq 3$ *is not e-positive. In particular,*

$$[e_{(n^3)}]X_{TT_{n,n,n}} = -n(n-1)^2(n-2).$$

Proof. By Observation 2.1 we can see that $e_{(n^3)}$ has a non-zero coefficient in p_λ only for $\lambda \vdash 3n$ with $\lambda \succeq_p (n^3)$. There are only three partitions $\lambda \vdash 3n$ with $\lambda \succeq_p (n^3)$, namely $(3n)$, $(2n, n)$ and (n^3).

Since

$$X_{TT_{n,n,n}} = \sum_{\lambda \vdash 3n} [p_\lambda]X_{TT_{n,n,n}} p_\lambda,$$

the coefficient of $e_{(n^3)}$ in $X_{TT_{n,n,n}}$ only arises from the $p_{(3n)}$, $p_{(2n,n)}$ and $p_{(n^3)}$ terms. In particular,

$$[e_{(n^3)}]X_{TT_{n,n,n}} = [e_{(n^3)}]p_{(3n)} \cdot [p_{(3n)}]X_{TT_{n,n,n}} + [e_{(n^3)}]p_{(2n,n)} \cdot [p_{(2n,n)}]X_{TT_{n,n,n}}$$
$$+ [e_{(n^3)}]p_{(n^3)} \cdot [p_{(n^3)}]X_{TT_{n,n,n}}.$$

Using Lemmas 5.2 and 5.3 we therefore have

$$[e_{(n^3)}]X_{TT_{n,n,n}} = n \cdot (12n^2 - 12n + 2) - n^2 \cdot (4n^2 + 6n - 7) + n^3 \cdot (3n - 2)$$
$$= -n(n-1)^2(n-2). \qquad\square$$

We can now identify our third family of graphs that are claw-contractible-free, are furthermore claw-free, and whose chromatic symmetric functions are not *e*-positive.

Theorem 5.5. *The graphs* $TT_{n,n,n}$ *for all* $n \geq 3$ *are claw-contractible-free, claw-free and not e-positive.*

Proof. This follows immediately from Lemmas 5.1 and 5.4. $\qquad\square$

Remark 5.6. One might ask whether triangular tower graphs are also incomparability graphs, so as to also potentially be a counterexample to Stanley's $(\mathbf{3} + \mathbf{1})$-free conjecture [31, Conjecture 5.1]. However, it is straightforward to check that triangular tower graphs are not incomparability graphs.

We conclude by conjecturing that the triangular tower graphs $TT_{n,n,n}$ for $n \geq 3$ are in some sense a minimal family of graphs that are claw-contractible-free, claw-free and whose chromatic symmetric functions are not *e*-positive. More precisely, we conjecture that there do not exist graphs with 10 or 11 vertices that are claw-contractible-free, claw-free and whose chromatic symmetric functions are not *e*-positive. One motivation for this conjecture is the scarcity of graphs that are claw-contractible-free and whose chromatic symmetric function's expansion into elementary symmetric functions has negative coefficients. For $N = 6$ only 4 of 112 connected graphs satisfy this. For $N = 7$ this becomes

7 of 853 and for $N = 8$ it is 27 of 11,117. Also the negative terms that we have identified are almost the *only* negative terms in the elementary symmetric function expansion. For example, of the 293 terms in $X_{TT_{7,7,7}}$ the only negative term is the one we identified, namely $-1260e_{(7^3)}$. Of the 564 terms in $X_{TT_{8,8,8}}$ the only negative term other than the one we identified is $-1944e_{(4^6)}$, while of the 1042 terms in $X_{TT_{9,9,9}}$ it is $-768e_{(3^9)}$.

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