

# LOLLIPOP AND LARIAT SYMMETRIC FUNCTIONS\*

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**Abstract.** We compute an explicit  $e$ -positive formula for the chromatic symmetric function of a lollipop graph,  $L_{m,n}$ . From here we deduce that there exist countably infinite distinct  $e$ -positive, and hence Schur-positive, bases of the algebra of symmetric functions whose generators are chromatic symmetric functions. Finally, we resolve 6 conjectures on the chromatic symmetric function of a lariat graph,  $L_{n+3}$ .

**Key words.** chromatic symmetric function, elementary symmetric function, lariat graph, lollipop graph, Schur function

**AMS subject classifications.** Primary 05E05; Secondary 05C15, 05C25

**1. Introduction.** A generalization of the chromatic polynomial of a graph that has recently been garnering much attention is the chromatic symmetric function [16]. As expected from the name, it generalizes many properties of the chromatic polynomial such as the number of acyclic orientations [16, Theorem 3.3] though not the property of deletion-contraction. However, recently a triple-deletion property has been established [14, Theorem 3.1] during the study of graphs with equal chromatic symmetric function, which we generalize to  $k$ -deletion in Proposition 5. Another property in which the chromatic symmetric function potentially differs from the chromatic polynomial is that it may distinguish nonisomorphic trees [16, p 170] and this has been one avenue of research [1, 2, 13, 14]. Due to connections to other areas such as representation theory and algebraic geometry, another active avenue of research is the positivity of the chromatic symmetric function of a given graph when expanded into elementary symmetric functions, known as  $e$ -positivity [4, 8, 9, 10, 11, 19], or when expanded into Schur functions, known as  $Schur$ -positivity [7, 15, 18]. Graphs of particular interest are indifference graphs due to their relation to imminants of Jacobi-Trudi matrices [17]. One specific well-known family of indifference graphs are lollipop graphs, which are important in the study of random walks as they are graphs that achieve the maximum possible hitting time [3], cover time [6] and commute time [12]. In [9] the chromatic symmetric function of lollipop graphs were confirmed to be  $e$ -positive indirectly, however the question still remains of an explicit combinatorial  $e$ -positive formula.

In this paper we directly prove the  $e$ -positivity of the chromatic symmetric function of lollipop graphs in Theorem 8, via a recurrence relation in Theorem 7, and consequently give an explicit combinatorial  $e$ -positive formula in Proposition 10. A useful technique is the aforementioned triple-deletion, which we generalize to  $k$ -deletion in Proposition 5. We subsequently prove the chromatic symmetric function distinguishes nonisomorphic lollipop graphs and then use  $e$ -positivity to prove there exist countably infinite distinct  $e$ -positive, and hence Schur-positive, bases of the algebra of symmetric functions whose generators are chromatic symmetric functions of connected graphs

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40 in Theorem 13. Finally in Section 4 we resolve 6 conjectures dating from 1998 on  
 41 the chromatic symmetric function of lariat graphs, which are special cases of lollipop  
 42 graphs. More precisely, 5 of these conjectures are proved in Theorem 14 and we  
 43 provide a counterexample to the sixth.

44 **2. Background.** Before we prove our results, let us recall useful combinatorial  
 45 concepts and the algebra of symmetric functions. A *partition*  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$  of  
 46  $N$ , denoted by  $\lambda \vdash N$ , is a list of positive integers whose *parts*  $\lambda_i$  satisfy  $\lambda_1 \geq \lambda_2 \geq$   
 47  $\dots \geq \lambda_\ell > 0$  and  $\sum_{i=1}^{\ell} \lambda_i = N$ . If  $\lambda$  has exactly  $a_i$  parts equal to  $i$  for  $1 \leq i \leq N$   
 48 we often denote  $\lambda$  by  $\lambda = (1^{a_1}, 2^{a_2}, \dots, N^{a_N})$ , from which we can obtain the partition  
 49 known as the *transpose* of  $\lambda$ , denoted by  $\lambda^t$  and given by  $\lambda^t = (a_1 + \dots + a_N, a_2 +$   
 50  $\dots + a_N, \dots, a_N)$  with zeros removed.

The algebra of symmetric functions is a subalgebra of  $\mathbb{Q}[[x_1, x_2, \dots]]$  and can be  
 defined as follows. We define the  *$i$ -th elementary symmetric function*  $e_i$  for  $i \geq 1$  to  
 be

$$e_i = \sum_{j_1 < j_2 < \dots < j_i} x_{j_1} \cdots x_{j_i}$$

and given a partition  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$  we define the *elementary symmetric function*  
 $e_\lambda$  to be

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_\ell}.$$

The *algebra of symmetric functions*,  $\Lambda$ , is then the graded algebra

$$\Lambda = \Lambda^0 \oplus \Lambda^1 \oplus \dots$$

where  $\Lambda^0 = \text{span}\{1 = e_0\} = \mathbb{Q}$  and for  $N \geq 1$

$$\Lambda^N = \text{span}\{e_\lambda \mid \lambda \vdash N\}.$$

In fact the elementary symmetric functions form a basis for  $\Lambda$  and the fundamental  
 theorem of symmetric functions states that

$$\Lambda = \mathbb{Q}[e_1, e_2, \dots].$$

51 Perhaps the most celebrated basis of  $\Lambda$  is the basis of Schur functions. For a partition  
 52  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell) \vdash N$ , we define the *Schur function*  $s_\lambda$  to be

$$53 \quad (1) \quad s_\lambda = \det \left( e_{\lambda_i^t - i + j} \right)_{1 \leq i, j \leq \lambda_1}$$

54 where if  $\lambda_i^t - i + j < 0$  then  $e_{\lambda_i^t - i + j} = 0$ .

55 If a symmetric function can be written as a nonnegative linear combination of  
 56 elementary symmetric functions then we say it is  *$e$ -positive*, and likewise if a symmetric  
 57 function can be written as a nonnegative linear combination of Schur functions then  
 58 we say it is *Schur-positive*. Although not clear from (1), it is a classical result that any  
 59 elementary symmetric function is Schur-positive. However, in general little is known  
 60 about the  $e$ -positivity and Schur-positivity of the symmetric function that will be our  
 61 main object of study, the chromatic symmetric function. One well-known result in  
 62 this direction is by Gasharov [7], who gave an explicit Schur-positive formula for the  
 63 chromatic symmetric function of the large class of graphs known as incomparability  
 64 graphs of  $(3+1)$ -free posets.

This function is dependent on a graph that is *finite* and *simple* and from now  
 on we will assume that all our graphs satisfy this property. This function is also

dependent on the concept of a proper colouring. Namely, given a graph,  $G$ , with vertex set  $V$  a *proper colouring*  $\kappa$  of  $G$  is a function

$$\kappa : V \rightarrow \{1, 2, \dots\}$$

65 such that if  $v_1, v_2 \in V$  are adjacent, then  $\kappa(v_1) \neq \kappa(v_2)$ . We are now ready to define  
66 the chromatic symmetric function.

DEFINITION 1. [16, Definition 2.1] For a graph  $G$  with vertex set  $V = \{v_1, v_2, \dots, v_N\}$  and edge set  $E$ , the chromatic symmetric function is defined to be

$$X_G = \sum_{\kappa} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_N)}$$

67 where the sum is over all proper colourings  $\kappa$  of  $G$ . If  $G = \emptyset$  then  $X_G = 1$ .

68 The chromatic symmetric function is a natural generalization of the *chromatic*  
69 *polynomial*,  $\chi_G(x)$ , of a graph  $G$ , which is the number of proper colourings with  $x$   
70 colours. More precisely, if  $X_G(1^x)$  means setting  $x$  variables equal to 1 and the rest  
71 equal to 0, then we have the following result.

PROPOSITION 2. [16, Proposition 2.2] Let  $G$  be a graph and  $x$  be a positive integer. Then

$$\chi_G(x) = X_G(1^x).$$

EXAMPLE 3. Let  $K_m$  for  $m \geq 1$  be the complete graph on  $m$  vertices, each pair of which are adjacent. Then when we compute  $X_{K_m}$ , since every vertex must be coloured a different colour, and given  $m$  colours this can be done in  $m!$  ways, we obtain

$$X_{K_m} = m! \sum_{i_1 < i_2 < \cdots < i_m} x_{i_1} x_{i_2} \cdots x_{i_m}$$

and hence

$$\chi_{K_m}(x) = X_{K_m}(1^x) = m! \binom{x}{m} = x(x-1)(x-2) \cdots (x-(m-1)).$$

72 The chromatic polynomial satisfies the key property of *deletion-contraction*, namely  
73 if  $G$  is a graph with edge  $\epsilon$ ,  $G - \{\epsilon\}$  denotes  $G$  with  $\epsilon$  deleted and  $G/\{\epsilon\}$  denotes  $G$   
74 with  $\epsilon$  contracted and its vertices at either end identified, then

$$75 \quad (2) \quad \chi_G(x) = \chi_{G-\{\epsilon\}}(x) - \chi_{G/\{\epsilon\}}(x).$$

76 This property is not satisfied by the chromatic symmetric function, however, recently  
77 Orellana and Scott proved the following beautiful *triple-deletion* property, which will  
78 be useful to us in the next section. It is reliant on the existence of a 3-cycle, that is,  
79 3 edges that form a triangle.

PROPOSITION 4. [14, Theorem 3.1] Let  $G$  be a graph with edge set  $E$  such that  $\epsilon_1, \epsilon_2, \epsilon_3 \in E$  form a triangle. Then

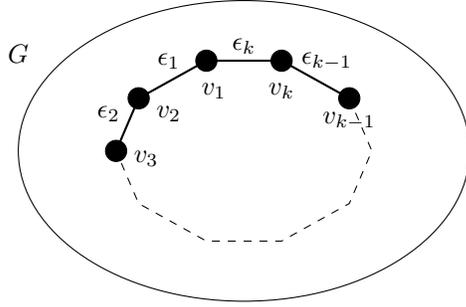
$$X_G = X_{G-\{\epsilon_1\}} + X_{G-\{\epsilon_2\}} - X_{G-\{\epsilon_1, \epsilon_2\}}.$$

80 In fact, we can generalize this to  $k$ -cycles, which we call *k-deletion*.

PROPOSITION 5. Let  $G$  be a graph with edge set  $E$  such that  $\epsilon_1, \epsilon_2, \dots, \epsilon_k \in E$  form a  $k$ -cycle for  $k \geq 3$ . Then

$$\sum_{S \subseteq [k-1]} (-1)^{|S|} X_{G - \cup_{i \in S} \{\epsilon_i\}} = 0.$$

81 *Proof.* We assume that  $G$  has a  $k$ -cycle for  $k \geq 3$ , with edges  $\epsilon_1, \epsilon_2, \dots, \epsilon_k$  and vertices  $v_1, v_2, \dots, v_k$  as below.



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We will prove the formula using a sign-reversing involution without any fixed points. Our signed set will be pairs  $(\kappa, S)$  where  $S \subseteq [k-1]$  and  $\kappa$  is a proper colouring on the graph  $G - \cup_{i \in S} \{\epsilon_i\}$ . The weight of this pair will be  $(-1)^{|S|} x_\kappa$ , that is, the monomial associated to the colouring  $\kappa$  with sign determined by  $|S|$ . Since the edge  $\epsilon_k$  is present in all graphs  $G - \cup_{i \in S} \{\epsilon_i\}$  for any  $S \subseteq [k-1]$  all colourings will have at least two colours on the vertices  $v_1, v_2, \dots, v_k$ . Hence, there will always exist a smallest  $j \in [k-1]$  where the two colours on  $v_j$  and  $v_{j+1}$  are different. We map  $(\kappa, S)$  to  $(\kappa, T)$  where  $T = S \cup \{j\}$  if  $j \notin S$  and otherwise  $T = S \setminus \{j\}$ . The colouring  $\kappa$  is a proper colouring on  $G - \cup_{i \in S} \{\epsilon_i\}$  and  $G - \cup_{i \in T} \{\epsilon_i\}$  since we are only including or excluding the edge  $\epsilon_j$ , which has adjacent vertices  $v_j$  and  $v_{j+1}$  with different colours. We can easily see that this is an involution that only switches the sign of the weight, and since there are no fixed points our desired sum is zero.  $\square$

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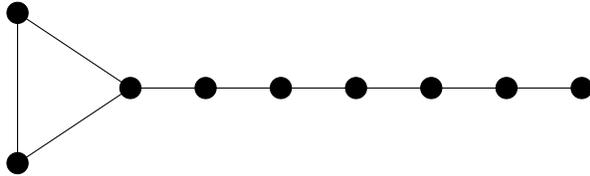
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**3. The chromatic symmetric function of lollipops.** We now come to our object of study, the chromatic symmetric function of lollipop graphs, or *lollipop symmetric functions*. However, before we define them we recall some necessary graphs, whose definitions we give for clarity. The *complete graph*  $K_m$  for  $m \geq 1$  has  $m$  vertices, each pair of which are adjacent. The *path graph*  $P_n$  for  $n \geq 1$  has  $n$  vertices and is the tree with 2 vertices of degree 1 and  $n-2$  vertices of degree 2 for  $n \geq 2$  and  $P_1 = K_1$ . From these we obtain the *lollipop graph*  $L_{m,n}$  for  $m \geq 1, n \geq 1$ , which we construct from the disjoint union of  $K_m$  and  $P_n$  by connecting with an edge one vertex of  $K_m$  and one vertex of degree 1 in  $P_n$ . For example,  $L_{3,6}$  is below.



For convenience we define

$$K_0 = P_0 = L_{0,0} = \emptyset$$

104 namely, the empty graph, and we define  $L_{m,0} = K_m$  and  $L_{0,n} = P_n$ . We are now  
105 ready to define lollipop symmetric functions.

106 DEFINITION 6. Let  $m \geq 0$  and  $n \geq 0$ . Then the lollipop symmetric function is  
107 given by  $X_{L_{m,n}}$ , that is, the chromatic symmetric function of  $L_{m,n}$ .

108 By definition  $X_{L_{m,0}} = X_{K_m}$  and  $X_{L_{0,n}} = X_{P_n}$  both of which are  $e$ -positive, and  
109 hence Schur-positive. More precisely by, for example, [5, Theorem 8]

$$110 \quad (3) \quad X_{K_m} = m!e_m$$

111 for  $m \geq 1$ , and by [19, Theorem 3.2] the coefficient of  $e_1^{a_1} e_2^{a_2} \cdots e_n^{a_n}$  in  $X_{P_n}$  for  $n \geq 1$   
112 is

$$113 \quad (4) \quad \binom{a_1 + \cdots + a_n}{a_1, \dots, a_n} \prod_{j=1}^n (j-1)^{a_j} + \sum_{i \geq 1} \left( \binom{(a_1 + \cdots + a_n) - 1}{a_1, \dots, a_i - 1, \dots, a_n} \left( \prod_{j=1, j \neq i}^n (j-1)^{a_j} \right) (i-1)^{a_i - 1} \right).$$

114 Hence a natural avenue to pursue is whether lollipop symmetric functions are  $e$ -  
115 positive, and hence Schur-positive, and if so what is an explicit combinatorial  $e$ -  
116 positive formula. The former was done implicitly in [9, Corollary 7.7], and we will  
117 now prove the former explicitly and give the latter in Proposition 10. In order to do  
118 this we need the following observation.

Note that for our results we will often have the bound  $m \geq 2$ , however, this does  
not restrict our results and is simply for clarity of exposition. This is because for  
 $n \geq 0$

$$L_{2,n} = L_{1,n+1} = L_{0,n+2} = P_{n+2}.$$

Also  $L_{1,0} = L_{0,1} = K_1 = P_1$  and we know that the chromatic symmetric function in  
this case is

$$x_1 + x_2 + \cdots = e_1 = s_1.$$

119 Finally, by definition, the chromatic symmetric function of  $L_{0,0}$  is  $1 = e_0 = s_0$ .

THEOREM 7. For  $m \geq 2$  and  $n \geq 0$  we have that

$$X_{L_{m,n}} = (m-1)X_{L_{m-1,n+1}} - (m-2)X_{K_{m-1}}X_{P_{n+1}}.$$

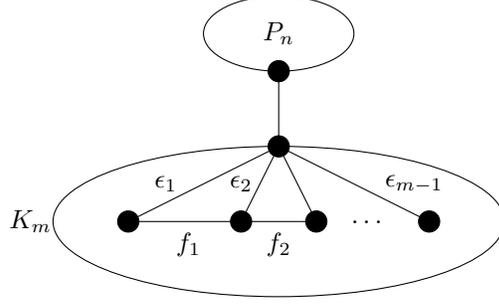
*Proof.* We will first focus on  $L_{m,n}$  in order to deduce our eventual result on  $X_{L_{m,n}}$ .  
First let  $m = 2$ . Then  $L_{2,n} = L_{1,n+1}$  and hence

$$X_{L_{2,n}} = X_{L_{1,n+1}}$$

120 as desired.

121 Now let  $m \geq 3$  and consider  $L_{m,n}$ . This graph has triangles, so we will name  
122 some edges in order to identify the triangles we want to focus on. There is a unique  
123 vertex in our graph of degree  $m$  where  $m-1$  edges belong to the copy of  $K_m$  and one  
124 edge is the bridge to the copy of  $P_n$ . Label these  $m-1$  edges  $\epsilon_1, \epsilon_2, \dots, \epsilon_{m-1}$ . Also  
125 let  $f_i$  be the edge, which together with  $\epsilon_i$  and  $\epsilon_{i+1}$  form a triangle. See the illustrative  
126 diagram below.

127 Note that if we remove any subsets  $S \subseteq \{\epsilon_1, \epsilon_2, \dots, \epsilon_{m-1}\}$  of equal size  $k$  from  
128  $L_{m,n}$  then these graphs are isomorphic, and hence without loss of generality we will  
129 focus on  $L_{m,n} - S_k$  where  $S_k = \{\epsilon_1, \epsilon_2, \dots, \epsilon_k\}$ . Three cases of note are as follows.  
130 First, if we remove  $S_0$ , then by removing no edges  $L_{m,n}$  remains unchanged, that is,



131  $L_{m,n} - S_0 = L_{m,n}$ . Second, if we remove  $S_{m-1}$ , then by removing all  $m-1$  edges  
 132 we separate  $L_{m,n}$  into two connected components, one being  $K_{m-1}$  and the other  
 133 being  $P_{n+1}$ , that is,  $L_{m,n} - S_{m-1} = K_{m-1} \cup P_{n+1}$ . Third, if we remove  $S_{m-2}$ , then  
 134 by removing all edges  $\epsilon_i$  except  $\epsilon_{m-1}$ , we obtain  $L_{m-1,n+1}$ , that is,  $L_{m,n} - S_{m-2} =$   
 135  $L_{m-1,n+1}$ .

Now let us apply Proposition 4 to  $X_{L_{m,n}-S_{k-1}}$  for  $1 \leq k \leq m-2$  using the triangle with edges  $\epsilon_k$ ,  $\epsilon_{k+1}$ , and  $f_k$ . We then get

$$X_{L_{m,n}-S_{k-1}} = X_{L_{m,n}-S_{k-1}-\{\epsilon_k\}} + X_{L_{m,n}-S_{k-1}-\{\epsilon_{k+1}\}} - X_{L_{m,n}-S_{k-1}-\{\epsilon_k, \epsilon_{k+1}\}} = 2X_{L_{m,n}-S_k} - X_{L_{m,n}-S_{k+1}}. \blacksquare$$

136 If we apply this continually to  $X_{L_{m,n}} = X_{L_{m,n}-S_0}$  we get

$$\begin{aligned} 137 \quad X_{L_{m,n}} &= 2X_{L_{m,n}-S_1} - X_{L_{m,n}-S_2} \\ 138 \quad &= 2(2X_{L_{m,n}-S_2} - X_{L_{m,n}-S_3}) - X_{L_{m,n}-S_2} \\ 139 \quad &= 3X_{L_{m,n}-S_2} - 2X_{L_{m,n}-S_3} \\ 140 \quad &= 3(2X_{L_{m,n}-S_3} - X_{L_{m,n}-S_4}) - 2X_{L_{m,n}-S_3} \\ 141 \quad &= 4X_{L_{m,n}-S_3} - 3X_{L_{m,n}-S_4} \\ 142 \quad &\vdots \\ 143 \quad &= kX_{L_{m,n}-S_{k-1}} - (k-1)X_{L_{m,n}-S_k} \\ 144 \quad &= k(2X_{L_{m,n}-S_k} - X_{L_{m,n}-S_{k+1}}) - (k-1)X_{L_{m,n}-S_k} \\ 145 \quad &= (k+1)X_{L_{m,n}-S_k} - kX_{L_{m,n}-S_{k+1}} \end{aligned}$$

147 for  $1 \leq k \leq m-2$ . In particular, at  $k = m-2$  we get

$$\begin{aligned} 148 \quad X_{L_{m,n}} &= (m-1)X_{L_{m,n}-S_{m-2}} - (m-2)X_{L_{m,n}-S_{m-1}} \\ 149 \quad &= (m-1)X_{L_{m-1,n+1}} - (m-2)X_{K_{m-1}}X_{P_{n+1}} \end{aligned}$$

151 and we are done.  $\square$

152 We can now affirm directly the  $e$ -positivity, and hence Schur-positivity, of lollipop  
 153 symmetric functions, which was proved indirectly in [9, Corollary 7.7].

154 **THEOREM 8.** *For all  $m \geq 0$  and  $n \geq 0$  we have that  $X_{L_{m,n}}$  is  $e$ -positive, and*  
 155 *hence Schur-positive.*

156 *Proof.* First note that when  $m = 0$  we have that  $X_{L_{m,n}} = X_{P_n}$ , which we know  
 157 is  $e$ -positive by (4) and Definition 1. We will now prove the result by induction on  
 158  $n$ . When  $n = 0$  we have  $X_{L_{m,0}} = X_{K_m}$ , which we know is  $e$ -positive by (3) and  
 159 Definition 1. Now assume that for any  $m \geq 1$  and  $0 < k < n$  that  $X_{L_{m,k}}$  is  $e$ -positive.

160 By Theorem 7 for  $n \geq 0$  and  $m \geq 2$  we have that

$$161 \quad X_{L_{m,n}} = (m-1)X_{L_{m-1,n+1}} - (m-2)X_{K_{m-1}}X_{P_{n+1}} \\ 162 \quad \Rightarrow X_{L_{m,n}} + (m-2)X_{K_{m-1}}X_{P_{n+1}} = (m-1)X_{L_{m-1,n+1}}.$$

164 By shifting the indices we get for  $m \geq 1$  and  $n \geq 1$  that

$$165 \quad mX_{L_{m,n}} = X_{L_{m+1,n-1}} + (m-1)X_{K_m}X_{P_n} \\ 166 \quad \Rightarrow X_{L_{m,n}} = \frac{1}{m} (X_{L_{m+1,n-1}} + (m-1)X_{K_m}X_{P_n}).$$

168 By our inductive assumption we know that  $X_{L_{m+1,n-1}}$  is  $e$ -positive. Plus, as  
169 stated earlier in the proof,  $X_{K_m}$  and  $X_{P_n}$  for all  $m, n \geq 1$  are  $e$ -positive, and since  
170 elementary symmetric functions are multiplicative  $X_{K_m}X_{P_n}$  is  $e$ -positive. Since we  
171 have now shown that  $X_{L_{m,n}}$  is a linear combination of two  $e$ -positive symmetric  
172 functions, we can conclude that  $X_{L_{m,n}}$  is  $e$ -positive, and hence Schur-positive.  $\square$

173 A natural question to ask is whether succinct formulas exist for  $X_{L_{m,n}}$  in terms  
174 of  $X_{K_m}$  and  $X_{P_n}$  since  $K_m$  and  $P_n$  are required to construct  $L_{m,n}$ . Such formulas are  
175 obtainable using Theorem 7 and the proof of Theorem 8, from which the expansion  
176 into elementary symmetric functions is immediate using (3) and (4).

PROPOSITION 9. For  $m \geq 2$  and  $n \geq 0$  we have that

$$X_{L_{m,n}} = (m-1)! \left( X_{P_{n+m}} - \sum_{i=1}^{m-2} \frac{(m-i-1)}{(m-i)!} X_{K_{m-i}} X_{P_{n+i}} \right).$$

*Proof.* We prove this by induction on  $m$  for  $m \geq 2$ . When  $m = 2$  and  $n \geq 0$  we  
have that

$$X_{L_{2,n}} = X_{P_{n+2}}$$

177 as desired. Now assume that the result holds for all  $2 < k < m$  and  $n \geq 0$ . Then by  
178 Theorem 7 we have that

$$179 \quad X_{L_{m,n}} = (m-1)X_{L_{m-1,n+1}} - (m-2)X_{K_{m-1}}X_{P_{n+1}} \\ 180 \quad = (m-1) \left( (m-2)! \left( X_{P_{n+m}} - \sum_{i=1}^{m-3} \frac{(m-i-2)}{(m-i-1)!} X_{K_{m-i-1}} X_{P_{n+i+1}} \right) \right) \\ 181 \quad \quad - (m-2)X_{K_{m-1}}X_{P_{n+1}} \\ 182 \quad = (m-1)! \left( X_{P_{n+m}} - \sum_{i=1}^{m-2} \frac{(m-i-1)}{(m-i)!} X_{K_{m-i}} X_{P_{n+i}} \right) \\ 183$$

184 by induction.  $\square$

185 This first formula will play a key role in the next section, while the second formula  
186 below yields an explicit  $e$ -positive formula for  $X_{L_{m,n}}$  using (3) and (4).

PROPOSITION 10. For  $m \geq 2$  and  $n \geq 0$  we have that

$$X_{L_{m,n}} = \frac{(m-1)!}{(m+n-1)!} X_{K_{m+n}} + \sum_{i=0}^{n-1} \frac{(m+i-1)}{m(m+1)\cdots(m+i)} X_{K_{m+i}} X_{P_{n-i}}.$$

*Proof.* We prove this by induction on  $n$  for  $n \geq 0$ . When  $n = 0$  and  $m \geq 2$  we have that

$$X_{L_{m,0}} = X_{K_m}$$

as desired. Now assume that the result holds for all  $0 < k < n$  and  $m \geq 2$ . Then by the last equation in the proof of Theorem 8 we have that

$$\begin{aligned} X_{L_{m,n}} &= \frac{1}{m} (X_{L_{m+1,n-1}} + (m-1)X_{K_m}X_{P_n}) \\ &= \frac{1}{m} \left( \frac{m!}{(m+n-1)!} X_{K_{m+n}} \right. \\ &\quad \left. + \sum_{i=0}^{n-2} \frac{(m+i)}{(m+1)\cdots(m+i+1)} X_{K_{m+i+1}}X_{P_{n-i-1}} + (m-1)X_{K_m}X_{P_n} \right) \\ &= \frac{(m-1)!}{(m+n-1)!} X_{K_{m+n}} + \sum_{i=0}^{n-1} \frac{(m+i-1)}{m(m+1)\cdots(m+i)} X_{K_{m+i}}X_{P_{n-i}} \end{aligned}$$

by induction.  $\square$

LEMMA 11. For  $m \geq 2$  and  $n \geq 0$  we have that

$$\chi_{L_{m,n}}(x) = x(x-1)^{n+1}(x-2)\cdots(x-(m-1)).$$

*Proof.* We prove this by induction on  $n$ . For  $n = 0$  and  $m \geq 2$  we have that  $L_{m,n} = K_m$  whose chromatic polynomial, which we computed in Example 3, is well known to be

$$x(x-1)(x-2)\cdots(x-(m-1)).$$

Now assume that the result holds for all  $0 < k < n$  and  $m \geq 2$ . Then by deletion-contraction where we delete an edge incident to a degree 1 vertex

$$\chi_{L_{m,n}}(x) = (x-1)\chi_{L_{m,n-1}} = (x-1)x(x-1)^n(x-2)\cdots(x-(m-1))$$

as desired.  $\square$

LEMMA 12. If  $L_{m,n}$  and  $L_{m',n'}$  satisfy  $(m,n) \neq (m',n')$  where  $m, m' \geq 2$  and  $n, n' \geq 0$ , then

$$X_{L_{m,n}} \neq X_{L_{m',n'}}.$$

*Proof.* First observe that by Lemma 11 that since  $m$  and  $n$  are recoverable from the chromatic polynomial, if  $(m,n) \neq (m',n')$  then  $\chi_{L_{m,n}}(x) \neq \chi_{L_{m',n'}}(x)$ . Furthermore by Proposition 2 the chromatic symmetric function reduces to the chromatic polynomial and hence if  $(m,n) \neq (m',n')$  then  $X_{L_{m,n}} \neq X_{L_{m',n'}}$ .  $\square$

We are now ready to prove that there exist countably infinite distinct bases arising from chromatic symmetric functions that are  $e$ -positive, and hence Schur-positive.

THEOREM 13. There exist countably infinite distinct  $e$ -positive, and hence Schur-positive, bases of the algebra of symmetric functions  $\Lambda$  whose generators are the chromatic symmetric functions of connected graphs. In particular, every distinct set of lollipops  $\{\mathcal{L}_1, \mathcal{L}_2, \dots\}$  where  $\mathcal{L}_i = L_{m_i, n_i}$  for some  $m_i + n_i = i$  gives rise to a distinct set of generators  $\{X_{\mathcal{L}_1}, X_{\mathcal{L}_2}, \dots\}$  such that

$$\Lambda = \mathbb{Q}[X_{\mathcal{L}_1}, X_{\mathcal{L}_2}, \dots].$$

*Proof.* To begin, first recall that  $L_{1,0} = L_{0,1} = K_1 = P_1$ , plus  $L_{2,0} = L_{1,1} = L_{0,2} = K_2 = P_2$ , and for a given integer  $k \geq 3$  there exist  $k - 1$  distinct lollipop graphs

$$L_{k,0}, \dots, L_{2,k-2}.$$

Hence there exist countably infinite distinct sets of lollipop graphs

$$\{L_{m_1, n_1}, L_{m_2, n_2}, \dots\}$$

where  $m_i + n_i = i$ . Given such a set  $\mathcal{L}$ , denote  $L_{m_i, n_i}$  by  $\mathcal{L}_i$ . Then by [5, Theorem 5]

$$\{X_{\mathcal{L}_{\lambda_1}} X_{\mathcal{L}_{\lambda_2}} \cdots X_{\mathcal{L}_{\lambda_\ell}} \mid (\lambda_1, \lambda_2, \dots, \lambda_\ell) \vdash N\}$$

is a  $\mathbb{Q}$ -basis for  $\Lambda^N$  and

$$\Lambda = \mathbb{Q}[X_{\mathcal{L}_1}, X_{\mathcal{L}_2}, \dots].$$

Moreover, since each  $X_{\mathcal{L}_i}$  is  $e$ -positive by Theorem 8 and elementary symmetric functions are multiplicative, it follows that each

$$X_{\mathcal{L}_{\lambda_1}} X_{\mathcal{L}_{\lambda_2}} \cdots X_{\mathcal{L}_{\lambda_\ell}}$$

above is  $e$ -positive, and hence Schur-positive. By Lemma 12 it follows that every set  $\mathcal{L}$  yields a distinct set of generators  $\{X_{\mathcal{L}_1}, X_{\mathcal{L}_2}, \dots\}$  for  $\Lambda$  and we are done.  $\square$

**4. The chromatic symmetric function of lariats.** In this final section we resolve 6 conjectures from 1998 [19, p656] regarding the chromatic symmetric function of lariat graphs, where the *lariat graph*  $L_{n+3}$  for  $n \geq 0$  is defined to be

$$L_{n+3} = L_{3,n}.$$

More precisely we prove 5 of the conjectures in Theorem 14 and conclude with a counterexample to the sixth.

Before we do, we note that setting  $m = 3$  in Proposition 9 yields the following expression for the chromatic symmetric function of  $L_{n+3}$  for  $n \geq 0$

$$(5) \quad X_{L_{n+3}} = 2 \left( X_{P_{n+3}} - \frac{1}{2} X_{K_2} X_{P_{n+1}} \right) = 2X_{P_{n+3}} - 2e_2 X_{P_{n+1}}$$

since  $X_{K_2} = 2e_2$  by (3). We are now ready to prove our theorem.

**THEOREM 14.** *Consider  $X_{L_{n+3}}$  for  $n \geq 0$  expanded into the basis of elementary symmetric functions.*

1. *If  $e_\lambda$  appears in  $X_{L_{n+3}}$  with nonzero coefficient, then  $e_\lambda$  appears in  $X_{P_{n+3}}$  with nonzero coefficient.*
2. *Let  $\lambda$  be a partition with no part equal to 2. Then if  $e_\lambda$  appears in  $X_{P_{n+3}}$  with coefficient  $c_\lambda$ , then  $e_\lambda$  appears in  $X_{L_{n+3}}$  with coefficient  $2c_\lambda$ .*
3. *The coefficient of  $e_{(n+1,2)}$  in  $X_{L_{n+3}}$  is  $4n$ .*
4. *The coefficient of  $e_{(n,2,1)}$  in  $X_{L_{n+3}}$  is  $2(n-1)$ .*
5. *The coefficient of  $e_{(n-1,2,2)}$  in  $X_{L_{n+3}}$  is  $4(n-2)$ .*

*Proof.* The first part follows from (5) and the fact that  $X_{L_{n+3}}$  is  $e$ -positive by Theorem 8 and  $X_{P_{n+3}}$  is  $e$ -positive by (4). The second part follows immediately from (5). For the third, fourth and fifth parts, respectively, we again use (5),

$$X_{L_{n+3}} = 2X_{P_{n+3}} - 2e_2 X_{P_{n+1}},$$

in conjunction with (4) to obtain the following coefficients.

- To compute the coefficient of  $e_{(n+1,2)}$  in  $X_{L_{n+3}}$ , note that its coefficient in  $2X_{P_{n+3}}$  is

$$2(2(n+1-1) + (n+1-1) + 1)$$

220 whereas in  $2e_2X_{P_{n+1}}$  it is  $2((n+1-1) + 1)$ .

- To compute the coefficient of  $e_{(n,2,1)}$  in  $X_{L_{n+3}}$ , note that its coefficient in  $2X_{P_{n+3}}$  is

$$2(2(n-1))$$

221 whereas in  $2e_2X_{P_{n+1}}$  it is  $2(n-1)$ .

- To compute the coefficient of  $e_{(n-1,2,2)}$  in  $X_{L_{n+3}}$ , note that its coefficient in  $2X_{P_{n+3}}$  is

$$2(3(n-1-1) + 2(n-1-1) + 1)$$

222 whereas in  $2e_2X_{P_{n+1}}$  it is  $2(2(n-1-1) + (n-1-1) + 1)$ . □

223 The final conjecture stated that if  $e_\lambda$ , where  $\lambda = (1^{a_1}, 2^{a_2}, \dots, (n+3)^{a_{n+3}})$ , ap-  
 224 pears with nonzero coefficient in  $X_{L_{n+3}}$  when expanded into the basis of elementary  
 225 symmetric functions, then  $a_i \leq 2$  for all  $i$ . This is false since  $L_9 = L_{3,6}$ , which is the  
 226 smallest counterexample and is shown earlier in Section 3, has chromatic symmetric  
 227 function

$$\begin{aligned} 228 \quad X_{L_9} &= 8e_{(3,2,2,2)} + 16e_{(3,3,2,1)} + 24e_{(3,3,3)} + 6e_{(4,2,2,1)} + 82e_{(4,3,2)} \\ 229 \quad &+ 18e_{(4,4,1)} + 16e_{(5,2,2)} + 32e_{(5,3,1)} + 62e_{(5,4)} + 10e_{(6,2,1)} \\ 230 \quad &+ 54e_{(6,3)} + 24e_{(7,2)} + 14e_{(8,1)} + 18e_{(9)}. \end{aligned}$$

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