DUAL GRAPHS FROM NONCOMMUTATIVE AND QUASISYMMETRIC SCHUR FUNCTIONS

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Abstract. By establishing relations between operators on compositions, we show that the posets of compositions arising from the right and left Pieri rules for noncommutative Schur functions can each be endowed with both the structure of dual graded graphs and dual filtered graphs when paired with the poset of compositions arising from the Pieri rules for quasisymmetric Schur functions and its deformation.

1. Introduction

Differential posets [19] and dual graded graphs [5, 6] were first developed in order to better understand the Robinson-Schensted-Knuth algorithm. However, since then they have developed into a research area in their own right, for example [16, 21], including rank variants [20] and signed analogues [13]. They also arise in the study of representations of towers of algebras [2, 8], have been generalized to planar binary trees [17], Kac-Moody algebras [1, 15], quantized versions [14], and most recently to related to K-theory via dual filtered graphs [18]. The classic example of dual graded graphs is Young's lattice paired with itself. Young's lattice appears in a variety of areas, such as being used to describe the Pieri rules for Schur functions. From this perspective, natural generalizations of Young's lattice exist arising from Pieri rules for the Schur function generalizations known as quasisymmetric Schur functions, and noncommutative Schur functions. In particular, quasisymmetric Schur functions [11] are a nonsymmetric generalization of Schur functions that form a basis for the increasingly ubiquitous Hopf algebra of quasisymmetric functions, for example [4, 10, 12]. Their Pieri rules [11, Theorem 6.3] give rise to the generalization of Young's lattice known as the quasisymmetric composition poset. Dual to this Hopf algebra is the Hopf algebra of noncommutative symmetric functions [9], whose basis dual to that of quasisymmetric Schur functions is the basis of noncommutative Schur functions [3], a noncommutative analogue of Schur functions. Due to noncommutativity, two sets of Pieri rules arise, one arising from multiplication on the right [22, Theorem 9.3] and one from multiplication on the left [3, Corollary 3.8]. These two sets of Pieri rules give rise to two generalizations of Young's lattice known as the right composition poset and the left composition poset. Therefore the

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question arises: Are these posets dual graded and dual filtered graphs? In this paper we answer this question in the affirmative.

More precisely, this paper is structured as follows. In Section 2 we review necessary notions on compositions in order to define operators on them. These operators are used to define three partially ordered sets in Subsection 2.1, $\mathcal{R}_c$ and $\mathcal{L}_c$ that arise in the right and left Pieri rules for noncommutative Schur functions, and $\mathcal{Q}_c$ that arises in the Pieri rules for quasisymmetric Schur functions. We then establish useful relations satisfied by these operators in Subsections 2.2 and 2.3. In Section 3 we show that $\mathcal{R}_c$ and $\mathcal{Q}_c$, plus $\mathcal{L}_c$ and $\mathcal{Q}_c$, are each a pair of dual graded graphs in Theorems 3.3 and 3.16. We define a strong filtered graph $\widetilde{\mathcal{Q}}_c$ on the set of compositions using the operators arising in the Pieri rules for quasisymmetric Schur functions in Definition 3.5, and establish that $\mathcal{R}_c$ and $\widetilde{\mathcal{Q}}_c$, plus $\mathcal{L}_c$ and $\widetilde{\mathcal{Q}}_c$, are each a pair of dual filtered graphs in Theorems 3.8 and 3.18.

2. Compositions and operators

A finite list of integers $\alpha = (\alpha_1, \dots, \alpha_\ell)$ is called a *weak composition* if $\alpha_1, \dots, \alpha_\ell$ are nonnegative, and is called a *composition* if $\alpha_1, \dots, \alpha_\ell$ are positive. Note that every weak composition has an underlying composition, obtained by removing all 0s. Given $\alpha = (\alpha_1, \dots, \alpha_\ell)$ we call the α_i the *parts* of α, and the sum of the parts of α the *size* of α.

Now we will recall four families of operators, each of which are indexed by positive integers, and have already contributed to the theory of quasisymmetric and noncommutative Schur functions. Although originally defined on compositions, we will define them in the natural way on weak compositions to simplify our proofs. Our first operator is the box removing operator $\mathfrak{d}$, which first appeared in the Pieri rules for quasisymmetric Schur functions [11]. Our second operator is the appending operator a. Together these give our third operator, the jeu de taquin or jdt operator $\mathfrak{u}$. This operator arises in jeu de taquin slides on semistandard reverse composition tableaux and in the right Pieri rules for noncommutative Schur functions [22]. Our fourth operator is the box adding operator $\mathfrak{t}$, which plays the same role in the left Pieri rules for noncommutative Schur functions [3] as $\mathfrak{u}$ does in the right Pieri rules. Each of these operators is defined on weak compositions for every integer $i \geq 0$ and we note that

$$\mathfrak{d}_0 = a_0 = \mathfrak{u}_0 = \mathfrak{t}_0 = Id$$

namely the identity map, which fixes the weak composition it is acting on. We now define the remaining operators for $i \geq 1$, after establishing some set notation. Let $\mathbb{N}$ be the set of positive integers. Anytime we refer to a set $I \subset \mathbb{N}$, we implicitly assume that I is finite. Also, if we are given such a set I, then $I - 1$ is the set obtained by subtracting 1 from all the elements in I and removing any 0s that might arise in so doing.

Example 2.1. *If $I = \{1, 2, 4\}$, then $I - 1 = \{1, 3\}$.*

By $[i]$ where $i \geq 1$, we mean the set $\{1, 2, \dots, i\}$. We furthermore define $[0]$ to be the empty set. We will denote the maximum element of a set A by $\max(A)$. If A is the empty set, by convention we have that $\max(A) = 0$.

The first *box removing operator* on weak compositions, $\mathfrak{d}_i$ for $i \geq 1$, is defined as follows. Let α be a weak composition. Then

$$\mathfrak{d}_i(\alpha) = \alpha'$$

where α' is the weak composition obtained by subtracting 1 from the rightmost part equalling i in α. If there is no such part then we define $\mathfrak{d}_i(\alpha) = 0$.

Example 2.2. *Let* $\alpha = (2,1,3)$. *Then* $\mathfrak{d}_1(\alpha) = (2,0,3)$, $\mathfrak{d}_2(\alpha) = (1,1,3)$, $\mathfrak{d}_3(\alpha) = (2,1,2)$ *and* $\mathfrak{d}_4(\alpha) = 0$. *In fact,* $\mathfrak{d}_i(\alpha) = 0$ *for all* $i \geq 4$.

Given a finite set $I = \{i_1 < \cdots < i_k\}$ of positive integers, we define

$$\mathfrak{d}_I = \mathfrak{d}_{i_1}\mathfrak{d}_{i_2}\cdots\mathfrak{d}_{i_k}.$$

For convenience, we define $\mathfrak{d}_\emptyset = \mathfrak{d}_0$. The empty product of box removing operators is also defined to be $\mathfrak{d}_0$.

Example 2.3.

$$\begin{aligned}
\mathfrak{d}_{[3]}((3,1,4,2,1)) &= \mathfrak{d}_1\mathfrak{d}_2\mathfrak{d}_3((3,1,4,2,1)) \\
&= \mathfrak{d}_1\mathfrak{d}_2((2,1,4,2,1)) \\
&= \mathfrak{d}_1((2,1,4,1,1)) \\
&= (2,1,4,1,0)
\end{aligned}$$

The second *appending operator* on weak compositions, a_i for $i \geq 1$, is defined as follows. Let $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ be a weak composition. Then

$$a_i(\alpha) = (\alpha_1, \ldots, \alpha_\ell, i)$$

namely, the weak composition obtained by appending a part i to the end of α. To simplify proofs later, we will abuse notation and also think of a_0 as adding 0 to the end of α that we will eventually remove.

Example 2.4. *Let* $\alpha = (2,1,3)$. *Then* $a_2(\alpha) = (2,1,3,2)$. *However,* $a_2\mathfrak{d}_4(\alpha) = 0$ *since* $\mathfrak{d}_4(\alpha) = 0$ *by Example 2.2.*

The third *jeu de taquin* or *jdt operator* on weak compositions, $\mathfrak{u}_i$ for $i \geq 1$, is defined as follows. Considering the box removing and appending operators,

$$\mathfrak{u}_i = a_i\mathfrak{d}_{[i-1]}.$$

Example 2.5. *Let us compute*

$$\mathfrak{u}_4((3,1,4,2,1)) = a_4\mathfrak{d}_{[3]}((3,1,4,2,1)).$$

By Example 2.3 $\mathfrak{d}_{[3]}((3,1,4,2,1)) = (2,1,4,1,0)$, *and hence* $\mathfrak{u}_4(3,1,4,2,1) = (2,1,4,1,0,4)$.

For any set of finite positive integers $I = \{i_1 < \cdots < i_k\}$, we define

$$\mathfrak{u}_I = \mathfrak{u}_{i_k}\cdots\mathfrak{u}_{i_1}.$$

For convenience, we define $\mathfrak{u}_\emptyset = \mathfrak{u}_0$. The empty product of jdt operators is also defined to be $\mathfrak{u}_0$. Note further that the order of indices in $\mathfrak{d}_I$ is the reverse of that in $\mathfrak{u}_I$.

Lastly, the fourth *box adding operator* on weak compositions, $\mathfrak{t}_i$ for $i \geq 1$, is defined as follows. Let $\alpha = (\alpha_1, \dots, \alpha_\ell)$ be a weak composition. Then

$$\mathfrak{t}_1(\alpha) = (1, \alpha_1, \dots, \alpha_\ell)$$

and for $i \geq 2$

$$\mathfrak{t}_i(\alpha) = (\alpha_1, \dots, \alpha_j + 1, \dots, \alpha_\ell)$$

where α_j is the leftmost part equalling $i-1$ in α. If there is no such part, then we define $\mathfrak{t}_i(\alpha) = 0$.

Example 2.6. *Let $\alpha = (3,1,4,2,1)$. Then $\mathfrak{t}_1(\alpha) = (1,3,1,4,2,1)$, $\mathfrak{t}_2(\alpha) = (3,2,4,2,1)$, $\mathfrak{t}_3(\alpha) = (3,1,4,3,1)$, $\mathfrak{t}_4(\alpha) = (4,1,4,2,1)$, $\mathfrak{t}_5(\alpha) = (3,1,5,2,1)$ and $\mathfrak{t}_i(\alpha) = 0$ for all $i \geq 6$.*

2.1. **Composition posets.** With our operators we will now define three partial orders on compositions noting that *if any parts of size 0 arise during computation, then they are ignored.* The adjectives right and left in the first two are not only used to distinguish between the posets, but also to refer to their roles in the right and left Pieri rules for noncommutative Schur functions in [22, Theorem 9.3] and [3, Corollary 3.8] respectively, and whose notation we follow now.

Definition 2.7. *The* right composition poset*, denoted by $\mathcal{R}_c$, is the poset consisting of all compositions with cover relation $\lessdot_r$ such that for compositions α, β*

$$\beta \lessdot_r \alpha \text{ if and only if } \alpha = \mathfrak{u}_i(\beta)$$

for some $i \geq 1$. Meanwhile the left composition poset*, denoted by $\mathcal{L}_c$, is the poset consisting of all compositions with cover relation $\lessdot_c$ such that for compositions α, β*

$$\beta \lessdot_c \alpha \text{ if and only if } \alpha = \mathfrak{t}_i(\beta)$$

for some $i \geq 1$.

The order relation $<_r$ in $\mathcal{R}_c$ (respectively, $<_c$ in $\mathcal{L}_c$) is obtained by taking the transitive closure of the cover relation $\lessdot_r$ (respectively, $\lessdot_c$).

Example 2.8. *Let $\beta = (3,1,4,2,1)$, $\alpha^R = (2,1,4,1,4)$ and $\alpha^L = (4,1,4,2,1)$. Then $\beta \lessdot_r \alpha^R = \mathfrak{u}_4(\beta)$ and $\beta \lessdot_c \alpha^L = \mathfrak{t}_4(\beta)$ by Examples 2.5 and 2.6, respectively.*

Our third poset, meanwhile, stems from the Pieri rules for quasisymmetric Schur functions [11, Theorem 6.3], hence its name.

Definition 2.9. *The* quasisymmetric composition poset*, denoted by $\mathcal{Q}_c$, is the poset consisting of all compositions with cover relation $\lessdot_q$ such that for compositions α, β*

$$\beta \lessdot_q \alpha \text{ if and only if } \mathfrak{d}_i(\alpha) = \beta$$

for some $i \geq 1$.

Again, the order relation $<_q$ in $\mathcal{Q}_c$ is obtained by taking the transitive closure of the cover relation $\lessdot_q$.

Example 2.10. *Let $\beta = (4,1,3,2,1)$ and $\alpha = (4,1,4,2,1)$. Then $\mathfrak{d}_4(\alpha) = \beta \lessdot_q \alpha$.*

2.2. **Relations satisfied by operators of type $\mathfrak{u}$ and $\mathfrak{d}$.** We will now give a variety of lemmas regarding the jdt operators and box removing operators, which will be useful in proving our main theorems later. Hence this subsection can be safely skipped for now and referred to later. In all the proofs we assume that α is a weak composition.

Lemma 2.11. *For $i \geq 0$ we have that $a_i = \mathfrak{d}_{i+1} a_{i+1}$.*

Proof. Let $\alpha = (\alpha_1, \ldots, \alpha_\ell)$. Then $a_{i+1}(\alpha) = (\alpha_1, \ldots, \alpha_\ell, i+1)$. This implies by definition that $\mathfrak{d}_{i+1} a_{i+1}(\alpha) = (\alpha_1, \ldots, \alpha_\ell, i) = a_i(\alpha)$. □

As a corollary we obtain the following relationship between any two appending operators.

Corollary 2.12. *For positive integers i and j satisfying $i \geq j$, we have that*

$$\mathfrak{d}_j \mathfrak{d}_{j+1} \cdots \mathfrak{d}_{i-1} \mathfrak{d}_i a_i = a_{j-1}.$$

Lemma 2.13. *Let $i \neq j$ be positive integers. Then*

$$\mathfrak{d}_i a_j = a_j \mathfrak{d}_i.$$

Proof. Let $\alpha = (\alpha_1, \ldots, \alpha_\ell)$. Let $\beta = a_j(\alpha) = (\alpha_1, \ldots, \alpha_\ell, j)$. If α does not have a part equalling i, then neither does β, as $i \neq j$. Thus in this case we have that $\mathfrak{d}_i a_j(\alpha) = \mathfrak{d}_i(\beta) = 0 = a_j \mathfrak{d}_i(\alpha)$. Now, assume that α_r is the rightmost part equalling i in α. Then $a_j \mathfrak{d}_i(\alpha) = (\alpha_1, \ldots, \alpha_{r-1}, \alpha_r - 1, \ldots, \alpha_\ell, j)$. Since $i \neq j$, we are guaranteed that $\mathfrak{d}_i(\beta) = (\alpha_1, \ldots, \alpha_{r-1}, \alpha_r - 1, \ldots, \alpha_\ell, j)$. Thus we have that $\mathfrak{d}_i a_j(\alpha) = a_j \mathfrak{d}_i(\alpha)$ in this case as well, and we are done. □

The proofs of the next three lemmas consist of case analyses that are similar in style to the proof of Lemma 2.13, however, they are more technical and hence we omit them for brevity.

Lemma 2.14. *Let i and j be distinct positive integers such that $|i-j| \geq 2$. Then*

$$\mathfrak{d}_i \mathfrak{d}_j = \mathfrak{d}_j \mathfrak{d}_i.$$

Lemma 2.15. *Let $i \geq 1$. Then $\mathfrak{d}_i^2 \mathfrak{d}_{i+1} = \mathfrak{d}_i \mathfrak{d}_{i+1} \mathfrak{d}_i$.*

Lemma 2.16. *Let $i \geq 1$. Then $\mathfrak{d}_i \mathfrak{d}_{i+1}^2 = \mathfrak{d}_{i+1} \mathfrak{d}_i \mathfrak{d}_{i+1}$.*

Lemma 2.17. *Let $i \neq j$ be positive integers. Then*

$$\mathfrak{u}_i \mathfrak{d}_j = \mathfrak{d}_j \mathfrak{u}_i.$$

Proof. Let us first consider the case $1 \leq i \leq j-1$. Then by Lemmas 2.13 and 2.14, we have that $\mathfrak{d}_j$ commutes with $a_i, \mathfrak{d}_1, \ldots, \mathfrak{d}_{i-1}$. Hence $\mathfrak{u}_i \mathfrak{d}_j = \mathfrak{d}_j \mathfrak{u}_i$ in this case.

Now consider the case where $i > j \geq 1$. Then $\mathfrak{d}_j \mathfrak{u}_i = \mathfrak{d}_j a_i \mathfrak{d}_1 \mathfrak{d}_2 \cdots \mathfrak{d}_{i-1}$. Again, using Lemmas 2.13 and 2.14, we can write this as

$$a_i \mathfrak{d}_1 \cdots \mathfrak{d}_{j-2} \mathfrak{d}_j \mathfrak{d}_{j-1} \mathfrak{d}_j \cdots \mathfrak{d}_{i-1}.$$

Using Lemma 2.16, we can write the above as

$$a_i\mathfrak{d}_1\cdots\mathfrak{d}_{j-2}\mathfrak{d}_{j-1}\mathfrak{d}_j\mathfrak{d}_j\mathfrak{d}_{j+1}\cdots\mathfrak{d}_{i-1}.$$

Notice at this stage, if we assume $j=i-1$, then we have shown that $\mathfrak{u}_i\mathfrak{d}_j=\mathfrak{d}_j\mathfrak{u}_i$. So let us assume $i-j\geq 2$. Using Lemma 2.15, we can transform the above expression to

$$a_i\mathfrak{d}_1\cdots\mathfrak{d}_{j-2}\mathfrak{d}_{j-1}\mathfrak{d}_j\mathfrak{d}_{j+1}\mathfrak{d}_j\mathfrak{d}_{j+2}\cdots\mathfrak{d}_{i-1}.$$

Now Lemma 2.14 easily establishes that the above expression equals

$$a_i\mathfrak{d}_1\cdots\mathfrak{d}_{j-2}\mathfrak{d}_{j-1}\mathfrak{d}_j\mathfrak{d}_{j+1}\mathfrak{d}_{j+2}\cdots\mathfrak{d}_{i-1}\mathfrak{d}_j$$

and we are done. □

Lemma 2.18. *Let $i\geq 1$. Then $\mathfrak{u}_i\mathfrak{d}_i=\mathfrak{d}_{i+1}\mathfrak{u}_{i+1}$.*

Proof. Notice that $\mathfrak{u}_i\mathfrak{d}_i=a_i\mathfrak{d}_{[i]}$. Furthermore, Lemma 2.11 states that $a_i=\mathfrak{d}_{i+1}a_{i+1}$, and hence $\mathfrak{u}_i\mathfrak{d}_i=\mathfrak{d}_{i+1}a_{i+1}\mathfrak{d}_{[i]}$. Since $\mathfrak{u}_{i+1}=a_{i+1}\mathfrak{d}_{[i]}$, by definition, the claim follows. □

2.3. **Relations satisfied by operators of type $\mathfrak{t}$ and $\mathfrak{d}$.** We now give two useful lemmas, but this time regarding the box adding and box removing operators. Again, if desired, this subsection can be safely skipped for now and referred to later. In all the proofs we assume that α is a weak composition.

Lemma 2.19. *Let $i\neq j$ be positive integers. Then*

$$\mathfrak{t}_i\mathfrak{d}_j=\mathfrak{d}_j\mathfrak{t}_i.$$

Proof. Let $\alpha=(\alpha_1,\ldots,\alpha_\ell)$. First consider the case $i=1$. If α does not have a part equalling j, then $\mathfrak{t}_1\mathfrak{d}_j(\alpha)=0$. Note now that, since $j\neq 1$, we have that $\mathfrak{d}_j\mathfrak{t}_1(\alpha)=\mathfrak{d}_j((1,\alpha_1,\ldots,\alpha_\ell))=0$ as well.

Hence we can assume that $i\geq 2$. If α does not have a part equalling $i-1$, then using the fact that $i\neq j$, we get that $\mathfrak{d}_j(\alpha)$ does not have a part equalling $i-1$ either (assuming it does not equal 0 already). This implies that $\mathfrak{t}_i\mathfrak{d}_j(\alpha)=0$. Our assumption that α has no part equalling $i-1$ also implies that $\mathfrak{d}_j\mathfrak{t}_i(\alpha)=0$.

Finally assume that α does have a part equalling $i-1$, and let α_r denote the leftmost such part. Then

$$\mathfrak{t}_i(\alpha)=(\alpha_1,\ldots,\alpha_r+1,\ldots,\alpha_\ell).$$

If α does not have a part equalling j, then neither does $\mathfrak{t}_i(\alpha)$. This follows from the fact that $i\neq j$. This immediately implies that $\mathfrak{t}_i\mathfrak{d}_j(\alpha)=\mathfrak{d}_j\mathfrak{t}_i(\alpha)=0$ in this case. If α does have a part equalling j, then let α_s denote the rightmost such part. Note that α_s continues to be the rightmost part equalling j in $\mathfrak{t}_i(\alpha)$ as well (unless there is a single part equalling $j=i-1$, in which case $\mathfrak{t}_i\mathfrak{d}_j(\alpha)=\mathfrak{d}_j\mathfrak{t}_i(\alpha)=0$). Again, this follows since $i\neq j$. Thus we get that

$$\mathfrak{t}_i\mathfrak{d}_j(\alpha)=\mathfrak{d}_j\mathfrak{t}_i(\alpha)=(\alpha_1,\ldots,\alpha_r+1,\ldots,\alpha_s-1,\ldots,\alpha_\ell)$$

if $r<s$ and

$$\mathfrak{t}_i\mathfrak{d}_j(\alpha)=\mathfrak{d}_j\mathfrak{t}_i(\alpha)=(\alpha_1,\ldots,\alpha_s-1,\ldots,\alpha_r+1,\ldots,\alpha_\ell)$$

if $s<r$. □

Lemma 2.20. *Let i be a positive integer. Then $\mathfrak{d}_i\mathfrak{t}_i(\alpha) = \mathfrak{t}_i\mathfrak{d}_i(\alpha)$ if one of the following conditions holds.*

(1) *We have $i \geq 2$ and α has parts equalling i and $i-1$.*
(2) *We have $i = 1$ and α has at least one part equalling 1.*

Proof. Assume first that $i \geq 2$ and that $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ has parts equalling i and $i-1$. Furthermore suppose that α_r is the leftmost part equalling $i-1$, and α_s is the rightmost part equalling i in α.

If $r < s$, then we have the following.

$$\begin{aligned}\mathfrak{d}_i\mathfrak{t}_i(\alpha) &= \mathfrak{d}_i\mathfrak{t}_i((\alpha_1, \ldots, \alpha_r, \ldots, \alpha_s, \ldots, \alpha_\ell))\\ &= \mathfrak{d}_i((\alpha_1, \ldots, \alpha_r + 1, \ldots, \alpha_s, \ldots, \alpha_\ell))\\ &= (\alpha_1, \ldots, \alpha_r + 1, \ldots, \alpha_s - 1, \ldots, \alpha_\ell)\end{aligned}$$

Also, the following sequence of equalities holds.

$$\begin{aligned}\mathfrak{t}_i\mathfrak{d}_i(\alpha) &= \mathfrak{t}_i\mathfrak{d}_i((\alpha_1, \ldots, \alpha_r, \ldots, \alpha_s, \ldots, \alpha_\ell))\\ &= \mathfrak{t}_i((\alpha_1, \ldots, \alpha_r, \ldots, \alpha_s - 1, \ldots, \alpha_\ell))\\ &= (\alpha_1, \ldots, \alpha_r + 1, \ldots, \alpha_s - 1, \ldots, \alpha_\ell)\end{aligned}$$

Thus if $r < s$ we have that $\mathfrak{d}_i\mathfrak{t}_i(\alpha) = \mathfrak{t}_i\mathfrak{d}_i(\alpha)$. The case $r > s$ is similar and yields $\mathfrak{d}_i\mathfrak{t}_i(\alpha) = \mathfrak{t}_i\mathfrak{d}_i(\alpha) = \alpha$.

Assume now that $i = 1$, and that α has at least one part equalling 1. Furthermore, suppose that α_r is the rightmost part equalling 1. Then it can be easily verified that $\mathfrak{d}_1\mathfrak{t}_1(\alpha) = \mathfrak{t}_1\mathfrak{d}_1(\alpha) = (1, \alpha_1, \ldots, \alpha_{r-1}, 0, \alpha_{r+1}, \ldots, \alpha_\ell)$. □

3. Dual graphs from composition posets

We now recall terminology pertaining to graded graphs and filtered graphs, and follow the notation of [18]. Let G be a graph consisting of a set of vertices P endowed with a rank function $\rho : P \to \mathbb{Z}$ with vertices $x, y \in P$ and y is of rank weakly greater than x. Then G is called a *graded graph* when the edge set E satisfies if $(x, y) \in E$ then $\rho(y) = \rho(x) + 1$. The graph G is called a *weak filtered graph* when the edge set E satisfies if $(x, y) \in E$ then $\rho(y) \geq \rho(x)$, and a *strong filtered graph* when the edge set E satisfies if $(x, y) \in E$ then $\rho(y) > \rho(x)$. Now given a field K of characteristic 0, the vector space KP is the space consisting of all formal linear combinations of vertices of G. Then we define the up and down operators $U, D \in End(KP)$ associated with G to be

$$U(x) = \sum_y m(x, y)y$$

$$D(y) = \sum_x m(x, y)x$$

where x and y are vertices of G, y is of weakly greater rank than x, and $m(x, y)$ is the number of edges connecting x and y. With this in mind, let G_1 be a graded graph with up operator

U and G_2 be a graded graph with down operator D such that G_1 and G_2 have a common vertex set P and rank function ρ. Then G_1 and G_2 are *dual graded graphs* if and only if on KP

$$DU - UD = Id \tag{3.1}$$

where Id is the identity operator on KP. Similarly let $\widetilde{G_1}$ be a weak filtered graph with up operator $\widetilde{U}$ and $\widetilde{G_2}$ be a strong filtered graph with down operator $\widetilde{D}$ such that $\widetilde{G_1}$ and $\widetilde{G_2}$ have a common vertex set P and rank function ρ. Then $\widetilde{G_1}$ and $\widetilde{G_2}$ are *dual filtered graphs* if and only if on KP

$$\widetilde{D}\widetilde{U} - \widetilde{U}\widetilde{D} = \widetilde{D} + Id. \tag{3.2}$$

3.1. **Dual graphs and the right composition poset.** Observe that our composition posets $\mathcal{R}_c$ and $\mathcal{Q}_c$ defined in Subsection 2.1 with vertex set being the set of all compositions, whose rank function is given by the size of a composition and whose edge sets are the respective cover relations, are both clear examples of graded graphs. By the definition of the cover relation $\lessdot_r$ it follows that the up operator associated with $\mathcal{R}_c$ is given by

$$U = \sum_{i\geq 1} \mathfrak{u}_i. \tag{3.3}$$

Example 3.1. *Let α be the composition $(2,1,3)$. Then*

$$U((2,1,3)) = (2,1,3,1)+(2,0,3,2)+(1,0,3,3)+(2,1,0,4) = (2,1,3,1)+(2,3,2)+(1,3,3)+(2,1,4).$$

Similarly, by the definition of the cover relation $\lessdot_q$ it follows that the down operator associated with $\mathcal{Q}_c$ is given by

$$D = \sum_{i\geq 1} \mathfrak{d}_i. \tag{3.4}$$

Example 3.2. *Let α be the composition $(2,1,3)$. Then by Example 2.2*

$$D((2,1,3)) = (2,0,3) + (1,1,3) + (2,1,2) = (2,3) + (1,1,3) + (2,1,2).$$

Moreover we have the following.

Theorem 3.3. *$\mathcal{R}_c$ and $\mathcal{Q}_c$ are dual graded graphs, that is, on compositions*

$$DU - UD = Id.$$

Proof. Notice that

$$DU = \sum_{\substack{i\neq j\\ i,j\geq 1}} \mathfrak{d}_j\mathfrak{u}_i + \sum_{k\geq 1} \mathfrak{d}_k\mathfrak{u}_k$$

and

$$UD = \sum_{\substack{i\neq j\\ i,j\geq 1}} \mathfrak{u}_i\mathfrak{d}_j + \sum_{k\geq 1} \mathfrak{u}_k\mathfrak{d}_k.$$

Using Lemma 2.17 and Lemma 2.18, we reach the conclusion that

$$DU - UD = \mathfrak{d}_1\mathfrak{u}_1.$$

By Lemma 2.11, $\mathfrak{d}_1\mathfrak{u}_1 = a_0 = Id$. This finishes the proof. □

Example 3.4. *Let $\alpha = (2,1,3)$. Then suppressing commas and parentheses for ease of comprehension, we have by Examples 3.1 and 3.2 that*

$$\begin{aligned} DU(\alpha) &= D(2131 + 2032 + 1033 + 2104) \\ &= 2130 + 1131 + 2121 + 2031 + 2022 + 0033 + 1032 + 2004 + 1104 + 2103 \end{aligned}$$

and

$$\begin{aligned} UD(\alpha) &= U(203 + 113 + 212) \\ &= 2031 + 0033 + 2004 + 1131 + 1032 + 1104 + 2121 + 2022 + 2103. \end{aligned}$$

Thus $(DU - UD)(\alpha) = 213 = Id(\alpha)$.

To describe our results in the context of dual filtered graphs, we need the following.

Definition 3.5. *Let $\widetilde{\mathcal{Q}}_c$ be the graded graph whose vertex set is the set of all compositions, whose rank function is given by the size of a composition, and whose edge set is as follows. Given compositions α and β such that the size of α is strictly greater than β, we have the edge*

$$(\beta, \alpha) \text{ if and only if } \mathfrak{d}_I(\alpha) = \beta$$

for some finite $\emptyset \neq I \subset \mathbb{N}$.

As before, when computing $\mathfrak{d}_I(\alpha)$ in Definition 3.5, we ignore all parts that equal 0.

Example 3.6. *We have an edge between $\beta = (4,1,3,1,1)$ and $\alpha = (4,1,4,2,1)$ in $\widetilde{\mathcal{Q}}_c$ since $\mathfrak{d}_{\{2,4\}}(\alpha) = \beta$.*

Remark 3.7. Observe that the relation $<_{\widetilde{q}}$ on compositions defined by $\beta <_{\widetilde{q}} \alpha$ if and only if $\beta = \mathfrak{d}_I(\alpha)$ does not give rise to a poset structure, since transitivity is not satisfied. For example, $\mathfrak{d}_{\{1,4\}}((4,1,4,1)) = (4,1,3)$ and $\mathfrak{d}_{\{1,4\}}((4,1,3)) = (3,3)$, but no I exists such that $\mathfrak{d}_I((4,1,4,1)) = (3,3)$.

Clearly, we have that $\widetilde{\mathcal{Q}}_c$ is an example of a strong filtered graph by definition. The associated down operator is given by

$$\widetilde{D} = \sum_{I \subset \mathbb{N}} \mathfrak{d}_I \tag{3.5}$$

where the sum is over all finite but nonempty subsets of $\mathbb{N}$. Hence we can relate $\mathcal{R}_c$ and $\widetilde{\mathcal{Q}}_c$ as follows, since any graded graph, such as $\mathcal{R}_c$, is also a weak filtered graph.

Theorem 3.8. *$\mathcal{R}_c$ and $\widetilde{\mathcal{Q}}_c$ are dual filtered graphs, that is, on compositions*

$$\widetilde{D}U - U\widetilde{D} = \widetilde{D} + Id.$$

Proof. First note that the operator $\widetilde{D}U$ has the following expansion.

$$\begin{aligned}\widetilde{D}U &= \sum_{\substack{i\geq 1\\ I\subset\mathbb{N}}} \mathfrak{d}_I\mathfrak{u}_i\\ &= \sum_{\substack{I\subset\mathbb{N}\\ i\in I}} \mathfrak{d}_I\mathfrak{u}_i + \sum_{\substack{I\subset\mathbb{N}\\ i\geq 1, i\notin I}} \mathfrak{d}_I\mathfrak{u}_i\end{aligned}$$

In a similar manner, we obtain the following expansion for $U\widetilde{D}$.

$$\begin{aligned}U\widetilde{D} &= \sum_{\substack{i\geq 1\\ I\subset\mathbb{N}}} \mathfrak{u}_i\mathfrak{d}_I\\ &= \sum_{\substack{I\subset\mathbb{N}\\ i\in I}} \mathfrak{u}_i\mathfrak{d}_I + \sum_{\substack{I\subset\mathbb{N}\\ i\geq 1, i\notin I}} \mathfrak{u}_i\mathfrak{d}_I\end{aligned}$$

Using Lemma 2.17, we obtain that

$$\widetilde{D}U - U\widetilde{D} = \sum_{\substack{I\subset\mathbb{N}\\ i\in I}} \mathfrak{d}_I\mathfrak{u}_i - \sum_{\substack{I\subset\mathbb{N}\\ i\in I}} \mathfrak{u}_i\mathfrak{d}_I.$$

Now consider a fixed set $I \subset \mathbb{N}$ and $i \in I$. We will next show that the operator $\mathfrak{d}_I\mathfrak{u}_i$ corresponds to either to a unique operator $\mathfrak{u}_{i'}\mathfrak{d}_{I'}$ where $i' \in I'$, or an operator $a_0\mathfrak{d}_{I'}$ where I' might be the empty set.

Let $j \in I$ be the smallest positive integer such that $j-1 \notin I$ but every integer k satisfying $j \leq k \leq i$ belongs to I. Consider the following sets.

$$\begin{aligned}A &= \{k \mid k \in I, k < j\}\\ B &= \{k \mid j \leq k \leq i\}\\ C &= \{k \mid k \in I, k > i\}\end{aligned}$$

Clearly, we have that $I = A \amalg B \amalg C$ where $\amalg$ denotes disjoint union. Define the set I' to be $A \amalg (B-1) \amalg C$. Notice that I' can be the empty set (precisely in the case where A and C are empty, while $B = \{1\}$). Now we have the following sequence of equalities using Lemma 2.17 and Lemma 2.18.

$$\begin{aligned}\mathfrak{d}_I\mathfrak{u}_i &= \mathfrak{d}_A\mathfrak{d}_B\mathfrak{d}_C\mathfrak{u}_i\\ &= \mathfrak{d}_A\mathfrak{d}_B\mathfrak{u}_i\mathfrak{d}_C\\ &= \mathfrak{d}_A\mathfrak{u}_{j-1}\mathfrak{d}_{B-1}\mathfrak{d}_C\\ &= \mathfrak{u}_{j-1}\mathfrak{d}_A\mathfrak{d}_{B-1}\mathfrak{d}_C\\ &= \mathfrak{u}_{j-1}\mathfrak{d}_{I'}\end{aligned}$$

Given the invertibility of our computation, it is clear how to recover $\mathfrak{d}_I\mathfrak{u}_i$ starting from $\mathfrak{u}_{j-1}\mathfrak{d}'_I$. Furthermore, if $j \neq 1$, then we clearly have that $j-1 \in I'$. The above thus implies that

$$\sum_{\substack{I\subset\mathbb{N}\\ i\in I}} \mathfrak{d}_I\mathfrak{u}_i - \sum_{\substack{I\subset\mathbb{N}\\ i\in I}} \mathfrak{u}_i\mathfrak{d}_I = a_0 + a_0\widetilde{D}$$

thereby finishing the proof. ☐

Example 3.9. *Let $\alpha = (1,2)$. Then suppressing commas and parentheses as before, we have that*

$$\widetilde{D}(\alpha) = (02 + 11 + 10).$$

Therefore

$$\begin{aligned}\widetilde{D}U(\alpha) &= \widetilde{D}(121 + 022 + 103)\\ &= 120 + 111 + 110 + 021 + 020 + 003 + 102 + 002 + 101 + 100\end{aligned}$$

and

$$\begin{aligned}U\widetilde{D}(\alpha) &= U(02 + 11 + 10)\\ &= 021 + 003 + 111 + 102 + 101 + 002.\end{aligned}$$

Thus $(\widetilde{D}U - U\widetilde{D})(\alpha) = 2 + 11 + 1 + 12 = (\widetilde{D} + Id)(\alpha)$.

Remark 3.10. It is worth noting the connection between our results here and Fomin's work in [7]. In particular, note that the relations [7, Equation 1.9] satisfied by his box adding and box removing operators on partitions (denoted therein by u and d, respectively) are the same as those satisfied by the jdt operators and box removing operators on compositions. The relations are easy to establish in the case of partitions, but as we have seen, deriving the same relations in the case of compositions is more delicate.

Fomin then uses these operators to define generating functions $A(x)$ and $B(y)$ that add or remove horizontal strips in all possible ways respectively, and then uses [7, Equation 1.9] to prove the following commutation relation [7, Theorem 1.2].

$$A(x)B(y) = B(y)A(x)(1 - xy)^{-1}$$

He later notes that the dual graded graph nature of Young's lattice is encoded in the aforementioned identity. More precisely it follows from comparing the coefficient of xy on either side [7, Equation 1.13]. In fact, one can obtain various identities by comparing coefficients and can verify that the relations describing dual filtered graphs can be obtained by setting $y = 1$ and then subsequently comparing the coefficient of x on either side. Thus in a sense, the relations uniformly establish both the dual graded graph and the dual filtered graph structures on Young's lattice and $\mathcal{R}_c$.

We now proceed to discuss $\mathcal{L}_c$ defined using box adding operators. We will establish that this poset can also be endowed with a structure of a dual graded graph and a dual filtered graph. But the relations satisfied in this case are different than the ones we have encountered, and we cannot use Fomin's commutation relation in this setting. In fact, as we will see, the cancellations in the case of $\mathcal{L}_c$ are more subtle despite the simplicity of the action of $\mathfrak{t}$ compared to the action of $\mathfrak{u}$.

3.2. **Dual graphs and the left composition poset.** Our composition poset $\mathcal{L}_c$ with vertex set being the set of all compositions, whose rank function is given by the size of a composition and whose edge set is the cover relations is clearly a graded graph and hence also a weak filtered graph. By the definition of the cover relation $\lessdot_c$ it follows that the up operator associated with $\mathcal{L}_c$ is given by

$$U_t = \sum_{i\geq 1} \mathfrak{t}_i. \tag{3.6}$$

Example 3.11. *Let α be the composition $(2,1,3)$. Then*

$$U_t((2,1,3)) = (1,2,1,3) + (2,2,3) + (3,1,3) + (2,1,4).$$

Again $\mathcal{Q}_c$ and $\widetilde{\mathcal{Q}}_c$ are respectively a graded graph and a strong filtered graph with respective down operators D and $\widetilde{D}$.

For the remainder of this section, we will fix a composition α. This given, define the sets X and Y as follows.

$$X = \{\mathfrak{d}_I \mathfrak{t}_i \mid I \subset \mathbb{N}, i \in I, \mathfrak{d}_I \mathfrak{t}_i(\alpha) \neq 0\}$$
$$Y = \{\mathfrak{t}_i \mathfrak{d}_I \mid I \subset \mathbb{N}, i \in I, \mathfrak{t}_i \mathfrak{d}_I(\alpha) \neq 0\}$$

Consider $w = \mathfrak{t}_i \mathfrak{d}_I \in Y$. Decompose $I = A \amalg \{i\} \amalg B$ where

$$A = \{j \in I \mid j < i\}$$
$$B = \{j \in I \mid j > i\}.$$

By Lemma 2.19, we have that $w = \mathfrak{d}_A \mathfrak{t}_i \mathfrak{d}_i \mathfrak{d}_B$. Let k denote the largest part of α that is strictly less than i. We will see in the following proof that $k \geq \max(A)$. If such a part does not exist, we define k to be 0. Let $i' = k+1$. Now let $I' = A \amalg \{i'\} \amalg B$ and

$$\Phi(w) = \mathfrak{d}_{I'} \mathfrak{t}_{i'} = \mathfrak{d}_A \mathfrak{d}_{i'} \mathfrak{t}_{i'} \mathfrak{d}_B.$$

Lemma 3.12. *Let $w = \mathfrak{t}_i \mathfrak{d}_I = \mathfrak{t}_i \mathfrak{d}_A \mathfrak{d}_i \mathfrak{d}_B = \mathfrak{d}_A \mathfrak{t}_i \mathfrak{d}_i \mathfrak{d}_B \in Y$ and let $w' = \Phi(w)$. Then the following statements hold.*

1. *$w'(\alpha) = w(\alpha)$ if $i = 1$.*
2. *$w'(\alpha) = w(\alpha)$ if $i \geq 2$ and i is not the smallest part of $\mathfrak{d}_B(\alpha)$.*
3. *$w'(\alpha) = (0, w(\alpha))$ if $i \geq 2$ and i is the smallest part of $\mathfrak{d}_B(\alpha)$.*

Proof. To prove this, we will perform a case analysis. Assume first that $i = 1$. Then it follows that A is the empty set, and thus $w = \mathfrak{t}_1 \mathfrak{d}_1 \mathfrak{d}_B$ and, by our definition of the map Φ, we have that $w' = \mathfrak{d}_1 \mathfrak{d}_B \mathfrak{t}_1$. Since $w \in Y$, we know that $\mathfrak{d}_B(\alpha)$ has a part equalling 1. But then Lemma 2.20 implies that $w(\alpha) = \mathfrak{d}_1 \mathfrak{t}_1 \mathfrak{d}_B(\alpha)$. Now, in view of Lemma 2.19, we have that $\mathfrak{t}_1 \mathfrak{d}_B(\alpha) = \mathfrak{d}_B \mathfrak{t}_1(\alpha)$, thereby establishing that $w(\alpha) = \mathfrak{d}_1 \mathfrak{d}_B \mathfrak{t}_1(\alpha) = w'(\alpha)$.

Now assume that $i \geq 2$ and is such that smallest part of $\mathfrak{d}_B(\alpha)$ does not equal i. Let k be the largest part of α strictly less than i. We know that it exists (that is, does not equal 0) by our hypothesis on i. First we will establish that $k \geq \max(A)$.

If $k = i-1$, then this is immediate. Hence assume that $k \leq i-2$. If $\max(A) > k$, consider the composition $\mathfrak{t}_i\mathfrak{d}_i\mathfrak{d}_B(\alpha) = \gamma$. By the definition of k, we know that this composition does not have parts r satisfying $k < r < i$. But then $\mathfrak{d}_A(\gamma) = 0$, contradicting our assumption that $w \in Y$. Thus we must have $k \geq \max(A)$.

Next we claim that $\mathfrak{t}_i\mathfrak{d}_i\mathfrak{d}_B(\alpha) = \mathfrak{d}_{k+1}\mathfrak{t}_{k+1}\mathfrak{d}_B(\alpha)$. If $k = i-1$, then this follows from Lemma 2.20 since $w \in Y$. Therefore we assume that $k \leq i-2$. But in this case, notice that $\mathfrak{t}_i\mathfrak{d}_i\mathfrak{d}_B(\alpha)$ is exactly $\mathfrak{d}_B(\alpha)$ since $\mathfrak{d}_B(\alpha)$ has no part equalling r where $k < r < i$. Also since $\mathfrak{d}_B(\alpha)$ has no part equalling r where $k < r < i$, but does have a part equalling k, we get that $\mathfrak{d}_{k+1}\mathfrak{t}_{k+1}\mathfrak{d}_B(\alpha)$ equals $\mathfrak{d}_B(\alpha)$. Thus we obtain that in all cases

$$\mathfrak{d}_A\mathfrak{t}_i\mathfrak{d}_i\mathfrak{d}_B(\alpha) = \mathfrak{d}_A\mathfrak{d}_{k+1}\mathfrak{t}_{k+1}\mathfrak{d}_B(\alpha).$$

Using Lemma 2.19, we get that the right hand side equals $\mathfrak{d}_A\mathfrak{d}_{k+1}\mathfrak{d}_B\mathfrak{t}_{k+1}(\alpha) = w'(\alpha)$. This finishes this case.

Finally assume that $i \geq 2$ and that i is the smallest part of $\mathfrak{d}_B(\alpha)$. Since i is the smallest part of $\mathfrak{d}_B(\alpha)$ and $w \in Y$ it follows that A is the empty set. Thus we have $w = \mathfrak{t}_i\mathfrak{d}_i\mathfrak{d}_B$. Since i is the smallest part of $\mathfrak{d}_B(\alpha)$, we have that $w(\alpha) = \mathfrak{d}_B(\alpha)$. On the other hand in this case, we have that $\Phi(w) = w' = \mathfrak{d}_1\mathfrak{d}_B\mathfrak{t}_1$. By Lemma 2.19 we get that $w'(\alpha) = \mathfrak{d}_1\mathfrak{t}_1\mathfrak{d}_B(\alpha)$, and this clearly equals $(0, \mathfrak{d}_B(\alpha)) = (0, w(\alpha))$. This finishes the proof. $\square$

Remark 3.13. Observe from the above lemma that it is implicit that $\Phi : Y \to X$, and that at the level of compositions we have that $w(\alpha) = \Phi(w)(\alpha)$ for all $w \in Y$.

Lemma 3.14. *Φ is an injection from Y to X.*

Proof. Let $w_1 = \mathfrak{t}_i\mathfrak{d}_I$ and $w_2 = \mathfrak{t}_j\mathfrak{d}_J$ be distinct elements of Y. We have that

$$w_1 = \mathfrak{t}_i\mathfrak{d}_A\mathfrak{d}_i\mathfrak{d}_B$$
$$w_2 = \mathfrak{t}_j\mathfrak{d}_C\mathfrak{d}_j\mathfrak{d}_D$$

and

$$\Phi(w_1) = \mathfrak{d}_A\mathfrak{d}_{k+1}\mathfrak{d}_B\mathfrak{t}_{k+1}$$
$$\Phi(w_2) = \mathfrak{d}_C\mathfrak{d}_{m+1}\mathfrak{d}_D\mathfrak{t}_{m+1}$$

where

$$k = \text{ largest part } < i \text{ in } \mathfrak{d}_B(\alpha) \text{ (or 0)}$$
$$m = \text{ largest part } < j \text{ in } \mathfrak{d}_D(\alpha) \text{ (or 0)}.$$

Assuming that $\Phi(w_1) = \Phi(w_2)$ we must have $k = m$. Furthemore, note that this also implies the following key fact.

$$A \amalg B = C \amalg D \tag{3.7}$$

The two facts together yield that $A = C$ and $B = D$. So all we need to finish the proof is to establish that $i = j$.

At this point assume without loss of generality that $i > j$. Then we have the following.

$$\begin{aligned} w_1 &= \mathfrak{t}_i \mathfrak{d}_A \mathfrak{d}_i \mathfrak{d}_B \\ w_2 &= \mathfrak{t}_j \mathfrak{d}_A \mathfrak{d}_j \mathfrak{d}_B \end{aligned}$$

Since both these words belong to Y, we know that they act on α and give a valid (nonzero) composition. However, $w_2 \in Y$ implies that the largest part of α that is strictly less than i is weakly greater than j. But this implies that $\Phi(w_1) \neq \Phi(w_2)$, which is contrary to our assumption. Hence i must equal j, and we are done. $\square$

The next step for us is to identify the image of Y under the map Φ. The image of Y is a very special subset of X, which has the following explicit description. Let the largest part of α be m. Define Z as follows.

$$Z = \{\mathfrak{d}_I \mathfrak{t}_i \in X \mid i \leq m\}$$

Thus in other words, Z is the subset comprising of words that never add a box to the largest part. Note that by the definition of Φ we have that $\Phi(Y) \subseteq Z$ since if $w \in Y$ and $\Phi(w)$ has rightmost operator $\mathfrak{t}_j$ then $j \leq m$. Our next aim is to show that Φ actually bijects Y onto Z, which by Lemma 3.14 only requires us to find the inverse of Φ.

Consider $w \in Z = \mathfrak{d}_I \mathfrak{t}_i$. Writing $I = A \amalg \{i\} \amalg B$ in the usual way, and using Lemma 2.19 allows us to write w as shown below.

$$w = \mathfrak{d}_A \mathfrak{d}_i \mathfrak{t}_i \mathfrak{d}_B$$

Let i'' be the smallest part of $\mathfrak{d}_B(\alpha)$ weakly greater than i. This always exists by our hypothesis that $w \in Z$ thereby implying that i is not the largest part of α. We define $\Psi(w)$ to be

$$\begin{aligned} \Psi(w) &= \mathfrak{d}_A \mathfrak{t}_{i''} \mathfrak{d}_{i''} \mathfrak{d}_B \\ &= \mathfrak{t}_{i''} \mathfrak{d}_{I''} \end{aligned}$$

where $I'' = A \amalg \{i''\} \amalg B$. It is straightforward to see that if k is the largest part of α strictly less than i, $i' = k+1$, and i'' is the smallest part of $\mathfrak{d}_B(\alpha)$ weakly greater than i', then $i'' = i$ and hence

$$\Psi(\Phi(w)) = \mathfrak{d}_A \mathfrak{t}_i \mathfrak{d}_i \mathfrak{d}_B = w$$

so Ψ is the inverse of Φ. Hence

$$\Phi(Y) = Z. \tag{3.8}$$

Example 3.15. *Consider the composition $\alpha = (2, 6, 1, 4)$ and let $w = \mathfrak{t}_4 \mathfrak{d}_{\{1,4,5,6\}}$. Then $w(\alpha) = (2, 4, 0, 4)$ so $w \in Y$. We have the following decomposition for w.*

$$w = \mathfrak{d}_{\{1\}} \mathfrak{t}_4 \mathfrak{d}_4 \mathfrak{d}_{\{5,6\}}$$

Then the corresponding A, B and i are $\{1\}$, $\{5,6\}$ and 4 respectively. Our method for constructing $\Phi(w)$ requires that first we find the largest part k strictly less than i in α. So it follows that $k=2$. This implies that

$$\Phi(w) = \mathfrak{d}_{\{1,3,5,6\}}\mathfrak{t}_3 = \mathfrak{d}_{\{1\}}\mathfrak{d}_3\mathfrak{t}_3\mathfrak{d}_{\{5,6\}}$$

and hence $\Phi(w)(\alpha) = (2,4,0,4) = w(\alpha)$ and $\Phi(w) \in Z$. Lastly note that since $\mathfrak{d}_B(\alpha) = (2,4,1,4)$ we have for $\Phi(w)$ that its $i'' = 4$ and

$$\Psi(\Phi(w)) = \Psi(\mathfrak{d}_{\{1\}}\mathfrak{d}_3\mathfrak{t}_3\mathfrak{d}_{\{5,6\}}) = \mathfrak{d}_{\{1\}}\mathfrak{t}_4\mathfrak{d}_4\mathfrak{d}_{\{5,6\}} = w$$

as desired.

Consider the sets P and Q defined as follows.

$$P = \{\mathfrak{d}_i\mathfrak{t}_i \mid i \geq 1, \mathfrak{d}_i\mathfrak{t}_i(\alpha) \neq 0\}$$
$$Q = \{\mathfrak{t}_i\mathfrak{d}_i \mid i \geq 1, \mathfrak{t}_i\mathfrak{d}_i(\alpha) \neq 0\}$$

Clearly, $P \subset X$ and $Q \subset Y$. Furthermore, we have that $\Phi(Q)$ maps into P. In fact, a stronger claim holds from the discussion prior to this:

$$\Phi(Q) = P \setminus \{\mathfrak{d}_{m+1}\mathfrak{t}_{m+1}\}$$

where m is the largest part of α.

Then utilising all of the above we have the following two theorems.

Theorem 3.16. *$\mathcal{L}_c$ and $\mathcal{Q}_c$ are dual graded graphs, that is, on compositions*

$$DU_t - U_tD = Id.$$

Proof. Firstly note that DU_t corresponds to the following expansion.

$$DU_t = \sum_{i,j\geq 1}\mathfrak{d}_i\mathfrak{t}_j = \sum_{i,j\geq 1, i\neq j}\mathfrak{d}_i\mathfrak{t}_j + \sum_{k\geq 1}\mathfrak{d}_k\mathfrak{t}_k$$

Also the operator U_tD corresponds to the expansion below.

$$U_tD = \sum_{i,j\geq 1}\mathfrak{t}_j\mathfrak{d}_i = \sum_{i,j\geq 1, i\neq j}\mathfrak{t}_j\mathfrak{d}_i + \sum_{k\geq 1}\mathfrak{t}_k\mathfrak{d}_k$$

Then, on using Lemma 2.19, we obtain the following.

$$DU_t - U_tD = \sum_{k\geq 1}\mathfrak{d}_k\mathfrak{t}_k - \sum_{k\geq 1}\mathfrak{t}_k\mathfrak{d}_k \tag{3.9}$$

Taking α into account we can rewrite the above equation as stating the following.

$$(DU_t - U_tD)(\alpha) = \sum_{w\in P} w(\alpha) - \sum_{w\in Q} w(\alpha) = \mathfrak{d}_{m+1}\mathfrak{t}_{m+1}(\alpha) + \sum_{w\in Q}(\Phi(w) - w)(\alpha)$$

Now at the level of compositions we have $\sum_{w\in Q}(\Phi(w) - w)(\alpha) = 0$ by Lemma 3.12, and $\mathfrak{d}_{m+1}\mathfrak{t}_{m+1}(\alpha) = \alpha$. This implies the claim. ☐

Example 3.17. *Let $\alpha = (2,1,3)$. Then suppressing commas and parentheses as before, we have that*

$$\begin{aligned} DU_t(\alpha) &= D(1213+223+313+214) \\ &= 1203+1113+1212+213+222+303+312+204+114+213 \end{aligned}$$

and

$$\begin{aligned} U_tD(\alpha) &= U_t(203+113+212) \\ &= 1203+303+204+1113+213+114+1212+222+312. \end{aligned}$$

Thus $(DU_t - U_tD)(\alpha) = 213 = Id(\alpha)$.

Theorem 3.18. *$\mathcal{L}_c$ and $\widetilde{\mathcal{Q}}_c$ are dual filtered graphs, that is, on compositions*

$$\widetilde{D}U_t - U_t\widetilde{D} = \widetilde{D} + Id.$$

Proof. The beginning of the proof is very similar to that in Theorem 3.8 but with $\mathfrak{t}_i$ instead of $\mathfrak{u}_i$. Using Lemma 2.19 we obtain the following equality.

$$\widetilde{D}U_t - U_t\widetilde{D} = \sum_{\substack{I\subset\mathbb{N}\\ i\in I}} \mathfrak{d}_I\mathfrak{t}_i - \sum_{\substack{I\subset\mathbb{N}\\ i\in I}} \mathfrak{t}_i\mathfrak{d}_I$$

Now for the fixed composition α, we can rewrite the above equation as follows.

$$\begin{aligned} (\widetilde{D}U_t - U_t\widetilde{D})(\alpha) &= \sum_{w\in X} w(\alpha) - \sum_{w\in Y} w(\alpha) \\ &= \sum_{w\in X\setminus Z} w(\alpha) + \sum_{w\in Z} w(\alpha) - \sum_{w\in Y} w(\alpha) \end{aligned} \tag{3.10}$$

At the level of compositions, Lemma 3.12 implies that

$$\sum_{w\in Y} (\Phi(w)(\alpha) - w(\alpha)) = 0.$$

Using the above and Equation (3.8) in Equation (3.10) at the level of compositions gives

$$(\widetilde{D}U_t - U_t\widetilde{D})(\alpha) = \sum_{w\in X\setminus Z} w(\alpha).$$

Observe now that every element of $X \setminus Z$ has the form $\mathfrak{d}_A\mathfrak{d}_{m+1}\mathfrak{t}_{m+1}$ where A consists only of instances of $\mathfrak{d}_i$ where $i \leq m$ and m is the largest part of α. Furthermore we do have the possibility that A is empty. Additionally, it is easy to see that $\mathfrak{d}_{m+1}\mathfrak{t}_{m+1}$ is the identity map. The preceding discussion allows us to conclude the following equality at the level of compositions, thereby finishing the proof.

$$(\widetilde{D}U_t - U_t\widetilde{D})(\alpha) = (\widetilde{D} + Id)(\alpha)$$

□

Example 3.19. *Let $\alpha = (1, 2)$. Then suppressing commas and parentheses as before, we have that*

$$\widetilde{D}(\alpha) = (02 + 11 + 10).$$

Therefore

$$\begin{aligned}\widetilde{D}U_t(\alpha) &= \widetilde{D}(112 + 22 + 13) \\ &= 102 + 111 + 110 + 21 + 20 + 03 + 12 + 02 + 11 + 10\end{aligned}$$

and

$$\begin{aligned}U_t\widetilde{D}(\alpha) &= U_t(02 + 11 + 10) \\ &= 102 + 03 + 111 + 21 + 110 + 20.\end{aligned}$$

Thus $(\widetilde{D}U_t - U_t\widetilde{D})(\alpha) = 2 + 11 + 1 + 12 = (\widetilde{D} + Id)(\alpha)$.

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References

[1] C. Berg, F. Saliola and L. Serrano, *The down operator and expansions of near rectangular k-Schur functions*, J. Combin. Theory Ser. A 120 (2013) 623–636.

[2] N. Bergeron, T. Lam and H. Li, *Combinatorial Hopf algebras and towers of algebras – dimension, quantization and functorality*, Algebr. Represent. Theory 15 (2012) 675–696.

[3] C. Bessenrodt, K. Luoto and S. van Willigenburg, *Skew quasisymmetric Schur functions and noncommutative Schur functions*, Adv. Math. 226 (2011) 4492–4532.

[4] G. Duchamp, D. Krob, B. Leclerc and J-Y. Thibon, *Fonctions quasi-symétriques, fonctions symétriques non-commutatives, et algèbres de Hecke à $q = 0$*, C. R. Math. Acad. Sci. Paris 322 (1996) 107–112.

[5] S. Fomin, *Duality of graded graphs*, J. Algebraic Combin. 3 (1994) 357–404.

[6] S. Fomin, *Schensted algorithms for dual graded graphs*, J. Algebraic Combin. 4 (1995) 5–45.

[7] S. Fomin, *Schur operators and Knuth correspondences*, J. Combin. Theory Ser. A 72 (1995) 277–292.

[8] C. Gaetz, *Dual graded graphs and Bratelli diagrams of towers of groups*, arXiv:1803.11168

[9] I. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V. Retakh and J.-Y. Thibon, *Noncommutative symmetric functions*, Adv. Math. 112 (1995) 218–348.

[10] I. Gessel, *Multipartite P-partitions and inner products of skew Schur functions*, Combinatorics and algebra, Proc. Conf., Boulder/CO 1983, Contemp. Math. 34 (1984) 289–301.

[11] J. Haglund, K. Luoto, S. Mason and S. van Willigenburg, *Quasisymmetric Schur functions*, J. Combin. Theory Ser. A 118 (2011) 463–490.

[12] P. Hersh and S. Hsiao, *Random walks on quasisymmetric functions*, Adv. Math. 222 (2009) 782–808.

[13] T. Lam, *Signed differential posets and sign-imbalance*, J. Combin. Theory Ser. A 115 (2008) 466–484.

[14] T. Lam, *Quantized dual graded graphs*, Electron. J. Combin. 17 (2010).

[15] T. Lam and M. Shimozono, *Dual graded graphs for Kac-Moody algebras*, Algebra Number Theory 1 (2007) 451–488.

[16] A. Miller, *Differential posets have strict rank growth: a conjecture of Stanley*, Order 30 (2013) 657–662.

[17] J. Nzeutchap, *Dual graded graphs and Fomin's r-correspondences associated to the Hopf algebras of planar binary trees, quasi-symmetric functions and noncommutative symmetric functions*, FPSAC 2006.
[18] R. Patrias and P. Pylyavskyy, *Dual filtered graphs*, arXiv:1410.7683
[19] R. Stanley, *Differential posets*, J. Amer. Math. Soc. 1 (1988) 919–961.
[20] R. Stanley, *Variations on differential posets*, IMA Vol. Math. Appl. 19 (1990) 145–165.
[21] R. Stanley and F. Zanello, *On the rank function of a differential poset*, Electron. J. Combin. 19 (2012).
[22] V. Tewari, *Backward jeu de taquin slides for composition tableaux and a noncommutative Pieri rule*, Electron. J. Combin. 22 (2015).

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