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Multiplicity Free Schur, Skew Schur, and Quasisymmetric Schur Functions*

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Abstract. In this paper we classify all Schur functions and skew Schur functions that are multiplicity free when expanded in the basis of fundamental quasisymmetric functions, termed F-multiplicity free. Combinatorially, this is equivalent to classifying all skew shapes whose standard Young tableaux have distinct descent sets. We then generalize our setting, and classify all F-multiplicity free quasisymmetric Schur functions with one or two terms in the expansion, or one or two parts in the indexing composition. This identifies composition shapes such that all standard composition tableaux of that shape have distinct descent sets. We conclude by providing such a classification for quasisymmetric Schur function families, giving a classification of Schur functions that are in some sense almost F-multiplicity free.

Keywords: compositions, multiplicity free, quasisymmetric function, Schur function, skew Schur function, tableaux

1. Introduction

The algebra of symmetric functions, *Sym*, is central to algebraic combinatorics and impacts many other areas of mathematics such as representation theory and algebraic geometry. A nonsymmetric generalization of *Sym* is the algebra of quasisymmetric functions, *QSym*. *QSym* was first introduced in the 1980s as weight enumerators for P-partitions [5]. Since then quasisymmetric functions have grown in importance, arising in many areas including the theory of Hopf algebras [13], chain enumeration [4], discrete geometry via the cd-index [3], combinatorics of random walks [9], representation theory [10], and affine Grassmannians [11].

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A natural question to study in symmetric function theory is when certain expansions of symmetric functions are multiplicity free. For example, the product of two Schur functions when expanded as a sum of Schur functions [16], or two Schur P-functions expanded as a sum of Schur P-functions [1], skew Schur functions expanded as a sum of Schur functions [6, 17], Schur P-functions as a sum of Schur functions [14].

In this paper we extend this study and consider the problem of when a (skew) Schur function expanded as a sum of fundamental quasisymmetric functions is multiplicity free, termed F-multiplicity free. Fundamental quasisymmetric functions are important to consider due to their connections to representation theory, poset enumeration and permutation enumeration. In our context, the property of a skew Schur function indexed by a shape D being F-multiplicity free is equivalent to the property that the set of standard Young tableaux of shape D consists of tableaux that all have distinct descent sets. We then generalize this question to consider recently discovered quasisymmetric functions that inherit many properties of Schur functions [2, 7, 8, 12], known as quasisymmetric Schur functions.

More precisely, this paper is structured as follows. In Section 2, we review necessary background material. Then in Section 3 we classify all Schur functions and skew Schur functions that are F-multiplicity free in Theorems 3.3 and 3.4, respectively. We focus our attention on quasisymmetric Schur functions in Section 4, classifying all those with one or two terms and observing that they are F-multiplicity free in Theorem 4.4. We classify F-multiplicity free quasisymmetric Schur functions whose indexing composition has two parts in Theorem 5.1. Finally in Theorem 6.1 we classify all families of quasisymmetric Schur functions each member of which is F-multiplicity free.

2. Background and Preliminaries

2.1. Compositions and Partitions

A *composition* $\alpha = (\alpha_1, \ldots, \alpha_k)$ of n, denoted $\alpha \vDash n$, is a list of positive integers such that $\sum_{i=1}^{k} \alpha_i = n$. We call the α_i the *parts* of α and n the *size* of α, denoted $|\alpha|$. The number of parts, k, is called the *length* of α, denoted $\ell(\alpha)$, and the magnitude of the largest part is called the *width* of α, denoted $w(\alpha)$. If $\alpha_j = \cdots = \alpha_{j+m-1} = i$ we often abbreviate this sublist to i^m. A *partition* is a composition whose parts are in weakly decreasing order, that is, $\alpha_1 \geq \cdots \geq \alpha_k$. Note that every composition α determines a partition $\lambda(\alpha)$ that is obtained by arranging the parts of α in weakly decreasing order. We denote by $\langle\lambda\rangle$ the set of all compositions that determine the partition λ.

Given a composition α, there are two compositions that are closely related to α. The first is the reverse of α, $\alpha^* = (\alpha_k, \ldots, \alpha_1)$, and the second is the complement of α. To define the complement of α, $\hat{\alpha}$, recall the natural bijection between compositions of n and subsets of $[n-1] = \{1, 2, \ldots, n-1\}$:

$$\alpha = (\alpha_1, \ldots, \alpha_k) \leftrightarrow \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}\} = \mathrm{set}(\alpha).$$

Then $\hat{\alpha} = \mathrm{set}^{-1}((\mathrm{set}(\alpha)^c)$. Note that forming the reverse and the complement commutes, that is, $\hat{\alpha}^* = \widehat{\alpha^*}$.

Given compositions α, β, we say that α is a *refinement* of β (or β is a *coarsening* of α), denoted $\alpha \preccurlyeq \beta$, if summing some consecutive parts of α gives β. We also define the *concatenation* of α and β, denoted $\alpha \cdot \beta$, to be the composition consisting of the parts of α followed by the parts of β. One final special composition is the empty partition of 0, denoted $\emptyset$.

Example 2.1. If $\alpha = (2, 1, 2, 2)$ and $\beta = (3, 4)$, then $\lambda = \lambda(\alpha) = (2, 2, 2, 1)$,

$$\langle\lambda\rangle = \{(1, 2, 2, 2), (2, 1, 2, 2), (2, 2, 1, 2), (2, 2, 2, 1)\}.$$

Note that $\alpha^* = (2, 2, 1, 2)$, $\hat{\alpha} = (1, 3, 2, 1)$, $\alpha \preccurlyeq \beta$ and $\alpha \cdot \beta = (2, 1, 2, 2, 3, 4)$.

2.2. Young Diagrams and Tableaux

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_{\ell(\lambda)})$, we say that its (*Young*) *diagram*, also denoted by λ, is the array of left-justified cells with λ_i cells in row i, from the top, for $1 \leq i \leq \ell(\lambda)$. We locate cells in the diagram by their row and column indices (i, j) where $1 \leq i \leq \ell(\lambda)$ and $1 \leq j \leq \lambda_i$. Given two partitions λ and μ we say μ is *contained* in λ, denoted $\mu \subset \lambda$ if $\ell(\mu) \leq \ell(\lambda)$ and $\mu_i \leq \lambda_i$ for $1 \leq i \leq \ell(\mu)$. If $\mu \subset \lambda$ then the *skew diagram* $D = \lambda/\mu$ is the array of cells whose indices satisfy

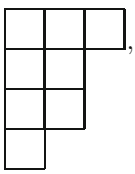
$$\lambda/\mu = \{(i, j) \mid (i, j) \in \lambda, (i, j) \notin \mu\}.$$

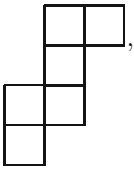
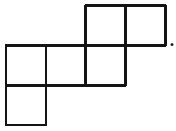
Note that every diagram λ can be considered as a skew diagram $\lambda/\emptyset$. The number of cells in a diagram D is called its *size*, denoted $|D|$. The *row lengths* of $D = \lambda/\mu$ are the numbers $\lambda_i - \mu_i$, $1 \leq i \leq \ell(\lambda)$, and $\mu_i = 0$ for $\ell(\mu) < i \leq \ell(\lambda)$; similarly for the *column lengths* of D we count the cells of D in the columns. Given two skew diagrams, D_1 and D_2, a disjoint union of them $D_1 \oplus D_2$ is obtained by placing D_2 strictly to the north and east of D_1 in such a way that D_1, D_2 occupy none of the same rows or columns. We say a skew diagram is *disconnected* if it can be written as a disjoint union of two skew diagrams, and *connected* if it cannot. Two useful operations on a skew diagram D are the *antipodal rotation* of 180° in the plane of D, denoted D°, and the *transpose* of D, denoted D^t, where D^t is the array of cells whose indices satisfy

$$D^t = \{(i, j) \mid (j, i) \in D\}.$$

If D is a Young diagram with corresponding partition λ then we call the partition λ^t corresponding to D^t the *transpose* of λ.

Example 2.2.

$\lambda = (3, 2, 2, 1) =$  , $\mu = (1, 1) =$,

$\lambda/\mu =$  , $(\lambda/\mu)^t =$  .

A filling of the cells of a (skew) diagram D with positive integers such that the entries weakly increase when read from left to right in a row, and strictly increase when read from top to bottom in a column, is called a *semi-standard Young tableau* (SSYT, plural SSYTx) T of *shape* D. If furthermore $|D| = n$ and each of $1, \ldots, n$ appears, then the filling is called a *standard Young tableau* (SYT, plural SYTx). Given an SYT T, we define its *descent set* to be the set of all i such that $i+1$ appears in a cell in a lower row than i. Since we will also consider different descent sets in the context of compositions below, we denote the descent set of an SYT by $\mathrm{des}_p(T)$. The composition $\mathrm{set}^{-1}(\mathrm{des}_p(T))$ associated to the descent set of an SYT T will be denoted by $\mathrm{com}(T)$. Given an SSYT T we say it is a *Littlewood-Richardson* (LR) tableau if its reverse reading word is a lattice permutation, i.e., as we read the entries right to left by row from top to bottom, then the number of i's we have read is always at least the number of $i+1$'s.

Example 2.3. For convenience we replace the cells by their entries when drawing tableaux:

$$\begin{array}{ll} 2\ 3 & 1\ 1 \\ 4 & 2 \\ 1\ 6 & 1\ 3 \\ 5 & 2 \end{array}$$

are respectively an SYT with descent set $\{3, 4\}$ and an SSYT that is an LR tableau.

2.3. Composition Diagrams and Tableaux

Given a composition $\alpha = (\alpha_1, \ldots, \alpha_{\ell(\alpha)})$ we say its composition diagram, also denoted α, is the array of left-justified cells with α_i cells in row i from the top, for $1 \leq i \leq \ell(\alpha)$. As with Young diagrams, cells are located by their row and column indices, and the number of cells is called the *size* of α, denoted $|\alpha|$. Also, as with Young diagrams, we will want to create tableaux, and for our subsequent proofs the most useful definition is via the following poset first introduced in [2].

Definition 2.4. *We say that the composition* γ covers β*, denoted* $\beta \lessdot \gamma$*, if* γ *can be obtained from* β *either by prepending* β *with a new part of size* 1 *(i.e.,* $\gamma = (1) \cdot \beta$*), or by adding* 1 *to the first* (*leftmost*) *part of* β *of size* k *for some* k*. The partial order* $\leq$ *defined on the set of all compositions is the transitive closure of these cover relations, and the resulting poset we denote* $\mathcal{L}_C$*.*

Example 2.5. The composition $\beta = (2, 2, 1, 3, 2, 3)$ is covered in $\mathcal{L}_C$ by the compositions $(1, 2, 2, 1, 3, 2, 3)$, $(3, 2, 1, 3, 2, 3)$, $(2, 2, 2, 3, 2, 3)$, and $(2, 2, 1, 4, 2, 3)$.

Standard composition tableaux can be created in the following way, which we can take to be our definition by [2, Proposition 2.11].

Definition 2.6. *Let* $\alpha \vDash n$ *and*

$$\emptyset = \alpha^{n+1} \lessdot \alpha^n \lessdot \alpha^{n-1} \lessdot \cdots \lessdot \alpha^2 \lessdot \alpha^1 = \alpha$$

be a sequence of consecutive cover relations in $\mathcal{L}_C$*. If we fill the cell that differs between* α^{i+1} *and* α^i *with* i *for* $1 \leq i \leq n$ *then we call the resulting filling a* standard composition tableau (*SCT, plural SCTx*) *of* shape α.

Example 2.7. The sequence

$$\emptyset \lessdot (1) \lessdot (1,1) \lessdot (2,1) \lessdot (1,2,1) \lessdot (2,2,1) \lessdot (1,2,2,1) \lessdot (1,3,2,1) \lessdot (2,3,2,1)$$

is a sequence of consecutive cover relations in $\mathcal{L}_C$ that corresponds to the SCT T' in Example 2.8 below.

As with SYTx we can define the descent set of an SCT. However, in this case we define the *descent set* of an SCT T, denoted $\mathrm{des}_c(T)$, to be the set of all i such that $i+1$ appears in a cell weakly to the right of i. It is straightforward to see that given a composition diagram $\alpha = (\alpha_1, \ldots, \alpha_{\ell(\alpha)})$, if the cells of row i are filled with

$$1 + \sum_{j=0}^{i-1} \alpha_j, 2 + \sum_{j=0}^{i-1} \alpha_j, \ldots, \alpha_i - 1 + \sum_{j=0}^{i-1} \alpha_j, \sum_{j=0}^{i} \alpha_j$$

(where $\alpha_0 = 0$) to form a filling T, then T is an SCT of shape α with $\mathrm{des}_c(T) = \mathrm{set}(\alpha)$. We call this filling the *canonical filling* of α and say that each row is *row filled*. The composition $\mathrm{set}^{-1}(\mathrm{des}_c(T))$ associated to the descent set of an SCT T will be denoted by $\mathrm{com}(T)$. Lastly, given an SCT T and a positive integer m, we denote by $T+m$ the filling that has m added to each entry of T.

Example 2.8. The composition tableau

$$T = \begin{array}{l} 2\ 1 \\ 5\ 4\ 3 \\ 7\ 6 \\ 8 \end{array}$$

is the canonical filling of $(2,3,2,1)$ and

$$T' = \begin{array}{l} 3\ 1 \\ 5\ 4\ 2 \\ 7\ 6 \\ 8 \end{array}$$

is an SCT with $des_c(T') = \{1,3,5,7\}$.

2.4. Quasisymmetric and Symmetric Functions

The algebra of quasisymmetric functions, *QSym*, is a graded algebra

$$QSym := QSym_0 \oplus QSym_1 \oplus \cdots \subseteq \mathbb{Q}[x_1, x_2, \ldots],$$

where $QSym_0$ is spanned by $M_0 = 1$ and all other $QSym_n$ are spanned by $\{M_\alpha\}_{\alpha=(\alpha_1,\ldots,\alpha_{\ell(\alpha)})\vDash n}$ where

$$M_\alpha = \sum_{i_1 < \cdots < i_{\ell(\alpha)}} x_{i_1}^{\alpha_1} \cdots x_{i_{\ell(\alpha)}}^{\alpha_{\ell(\alpha)}}.$$

This basis is called the basis of *monomial quasisymmetric functions*. A second basis for *QSym*, called the basis of *fundamental quasisymmetric functions*, consists of $F_0 = 1$ and

$$F_\alpha = \sum_{\beta \preccurlyeq \alpha} M_\beta.$$

A third, recently discovered, basis is the basis of *quasisymmetric Schur functions*, which consists of $\mathcal{S}_0 = 1$ and $\{\mathcal{S}_\alpha\}_{\alpha \vDash n}$ that are defined as follows.

Definition 2.9. *For $\alpha \vDash n$ let the* quasisymmetric Schur function $\mathcal{S}_\alpha$ *be*

$$\mathcal{S}_\alpha = \sum_{\beta \vDash n} d_{\alpha\beta} F_\beta,$$

where $d_{\alpha\beta}$ = the number of SCTx T of shape α and $\mathrm{com}(T) = \beta$.

Example 2.10. If $n = 4$,

$$M_{(1,3)} = x_1x_2^3 + x_1x_3^3 + x_2x_3^3 + \cdots,$$
$$F_{(1,3)} = M_{(1,3)} + M_{(1,2,1)} + M_{(1,1,2)} + M_{(1,1,1,1)},$$
$$\mathcal{S}_{(1,3)} = F_{(1,3)} + F_{(2,2)},$$

from the SCTx

$$\begin{array}{ll} 1 & 2 \\ 4\ 3\ 2 & 4\ 3\ 1 \end{array}.$$

The algebra of symmetric functions, *Sym*, is a graded subalgebra of *QSym*

$$Sym := Sym_0 \oplus Sym_1 \oplus \cdots$$

with a variety of bases, the most renowned of which is the basis consisting of Schur functions, whose connection with quasisymmetric Schur functions can be described very simply as follows. Let λ be a partition, then the *Schur function* s_λ is

$$s_\lambda = \sum_{\alpha \in \langle\lambda\rangle} \mathcal{S}_\alpha. \tag{2.1}$$

A more familiar description is the following, which we state more generally for the case of skew Schur functions, and which will be crucial in this paper. We will take this to be our definition of skew Schur functions.

Definition 2.11. *[15, Theorem 7.19.7] For D a skew diagram with $|D| = n$, let the* skew Schur function s_D *be*

$$s_D = \sum_{\beta \vDash n} d_{D\beta} F_\beta,$$

where $d_{D\beta}$ = the number of SYTx T of shape D and $\mathrm{com}(T) = \beta$. *When $D = \lambda$ is a partition, we call s_λ a* Schur function.

Another useful result for us will be the expansion of a skew Schur function in terms of Schur functions, known as the *Littlewood-Richardson* (*LR*) *rule*: For D a skew diagram, the expansion of the skew Schur function s_D in the basis of Schur functions is given by

$$s_D = \sum c_{D\lambda} s_\lambda,$$

where $c_{D\lambda}$ is the number of LR tableaux of shape D with λ_1 ones, λ_2 twos, etc., where $\lambda = (\lambda_1, \lambda_2, \ldots)$.

We are now ready to present our final definition, which will be the focus of our study.

Definition 2.12. *Let G be a quasisymmetric function. Then we say*

$$G = \sum_\alpha c_\alpha F_\alpha$$

*is F-*multiplicity free *if $c_\alpha = 0$ or 1 for all compositions α.*

Remark 2.13. Note that in the case of quasisymmetric Schur functions and skew Schur functions we may rephrase this condition combinatorially in the following way. By Definition 2.9, a quasisymmetric Schur function $\mathcal{S}_\alpha$ is F-multiplicity free if and only if the SCTx of shape α all have different descent sets, and by Definition 2.11, a skew Schur function s_D is F-multiplicity free if and only if the SYTx of shape D all have different descent sets.

3. Multiplicity Free Schur and Skew Schur Functions

In this section we determine necessary and sufficient conditions for when

$$s_D = \sum_\beta d_{D\beta} F_\beta$$

is F-multiplicity free for some (skew) diagram D. We begin by resolving the situation when D is a partition. However, we first reduce the number of cases we need to consider with the following lemma.

Lemma 3.1. *Let λ be a partition. Then s_λ is F-multiplicity free if and only if s_{λ^t} is F-multiplicity free.*

Proof. Using the involution ω on quasisymmetric functions that satisfies $\omega(F_{\alpha^*}) = F_{\hat{\alpha}}$ and $\omega(s_\lambda) = s_{\lambda^t}$ [4, Section 5], if $s_\lambda = \sum_\alpha d_{\lambda\alpha} F_\alpha$ is F-multiplicity free then

$$s_{\lambda^t} = \omega(s_\lambda) = \omega\left(\sum_\alpha d_{\lambda\alpha} F_\alpha\right) = \sum_\alpha d_{\lambda\alpha} F_{\hat{\alpha}^*}$$

is F-multiplicity free. The converse then holds since transposing is an involution. ∎

We now identify two families of Schur functions that are F-multiplicity free by giving explicit formulae for them. Below, and elsewhere when the meaning is clear from the context, we will also index quasisymmetric functions by the sets $\mathrm{set}(\alpha)$ instead of the compositions α, when the formulae can be stated more succinctly in this way.

Lemma 3.2. (i) *For $n \geq 1, 0 \leq k \leq n-1$,*

$$s_{(n-k,1^k)} = \sum_{\substack{R \subseteq [n-1] \\ |R|=k}} F_R \,.$$

(ii) *For $n \geq 4$,*

$$s_{(n-2,2)} = \sum_{i=2}^{n-2} F_{\{i\}} + \sum_{j=3}^{n-1} \sum_{i=1}^{j-2} F_{\{i,j\}} \,.$$

Proof. For the first part, observe that for an SYT of shape $(n-k,1^k)$, with $t_1, \ldots, t_k$ in the first column but not in the first row, its descent set is the k-element set $R = \{t_1 - 1, \ldots, t_k - 1\} \subseteq [n-1]$. Since the entries in an SYT of shape $(n-k,1^k)$ increase along the first row and first column, for any k-element set $R \subseteq [n-1]$ we get a unique SYT of shape $(n-k,1^k)$ with descent set R. The result now follows from Definition 2.11.

For the second part, observe that an SYT of shape $(n-2,2)$ with entries $i+1, i+2$ in the second row has only one descent i, and this can only take values in $[2, \ldots, n-2]$. Meanwhile, an SYT of this shape with entries $i+1, j+1$ in the second row with $i+1 < j$ has descent set $\{i, j\}$, but this only occurs for $i \in [1, \ldots, n-3]$, $j \in [i+2, \ldots, n-1]$. Thus, note that any SYT of shape $(n-2,2)$ is determined by its descent set. The result now follows from Definition 2.11. ∎

Theorem 3.3. *Let λ be a partition of n. Then s_λ is F-multiplicity free if and only if λ or λ^t is one of*

(i) $(3,3)$ *if* $n = 6$,
(ii) $(4,4)$ *if* $n = 8$,
(iii) $(n-2,2)$ *if* $n \geq 4$,
(iv) $(n-k,1^k)$ *if* $n \geq 1$ *and* $0 \leq k \leq n-1$.

Proof. Note that by direct computation $s_{(3,3)}$ and $s_{(4,4)}$ are F-multiplicity free. Thus by Lemmas 3.1 and 3.2, the Schur functions listed are F-multiplicity free. Now all that remains is to establish that all other Schur functions are not F-multiplicity free. To do this we first note that if λ (or λ^t) is not one of those listed then the diagram λ (or λ^t) contains $(3,2,1)$ or $(4,3)$.

Observe that $s_{(3,2,1)}$ is not F-multiplicity free since it contains $F_{\{2,4\}}$ with multiplicity two, arising from the SYTx

$$\begin{array}{lll} 1\ 2\ 4 & & 1\ 2\ 6 \\ 3\ 6 & , & 3\ 4 \\ 5 & & 5 \end{array} \,. \tag{3.1}$$

Similarly, $s_{(4,3)}$ is not F-multiplicity free since it contains $F_{\{2,5\}}$ with multiplicity two, arising from the SYTx

$$\begin{array}{lll} 1\ 2\ 5\ 7 & & 1\ 2\ 4\ 5 \\ 3\ 4\ 6 & , & 3\ 6\ 7 \end{array} \,. \tag{3.2}$$

Now, if λ strictly contains $(3,2,1)$, we are either in the case $\lambda=(3,3,1)$ where we have the SYTx

$$\begin{array}{lll}1&2&4\\3&5&6\\7\end{array}, \qquad \begin{array}{lll}1&2&6\\3&4&7\\5\end{array},$$

or we can extend the SYTx in (3.1) to SYTx of shape λ with the same descent set by placing 7 in the top, third or fourth row, and then extend these by filling the remaining cells in both diagrams identically. Thus by Definition 2.11, s_λ is not F-multiplicity free.

On the other hand, if $\lambda\neq(4,4)$ strictly contains $(4,3)$, then we can extend the SYTx in (3.2) to SYTx of shape λ with the same descent set by placing 8 in the top or third row, and a further identical extension of both tableaux yields again that s_λ is not F-multiplicity free.

If $\lambda^t\neq(4,4)$ contains $(4,3)$ then similarly s_{λ^t} is not F-multiplicity free, and s_λ is not F-multiplicity free, by Lemma 3.1. ∎

Combining this theorem with the LR rule yields the following more general result.

Theorem 3.4. *Let D be a skew diagram of size n. Then s_D is F-multiplicity free if and only if up to transpose, D or D° is one of*

(i) $(3,3)$ *if* $n=6$,
(ii) $(4,4)$ *if* $n=8$,
(iii) $(n-2,2)$ *if* $n\geq 4$,
(iv) $\left(n-k,1^k\right)$ *if* $n\geq 1$ *and* $0\leq k\leq n-1$,
(v) $(n-k)\oplus\left(1^k\right)$, $0<k<n$.

Thus this is a complete list of skew diagrams D such that all SYTx of shape D have distinct descent sets.

Proof. It is an easy consequence of the LR rule that if R_D is the partition consisting of the row lengths of D listed in weakly decreasing order, and C_D is the partition consisting of the column lengths of D listed in weakly decreasing order, then s_D contains both s_{R_D} and $s_{(C_D)^t}$ as summands.

More precisely, consider the SSYT where each column c is filled with $1,2,\ldots,\ell_c$, and ℓ_c is the number of cells in column c. This is clearly an LR tableau contributing 1 to the coefficient of $s_{(C_D)^t}$. Meanwhile, to obtain the term s_{R_D} consider the SSYT formed as follows. Fill the rightmost cell of the highest not completely filled row with a 1. Then fill the rightmost cell of the second highest not completely filled row with a 2. Continue until the rightmost cell of the lowest not completely filled row is filled. Repeat with remaining unfilled cells until an SSYT is produced, which is easily checked to be an LR tableau contributing 1 to the coefficient of s_{R_D}.

It also follows from the LR rule that $s_D=s_{D^\circ}$. Therefore, by Theorem 3.3 it is sufficient to check only those skew Schur functions s_D such that R_D is one of the partitions listed there but D or D° is not one of the diagrams listed therein.

Case $R_D=(n-2,2)$ $n\geq 4$. Note that $F_{(2,n-2)}$ appears with multiplicity.

Case $R_D = \left(4^2\right)$. Using the LR rule this contains $s_{(5,3)}$ as a term in the Schur function expansion of s_D, which is not F-multiplicity free.

Case $R_D = \left(2^4\right)$. Contains either $s_{(3,2,2,1)}$ or $s_{(3,3,1,1)}$ as a term in the Schur function expansion of s_D, which is not F-multiplicity free.

Case $R_D = \left(3^2\right)$. Contains $s_{(4,2)} + s_{(3,3)}$ as a term in the Schur function expansion of s_D, so by the proof of Theorem 3.3 and Lemma 3.2 we have that $F_{(3^2)}$ appears with multiplicity.

Case $R_D = \left(2^3\right)$. Contains $s_{(3,2,1)}$ as a term in the Schur function expansion of s_D, which is not F-multiplicity free.

Case $R_D = \left(2^2, 1^{(n-4)}\right), n \geq 5$. Note that $F_{(1,2^2,1^{n-5})}$ appears with multiplicity.

Case $R_D = \left(n-k, 1^k\right), n > k$. If $D = (n-k) \oplus \left(1^k\right)$ then $s_D = s_{(n-k+1,1^{k-1})} + s_{(n-k,1^k)}$. Otherwise, if $n = 3$ then $F_{(2,1)}$ appears with multiplicity. If $n \geq 4$ then $F_{(1,2,1)\cdot\beta}$ for $\beta \vDash n-4$ will appear with multiplicity.

This concludes the proof. ∎

4. Multiplicity Free Quasisymmetric Schur Functions with One or Two Terms

A natural avenue to pursue is to establish when quasisymmetric Schur functions are F-multiplicity free. Classifying when quasisymmetric (skew) Schur functions are F-multiplicity free seems more subtle than the symmetric situation, however, we can classify certain special cases. To this end we begin by classifying all quasisymmetric Schur functions with one or two terms that are F-multiplicity free. This task is simplified by the following lemma. For simplicity of exposition we denote the shape of an SCT T by $\mathrm{sh}(T)$.

Lemma 4.1. *Let* $\alpha = (\alpha_1, \ldots, \alpha_k)$ *be a composition.*

(i) *There exists a unique SCT* T *with* $\mathrm{sh}(T) = \alpha$ *and* $\mathrm{com}(T) = \alpha$.
(ii) *If* $(\alpha_i, \alpha_{i+1}) \notin \{(m,1), (1,2) \mid m \geq 1\}$ *for some* i*, then there exists an SCT* T' *with* $\mathrm{sh}(T') = \alpha$ *and* $\mathrm{com}(T') \neq \alpha$.
(iii) *If* $(\alpha_i, \alpha_{i+1}) \notin \{(m,1), (1,2), (2,2), (2,3), (1,3) \mid m \geq 1\}$ *for some* i*, then there exist at least two SCTx* T', T''*, with* $\mathrm{sh}(T') = \mathrm{sh}(T'') = \alpha$ *and* $\mathrm{com}(T')$, $\mathrm{com}(T'') \neq \alpha$ *and* $\mathrm{com}(T') \neq \mathrm{com}(T'')$.

Proof. (i) This follows immediately by considering the canonical filling of α, which is the unique SCT T with $\mathrm{sh}(T) = \alpha$ and $\mathrm{com}(T) = \alpha$.

(ii) Now consider a composition diagram α such that there exists a pair of rows with $(\alpha_i, \alpha_{i+1}) \notin \{(m,1), (1,2)\}$. Then either $\alpha_i = 1$, $\alpha_{i+1} \geq 3$, or $\alpha_i, \alpha_{i+1} \geq 2$.

We set $x = \sum_{j=1}^{i+1} \alpha_j$ and describe the filling of rows $i, i+1$, while the remaining rows are row filled.

If $\alpha_i = 1$, $\alpha_{i+1} \geq 3$, we consider the SCT T' with rows $i, i+1$ filled

$$\begin{array}{cccccc} x-2 & & & & & \\ x & x-1 & x-3 & \cdots & x-\alpha_{i+1} & . \end{array}$$

Now assume $\alpha_i \geq 2$; if $\alpha_{i+1} = 2$, we consider

$$\begin{array}{llll} x-1 & x-2 & x-4 \cdots & x-\alpha_i-1 \\ x & x-3 & & \end{array},$$

or otherwise, if $\alpha_{i+1} > 2$ we consider

$$\begin{array}{llllll} x-2 & x-\alpha_{i+1}-1 & \cdots & x-\alpha_{i+1}-\alpha_i+1 & & \\ x & x-1 & x-3 & \cdots & x-\alpha_{i+1} & \end{array}.$$

Then in all cases we have $\mathrm{sh}(T') = \alpha$ and $\mathrm{com}(T') \neq \alpha$.

(iii) Now consider a composition diagram α such that there exists a pair of rows with

$$(\alpha_i, \alpha_{i+1}) \notin \{(m,1),(1,2),(2,2),(2,3),(1,3)\}.$$

Then either $\alpha_i = 1$, $\alpha_{i+1} \geq 4$, or $\alpha_i = 2$, $\alpha_{i+1} \geq 4$, or $\alpha_i \geq 3$, $\alpha_{i+1} \geq 2$.

Again, we set $x = \sum_{j=1}^{i+1} \alpha_j$ and focus on the filling of rows $i, i+1$.

First assume $\alpha_i = 1$, $\alpha_{i+1} \geq 4$. We consider the SCTx T', T'', with rows $i, i+1$ respectively filled

$$\begin{array}{llllll} x-2 & & & & & \\ x & x-1 & x-3 & \cdots & x-\alpha_{i+1} & \end{array}$$

and

$$\begin{array}{llllll} x-3 & & & & & \\ x & x-1 & x-2 & \cdots & x-\alpha_{i+1} & \end{array}.$$

In the case $\alpha_i = 2$, $\alpha_{i+1} \geq 4$, consider

$$\begin{array}{llllll} x-2 & x-4 & & & & \\ x & x-1 & x-3 & x-5 & \cdots & x-\alpha_{i+1}-1 \end{array}$$

and

$$\begin{array}{llllll} x-3 & x-4 & & & & \\ x & x-1 & x-2 & x-5 & \cdots & x-\alpha_{i+1}-1 \end{array}.$$

Finally, assume $\alpha_i \geq 3$; if $\alpha_{i+1} = 2$ we consider

$$\begin{array}{llll} x-1 & x-2 & x-4 \cdots & x-\alpha_i-1 \\ x & x-3 & & \end{array}$$

and

$$\begin{array}{llll} x-1 & x-2 & x-3 \cdots & x-\alpha_i-1 \\ x & x-4 & & \end{array},$$

or otherwise, if $\alpha_{i+1} > 2$, consider

$$\begin{array}{lll} x-3 & x-4 & \cdots \\ x & x-1 & x-2 \cdots \end{array}$$

and

$$\begin{array}{lll} x-2 & x-4 & \cdots \\ x & x-1 & x-3 \cdots \end{array},$$

and the remaining cells of these rows are filled identically. In all cases the remaining rows are row filled. Then in all cases we have $\mathrm{sh}(T') = \mathrm{sh}(T'') = \alpha$ with $\mathrm{com}(T'), \mathrm{com}(T'') \neq \alpha$ and $\mathrm{com}(T') \neq \mathrm{com}(T'')$. ∎

Note that the first part of Lemma 4.1 says that F_α is always a summand of $\mathcal{S}_\alpha$ with coefficient 1, and was also established in [7, Lemma 5.4].

The following result implies that being F-multiplicity free is inherited in certain cases.

Lemma 4.2. *Let α be a composition and $\mathcal{S}_\alpha = \sum_\beta d_{\alpha\beta} F_\beta$. Then*

$$\mathcal{S}_{\alpha\cdot(1)} = \sum_\beta d_{\alpha\beta} F_{\beta\cdot(1)}$$

and

$$\mathcal{S}_{\alpha\cdot(1,2)} = \sum_\beta d_{\alpha\beta} F_{\beta\cdot(1,2)}.$$

Proof. Note that there exists a bijection between SCTx of shape α and $\alpha\cdot(1)$: Simply append (for the inverse, remove) the cell containing $|\alpha|+1$. Similarly, there exists a bijection between SCTx of shape α and $\alpha\cdot(1,2)$: Simply append (for the inverse, remove) the cells containing $|\alpha|+1, |\alpha|+2, |\alpha|+3$ by row filling the final two rows. ∎

This motivates us to define the following special set of compositions, which also contains the empty composition:

$$C_2 = \{(1^{e_1}, 2, 1^{e_2}, \ldots, 2, 1^{e_k}) \mid k \in \mathbb{N}_0, e_i \in \mathbb{N} \text{ for } i \in [k-1], e_k \in \mathbb{N}_0\}.$$

Corollary 4.3. *Let α be a composition and $\mathcal{S}_\alpha = \sum_\beta d_{\alpha\beta} F_\beta$. Then for any $\gamma \in C_2$,*

$$\mathcal{S}_{\alpha\cdot\gamma} = \sum_\beta d_{\alpha\beta} F_{\beta\cdot\gamma}.$$

We define the following set of compositions C_2', which is a subset of C_2:

$$C_2' = \{(1^{e_1}, 2, 1^{e_2}, \ldots, 2, 1^{e_{k-1}}, 2) \mid k-1 \in \mathbb{N}, e_i \in \mathbb{N} \text{ for } i \in [k-1]\}.$$

When $\mathcal{S}_\alpha = \sum_\beta d_{\alpha\beta} F_\beta$, the number $|\{\beta \mid d_{\alpha\beta} \neq 0\}|$ is the *number of F-components* of $\mathcal{S}_\alpha$.

Theorem 4.4. *Let α be a composition.*

(i) *$\mathcal{S}_\alpha$ has only one F-component, and moreover $\mathcal{S}_\alpha = F_\alpha$, if and only if $\alpha = (m)\cdot\gamma$ for some $\gamma \in C_2$, $m \in \mathbb{N}_0$. (Here $m = 0$ should be understood as non-appearing in the composition.)*

(ii) *$\mathcal{S}_\alpha$ has two F-components, and moreover $\mathcal{S}_\alpha = F_\alpha + F_\beta$ where $\beta \neq \alpha$, if and only if*

(I) *$\alpha = (m)\cdot\gamma'\cdot(2)\cdot\gamma$ and $m \in \mathbb{N}_0$, $\gamma' \in C_2'$, $\gamma \in C_2$ or*
(II) *$\alpha = (1,3)\cdot\gamma$, $\gamma \in C_2$ or*
(III) *$\alpha = (m)\cdot\gamma'\cdot(3)\cdot\gamma$ and $m \in \mathbb{N}_0$, $\gamma' \in C_2'$, $\gamma \in C_2$.*

In particular, for all the compositions listed above, $\mathcal{S}_\alpha$ is F-multiplicity free.

Proof. The first part follows from Lemma 4.1 and Corollary 4.3, and alternatively is proved directly in [7, Corollary 6.9].

For the second part, by considering all consecutive pairs of parts in α, we immediately deduce that if α is not one of the compositions listed in the theorem, then $\mathcal{S}_\alpha$ consists of more than two different terms F_β by the SCTx created in the proof of Lemma 4.1, unless we have three consecutive rows $i, i+1, i+2$ of length $m, 1, 3$ for some $m \geq 1$; or $2, 2, 2$; or $2, 2, 3$.

In the case $m, 1, 3$ for some $m \geq 1$, we can obtain three fillings for these rows:

$$\begin{array}{ccc} x & x-1 & x-2 \cdots \\ x+1 & & \\ x+4 & x+3 & x+2 \end{array} \qquad \begin{array}{ccc} x & x-1 & x-2 \cdots \\ x+2 & & \\ x+4 & x+3 & x+1 \end{array} \qquad \begin{array}{ccc} x+1 & x-1 & x-2 \cdots \\ x+2 & & \\ x+4 & x+3 & x \end{array},$$

where $x = \sum_{j=1}^{i} \alpha_j$; when $m < 3$, the first row is accordingly shortened.

Similarly, in the case $2, 2, 2$, we can obtain three fillings for these rows:

$$\begin{array}{cc} x & x-1 \\ x+2 & x+1 \\ x+4 & x+3 \end{array} \qquad \begin{array}{cc} x & x-1 \\ x+3 & x+2 \\ x+4 & x+1 \end{array} \qquad \begin{array}{cc} x+2 & x+1 \\ x+3 & x \\ x+4 & x-1 \end{array},$$

where $x = \sum_{j=1}^{i} \alpha_j$.

Also, in the case $2, 2, 3$, we can obtain three fillings for these rows:

$$\begin{array}{ccc} x & x-1 & \\ x+2 & x+1 & \\ x+5 & x+4 & x+3 \end{array} \qquad \begin{array}{ccc} x & x-1 & \\ x+3 & x+1 & \\ x+5 & x+4 & x+2 \end{array} \qquad \begin{array}{ccc} x+1 & x & \\ x+3 & x-1 & \\ x+5 & x+4 & x+2 \end{array},$$

where $x = \sum_{j=1}^{i} \alpha_j$.

We still have to check that the $\mathcal{S}_\alpha$ for the α listed above expand into exactly two fundamental quasisymmetric functions. Note that by the cover relations in $\mathcal{L}_C$, SCTx of shape listed in the cases above will be created almost uniquely by being row filled, apart from the rows corresponding to $(\alpha_i, \alpha_{i+1}) = (2,2), (1,3), (2,3)$. The cover relations on $\mathcal{L}_C$ yield that there are exactly two ways to fill these rows, respectively, where $x = \sum_{j=1}^{i+1} \alpha_j$:

$$\begin{array}{cc} x-2 & x-3 \\ x & x-1 \end{array} \quad \text{and} \quad \begin{array}{cc} x-1 & x-2 \\ x & x-3 \end{array},$$

$$\begin{array}{ccc} x-3 & & \\ x & x-1 & x-2 \end{array} \quad \text{and} \quad \begin{array}{ccc} x-2 & & \\ x & x-1 & x-3 \end{array},$$

$$\begin{array}{ccc} x-3 & x-4 & \\ x & x-1 & x-2 \end{array} \quad \text{and} \quad \begin{array}{ccc} x-2 & x-4 & \\ x & x-1 & x-3 \end{array}.$$

∎

5. Multiplicity Free Quasisymmetric Schur Functions with Two Parts

In this section we continue our classification of F-multiplicity free quasisymmetric Schur functions $\mathcal{S}_\alpha$. The case when α has only one part we classified in the previous section; when α has exactly two parts we have the following theorem.

Theorem 5.1. *Let α be a composition of n with two parts. Then $\mathcal{S}_\alpha$ is F-multiplicity free if and only if α is one of*

(i) $\alpha = (n-1,1), (1,n-1)$, *for* $n \geq 2$,
(ii) $\alpha = (n-2,2), (2,n-2)$, *for* $n \geq 4$,
(iii) $\alpha = (n-3,3)$, *for* $n \geq 6$,
(iv) $\alpha = (3,4), (4,4), (4,5)$.

We will prove this theorem via the following lemmas, although parts of the result could also be obtained from Theorem 3.3 and Lemma 3.2.

Lemma 5.2. (i) *For* $n \geq 3$,

$$\mathcal{S}_{(n-1,1)} = F_{\{n-1\}},$$

$$\mathcal{S}_{(1,n-1)} = \sum_{i=1}^{n-2} F_{\{i\}}.$$

(ii) *For* $n \geq 5$,

$$\mathcal{S}_{(n-2,2)} = F_{\{n-2\}} + \sum_{i=1}^{n-3} F_{\{i,n-1\}},$$

$$\mathcal{S}_{(2,n-2)} = \sum_{i=2}^{n-3} F_{\{i\}} + \sum_{j=3}^{n-2} \sum_{i=1}^{j-2} F_{\{i,j\}}.$$

(iii) *For* $n \geq 7$,

$$\mathcal{S}_{(n-3,3)} = F_{\{n-3\}} + F_{\{n-4,n-2\}} + \sum_{i=1}^{n-5} F_{\{i,n-2\}} + \sum_{i=2}^{n-4} F_{\{i,n-1\}} + \sum_{j=3}^{n-2} \sum_{\substack{i,j \in [n-3] \\ j \neq i+1}} F_{\{i,j,n-1\}}.$$

Proof. (i) The expression for $\mathcal{S}_{(n-1,1)}$ is already contained in Theorem 4.4. The expression for $\mathcal{S}_{(1,n-1)}$ then follows from Lemma 3.2 and the formula for $\mathcal{S}_{(n-1,1)}$.

(ii) Consider a composition diagram of shape $(n-2,2)$. In order to create an SCT of shape α, if the second row is filled with

$$n\ i$$

then the first row is filled with $\{1,\ldots,n-1\}\backslash\{i\}$ in decreasing order from left to right. Thus, all we need to do is to determine which values i can take using the cover relations in $\mathcal{L}_C$. This now yields that $i \neq n-2$, and the expression for $\mathcal{S}_{(n-2,2)}$ immediately follows. The expression for $\mathcal{S}_{(2,n-2)}$ then follows from Lemma 3.2 and the formula for $\mathcal{S}_{(n-2,2)}$.

(iii) Finally, consider a composition diagram of shape $(n-3,3)$. In order to create an SCT of this shape, the second row must be filled with $\{i, j, n\}$ and the first row filled with $\{1,\ldots,n-1\}\backslash\{i,j\}$ in decreasing order from left to right. For the cover relations of $\mathcal{L}_C$ to be satisfied, $\{i, j\}$ must be one of

(1) $\{n-2, n-1\}$,
(2) $\{n-3, n-1\}$,
(3) $\{i, n-1\}$ for $1 \le i \le n-5$,
(4) $\{i, i+1\}$ for $1 \le i \le n-5$,
(5) $\{i, j\}$ for $1 \le i, j \le n-3$ and $i < j \ne i+1$,

which results in the five types of summands listed above, respectively. ∎

Lemma 5.3. *Let $n \ge 8$. Then $F_{\{2,5\}}$ appears in the expansion of $\mathcal{S}_{(3,n-3)}$ with multiplicity.*

Proof. Consider the SCT T' of shape $(3, n-3)$, whose first row is filled with $\{1, 2, 5\}$ in decreasing order from left to right, and second row is filled with $\{3, 4, 6, \ldots, n\}$ in decreasing order from left to right. Also consider the SCT T'' of shape $(3, n-3)$, whose first row is filled with $\{2, 4, 5\}$ in decreasing order from left to right, and second row is filled with $\{1, 3, 6, \ldots, n\}$ in decreasing order from left to right. Then $\mathrm{des}(T') = \mathrm{des}(T'') = \{2, 5\}$, and the result follows. ∎

Lemma 5.4. *Let $n \ge 9$ and $n-m > m \ge 4$. Then $F_{\{2,5,n-m+3\}}$ appears in the expansion of $\mathcal{S}_{(n-m,m)}$ with multiplicity.*

Proof. As in the proof of Lemma 5.3, we give two sets of integers to fill a row, thus determining two SCTx T', T'' with $\mathrm{des}(T') = \mathrm{des}(T'') = \{2, 5, n-m+3\}$. In this case, the two sets to fill the first row with are

$$\{1, 3, 6, 7, \ldots, n-m+3\} \quad \text{and} \quad \{3, 4, 6, 7, \ldots, n-m+3\}.$$ ∎

Lemma 5.5. *Let $n \ge 10$ and $n-m > m \ge 4$. Then $F_{\{2,5,n-m+2\}}$ appears in the expansion of $\mathcal{S}_{(m,n-m)}$ with multiplicity.*

Proof. As in the proof of Lemma 5.3 we give two sets of integers to fill a row, thus determining two SCTx T', T'' with $\mathrm{des}(T') = \mathrm{des}(T'') = \{2, 5, n-m+2\}$. If $n-m > m+1$, the two sets to fill the first row with are

$$\{1, 2, 5, n-2m+6, \ldots, n-m+2\} \quad \text{and} \quad \{2, 4, 5, n-2m+6, \ldots, n-m+2\}.$$

However, if $n-m = m+1$ the two sets to fill the first row with are

$$\{1, 2, 5, 7, \ldots, n-m+2\} \text{ and } \{1, 2, 4, 5, 8, \ldots, n-m+2\}.$$ ∎

We are now ready to prove Theorem 5.1.

Proof of Theorem 5.1. Let α be a composition with two parts. By direct calculation we see that $\mathcal{S}_{(4,5)}$ and $\mathcal{S}_{(3,4)}$ are F-multiplicity free. Since $s_\lambda = \sum_{\lambda(\alpha)=\lambda} \mathcal{S}_\alpha$, it follows that $s_{(m,m)} = \mathcal{S}_{(m,m)}$, and hence $\mathcal{S}_{(m,m)}$ is F-multiplicity free if and only if $(m,m) \in \{(1,1), (2,2), (3,3), (4,4)\}$ by Theorem 3.3. The remaining $\mathcal{S}_\alpha$ stated in the theorem are F-multiplicity free by Lemma 5.2. For all other α, $\mathcal{S}_\alpha$ is not F-multiplicity free by Lemmas 5.3, 5.4, and 5.5. ∎

6. Multiplicity Free Quasisymmetric Schur Function Families

Refining our results from Section 3, we can also identify partitions λ such that $\mathcal{S}_\alpha$ is F-multiplicity free for all $\alpha \in \langle\lambda\rangle$. This gives us a classification, in some sense, of Schur functions that are almost F-multiplicity free, since

$$s_\lambda = \sum_{\alpha\in\langle\lambda\rangle} \mathcal{S}_\alpha.$$

Note that in general it is not true that the quasisymmetric Schur functions indexed by $\alpha \in \langle\lambda\rangle$ all behave in the same way; for example, $\mathcal{S}_{(2,3,3)}$ is F-multiplicity free while $\mathcal{S}_{(3,3,2)}$ is not. Additionally, the classification we want to obtain yields further quasisymmetric Schur functions that are F-multiplicity free. We accomplish this task by the following theorem, whose proof we devote the rest of this section to.

Theorem 6.1. *Let λ be a partition of $n \in \mathbb{N}$. Then $\mathcal{S}_\alpha$ is F-multiplicity free for all $\alpha \in \langle\lambda\rangle$ if and only if λ is one of*

(i) $\left(n-k, 1^k\right)$ *if* $0 \le k \le n-1$,
(ii) $\left(n-2-k, 2, 1^k\right)$ *if* $0 \le k \le n-5$,
(iii) $\left(2^a, 1^{n-2a}\right)$ *if* $2 \le a \le 4$ *and* $2a \le n$,
(iv) $\left(3, 2, 2, 1^{n-7}\right)$ *if* $n \ge 7$,
(v) $(3,3), (4,3), (4,4)$.

Observe that from the cover relations on $\mathcal{L}_C$ we obtain the following result.

Lemma 6.2. *Let $\alpha = \left(1^{f_1}, 2^{e_1}, 1^{f_2}, \ldots, 1^{f_{k-1}}, 2^{e_{k-1}}, 1^{f_k}\right)$, where $f_1, f_k \in \mathbb{N}_0$, $f_2, \ldots, f_{k-1} \in \mathbb{N}$, $e_1, \ldots, e_{k-1} \in \mathbb{N}$. Then*

$$\mathcal{S}_\alpha = \sum_{(\gamma_1,\ldots,\gamma_{k-1})} \left(\prod_i d_{(2^{e_i})\gamma_i}\right) F_{(1^{f_1})\cdot\gamma_1\cdot(1^{f_2})\cdot\gamma_2\cdot\cdots\cdot(1^{f_{k-1}})\cdot\gamma_{k-1}\cdot(1^{f_k})},$$

where the sum runs over all $(k-1)$-tuples of compositions $(\gamma_1, \ldots, \gamma_{k-1})$ with $\gamma_i \vDash 2e_i$, $i = 1, \ldots, k-1$.

Proof. We begin by considering $\alpha = \left(1^f, 2^e\right)$ for $f \in \mathbb{N}_0$, $e \in \mathbb{N}$. Note that in any SCT of shape α the numbers $1, 2, \ldots, f$ appear in the top f rows in increasing order, and the numbers $f+1, \ldots, f+2e$ appear in the bottom e rows. Otherwise, some number $m > f$ appears in the top f rows and hence one of the numbers $1, 2, \ldots, f$ appears in a row $f+1, \ldots, f+e$, which is impossible by the cover relations on $\mathcal{L}_C$. Thus only SCTx with descent composition $\left(1^f\right)\cdot\gamma$, $\gamma \vDash 2e$, can appear, with multiplicity $d_{(2^e)\gamma}$.

As a consequence, if $\alpha = \left(1^{f_1}, 2^{e_1}, 1^{f_2}, \ldots, 1^{f_{k-1}}, 2^{e_{k-1}}, 1^{f_k}\right) \vDash n$ then an SCT of shape α can only be constructed by placing $n, n-1, \ldots, n-f_k+1$ in increasing order in the bottom f_k rows in the first column, then placing $n-f_k, \ldots, n-f_k-2e_{k-1}+1$ in one of $d_{(2^{e_{k-1}})\gamma_{k-1}}$ ways in the next e_{k-1} rows and first and second columns, with $\gamma_{k-1} \vDash 2e_{k-1}$. Similarly fill the $f_{k-i}+e_{k-i-1}$ rows, until the final f_1 rows are filled uniquely with $1, \ldots, f_1$.

In this way we construct $\prod_i d_{(2^{e_i})\gamma_i}$ SCTx of shape α and descent composition $\gamma = \left(1^{f_1}\right)\cdot\gamma_1\cdot\left(1^{f_2}\right)\cdot\gamma_2\cdot\cdots\cdot\left(1^{f_{k-1}}\right)\cdot\gamma_{k-1}\cdot\left(1^{f_k}\right)$, $\gamma_i \vDash 2e_i$ for $i = 1, \ldots, k-1$, and we are done. ∎

Corollary 6.3. *Let* $\alpha = \left(1^{f_1}, 2^{e_1}, 1^{f_2}, \ldots, 1^{f_{k-1}}, 2^{e_{k-1}}, 1^{f_k}\right)$ *be as above. Then* $\mathcal{S}_\alpha$ *is* F*-multiplicity free if and only if* $e_i \leq 4$ *for all* i.

Proof. By Theorem 3.3, $\mathcal{S}_{(2^a)} = s_{(2^a)}$ is F-multiplicity free if and only if $0 \leq a \leq 4$. Hence the claim follows by Lemma 6.2. ∎

Corollary 6.4. *Let* $\lambda = \left(2^a, 1^{n-2a}\right)$*, with* $0 \leq a \leq n/2$*. Then* $\mathcal{S}_\alpha$ *is* F*-multiplicity free for all* $\alpha \in \langle \lambda \rangle$ *if and only if* $a \leq 4$.

Lemma 6.5. *Let* $\alpha \in \left\langle \left(n-2-k, 2, 1^k\right) \right\rangle$ *for* $0 \leq k \leq n-5$*. Then* $\mathcal{S}_\alpha$ *is* F*-multiplicity free.*

Proof. By Corollary 4.3 we may drop a trailing sequence of ones from α; thus we only have to consider the following three cases.

Case $\alpha = \left(1^{k_1}, n-2-k, 1^{k_2}, 2\right)$*, where* $k_1 + k_2 = k, k_2 > 0$. By Lemma 3.2, $s_{\left(n-2-k, 1^{k_1}\right)}$ is F-multiplicity free. Thus $\mathcal{S}_{\left(1^{k_1}, n-2-k\right)}$ is F-multiplicity free by Equation (2.1), and $\mathcal{S}_\alpha$ is F-multiplicity free by Corollary 4.3.

Case $\alpha = \left(1^k, n-2-k, 2\right)$. Consider an SCT T of shape α. Let r be maximal such that $n, n-1, \ldots, n-(r-1)$ are in the last two rows of T. By the cover relations in $\mathcal{L}_C$ it follows that $r \geq 4$ and that the last row is filled by two of the r largest numbers. Then $n-r$ is the entry in row k, and the SCT is determined by the $k-1$ numbers in $\{1, \ldots, n-r-1\}$ which appear in rows 1 to $k-1$; the remaining entries appear in row $k+1$. The descent set of the SCT is then given by the entries in the first k rows together with the descent set of the subtableaux of shape $(r-2, 2)$ filled by the r largest numbers. As the Schur function $s_{(r-2,2)}$ is F-multiplicity free, then so is $\mathcal{S}_\alpha$.

Case $\alpha = \left(1^{k_1}, 2, 1^{k_2}, n-2-k\right)$*, where* $k_1 + k_2 = k$*,* $n-2-k > 2$. For T an SCT of shape α, for $k_2 > 0$ ($k_2 = 0$, respectively), the cover relations in $\mathcal{L}_C$ imply that $n, n-1$ ($n, n-1, n-2$, respectively) must appear in the last row. Let X be the set of entries in rows $k_1+1, \ldots, k_1+1+k_2$, then $X \subset \{k_1+1, \ldots, n-2\}$ ($X \subset \{k_1+1, \ldots, n-3\}$, respectively) and its two smallest elements $z > x$ are the entries in row k_1+1. Let X' be the set of entries in the first k_1 rows; note that $\max X' < x = \min X$. The descent set of T is then $X' \cup X \setminus \{x\}$ if $z = x+1$ and $X \cup X'$ otherwise. Thus the descent set determines T, and hence $\mathcal{S}_\alpha$ is F-multiplicity free. ∎

Lemma 6.6. *Let* $\alpha \in \left\langle \left(3, 2, 2, 1^{n-7}\right) \right\rangle$ *for* $n \geq 7$*. Then* $\mathcal{S}_\alpha$ *is* F*-multiplicity free.*

Proof. Note that α can be written as $\alpha = \beta \cdot \left(1^k\right) \cdot (3) \cdot \gamma$ for some suitable composition $\beta \vDash m_1$ where the last part of β is 2 (or possibly $\beta = \emptyset$), and $\gamma \vDash m_2$ such that $m_1 + m_2 + k + 3 = n$. Then by the cover relations on $\mathcal{L}_C$ it follows that to make an SCT of shape α we first need to make an SCT T of shape $(2) \cdot \gamma$.

Then we create an SCT T' of shape $1^k \cdot (3) \cdot \gamma$ from T by placing $x \in [k+1]$ in the top row of $T + (k+1)$ and $[k+1] \setminus \{x\}$ in the first column in increasing order. Then extend this to an SCT T'' of shape α by appending an SCT of shape β on top of $T' + m_1$. All SCTx created this way have distinct descent sets since $\beta, \gamma \in \langle (2, 2, 1^{r_1}) \rangle_{r_1 \geq 0} \cup \langle (2, 1^{r_2}) \rangle_{r_2 \geq 0} \cup \langle (1^{r_3}) \rangle_{r_3 \geq 0}$.

The only SCTx of shape α that are not created in this way by extending $(2)\cdot\gamma$ are those when we extend $T+(m_1+k+1)$

$$\begin{array}{ll} m_1+1 & m_1-1 \\ m_1+2 & \\ \vdots & \\ m_1+k+1 & \\ m_1+k+3\ m_1+k+2\ m_1 & \end{array}$$

or the last two parts of β are both 2 and we extend $T+(m_1+k+1)$ by one of

$$\begin{array}{ll} m_1 & m_1-2 \\ m_1+1 & m_1-3 \\ m_1+2 & \\ \vdots & \\ m_1+k+1 & \\ m_1+k+3\ m_1+k+2\ m_1-1 & \end{array} \qquad \begin{array}{ll} m_1-1 & m_1-2 \\ m_1+1 & m_1-3 \\ m_1+2 & \\ \vdots & \\ m_1+k+1 & \\ m_1+k+3\ m_1+k+2\ m_1 & \end{array}.$$

Note that in each of these additional cases the descent sets are unique and differ from those created earlier since now $\{m_1+1,\ldots,m_1+k+1\}$ are all descents. Hence $\mathcal{S}_\alpha$ is F-multiplicity free. ∎

Lemma 6.7. *If* $\mathcal{S}_\alpha=\sum_\beta d_{\alpha\beta}F_\beta$ *then*

$$d_{\alpha\beta}\leq d_{(\alpha\cdot\gamma)(\beta\cdot\gamma)}$$

and

$$d_{\alpha\beta}\leq d_{(\gamma\cdot\alpha)(\gamma\cdot\beta)},$$

for any composition γ.

Proof. Consider an SCT of shape α and descent set set(β) and extend this to an SCT of shape $\alpha\cdot\gamma$ and descent set set$(\beta\cdot\gamma)$ by letting the additional rows be row filled. The first result now follows. The second result follows by a similar argument. ∎

We can now prove the main result of this section.

Proof of Theorem 6.1. By Lemmas 3.2, 6.5, 6.6, Corollary 6.4, and direct computation, we know that the families listed in Theorem 6.1 are F-multiplicity free. All we need to do is to show that no other families are F-multiplicity free. In order to do this we identify family members that are not F-multiplicity free by noting certain quasisymmetric Schur functions $\mathcal{S}_\beta$ that are not F-multiplicity free and then applying Lemma 6.7. We sort the families $\langle\lambda\rangle$ for partitions $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_{\ell(\lambda)})$ by width $w(\lambda)=\lambda_1$.

Case $\lambda_1=m\geq 5$ *and* $\lambda_2\geq 3$. Note that $\mathcal{S}_\beta$ for $\ell(\beta)=2$ and $\beta\notin\{(m,3),(4,5)\}$ is not F-multiplicity free by Theorem 5.1.

Case $\lambda_1 = m \geq 4$ *and* $\lambda_2 = \lambda_3 = 2$. Note that $\mathcal{S}_{(2,2,m)}$ is not F-multiplicity free by the following two SCTx.

$$\begin{array}{cccccc} 3 & 1 & & & & \\ 5 & 4 & & & & \\ m & m-1 & \cdots & 6 & 2 & \end{array}, \qquad \begin{array}{cccccc} 3 & 2 & & & & \\ 5 & 1 & & & & \\ m & m-1 & \cdots & 6 & 4 & \end{array}.$$

Case $\lambda_1 = 4, \lambda_2 = 3, \lambda_3 \geq 1$. Note that $\mathcal{S}_\beta$ for $\beta \in \{(1,4,3), (2,4,3), (3,4,3)\}$ is not F-multiplicity free.

Case $\lambda_1 = \lambda_2 = 4, \lambda_3 \geq 1$. Note that $\mathcal{S}_\beta$ for $\beta \in \{(1,4,4), (2,4,4), (3,4,4), (4,4,4)\}$ is not F-multiplicity free.

Case $\lambda_1 = \lambda_2 = 3, \lambda_3 \geq 1$. Note that $\mathcal{S}_\beta$ for $\beta \in \{(1,3,3), (3,3,2), (3,3,3)\}$ is not F-multiplicity free.

Case $\lambda_1 = 3, \lambda_2 = \lambda_3 = \lambda_4 = 2$. In this case $\mathcal{S}_{(2,2,3,2)}$ is not F-multiplicity free.

Case $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 2$. In this case $\mathcal{S}_{(2^5)}$ is not F-multiplicity free. ∎

7. Further Directions

We conclude by outlining three possible directions to pursue.

As discussed in the beginning of Section 4, one natural goal would be to generalize Theorems 4.4, 5.1, and 6.1 to obtain a classification of F-multiplicity free quasisymmetric Schur functions, and then generalize this classification to encompass *skew* quasisymmetric Schur functions.

Related to this latter classification, a second goal could be to determine when a skew quasisymmetric Schur function is multiplicity free when expanded as a linear combination of quasisymmetric Schur functions.

Finally, it is possible to return to our original focus of symmetric functions. In particular, using the expansion of integral Macdonald polynomials in terms of fundamental quasisymmetric functions [7, Equation (7.11)], one could investigate when Macdonald and Hall-Littlewood polynomials are F-multiplicity free.

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References

1. Bessenrodt, C.: On multiplicity-free products of Schur *P*-functions. Ann. Combin. 6(2), 119–124 (2002)
2. Bessenrodt, C., Luoto, K., van Willigenburg, S.: Skew quasisymmetric Schur functions and noncommutative Schur functions. Adv. Math. 226(5), 4492–4532 (2011)
3. Billera, L.J., Hsiao, S.K., van Willigenburg, S.: Peak quasisymmetric functions and Eulerian enumeration. Adv. Math. 176(2), 248–276 (2003)
4. Ehrenborg, R.: On posets and Hopf algebras. Adv. Math. 119(1), 1–25 (1996)

5. Gessel, I.M.: Multipartite P-partitions and inner products of skew Schur functions. In: Greene, C. (Ed.) Combinatorics and Algebra, Contemp. Math. 34, pp. 289–301. Amer. Math. Soc., Providence, RI (1984)
6. Gutschwager, C.: On multiplicity-free skew characters and the Schubert calculus. Ann. Combin. 14(3), 339–353 (2010)
7. Haglund, J., Luoto, K., Mason, S., van Willigenburg, S.: Quasisymmetric Schur functions. J. Combin. Theory Ser. A 118(2), 463–490 (2011)
8. Haglund, J., Luoto, K., Mason, S., van Willigenburg, S.: Refinements of the Littlewood-Richardson rule. Trans. Amer. Math. Soc. 363(3), 1665–1686 (2011)
9. Hersh, P., Hsiao, S.K.: Random walks on quasisymmetric functions. Adv. Math. 222(3), 782–808 (2009)
10. Hivert, F.: Hecke algebras, difference operators, and quasi-symmetric functions. Adv. Math. 155(2), 181–238 (2000)
11. Lam, T., Lapointe, L., Morse, J., Shimozono, M.: Affine Insertion and Pieri Rules for the Affine Grassmannian. Amer. Math. Soc., Providence (2010)
12. Lauve, A., Mason, S.K.: QSym over Sym has a stable basis. J. Combin. Theory Ser. A 118(5), 1661–1673 (2011)
13. Malvenuto, C., Reutenauer, C.: Duality between quasi-symmetric functions and the Solomon descent algebra. J. Algebra 177(3), 967–982 (1995)
14. Shaw, K.M., van Willigenburg, S.: Multiplicity free expansions of Schur *P*-functions. Ann. Combin. 11(1), 69–77 (2007)
15. Stanley, R.: Enumerative Combinatorics Vol. 2. Cambridge University Press, Cambridge (1999)
16. Stembridge, J.R.: Multiplicity-free products of Schur functions. Ann. Combin. 5(2), 113–121 (2001)
17. Thomas, H., Yong, A.: Multiplicity-free Schubert calculus. Canad. Math. Bull. 53(1), 171–186 (2010)