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Chromatic posets ☆

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A B S T R A C T

In 1995 Stanley introduced the chromatic symmetric function X_G of a graph G, whose e-positivity and Schur-positivity has been of large interest. In this paper we study the relative e-positivity and Schur-positivity between connected graphs on n vertices. We define and investigate two families of posets on distinct chromatic symmetric functions. The relations depend on the e-positivity or Schur-positivity of a weighed subtraction between X_G and X_H. We find a biconditional criterion between e-positivity or Schur-positivity and the relation to the complete graph. This gives a new paradigm for e-positivity and for Schur-positivity. We show many other interesting properties of these posets including that the family of trees forms an independent set and are maximal elements. Additionally, we find that stars are independent elements, the independence number increases as we increase in the poset and that the family of lollipop graphs forms a chain.

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1. Introduction

The chromatic symmetric function defined by Stanley [24] has received a lot of attention lately, and is a generalization of the chromatic polynomial by Birkhoff [3]. Many properties of the chromatic polynomial are generalized by the chromatic symmetric function including number of acyclic orientations [24, Theorem 3.3], but not the property of deletion-contraction. The study of these symmetric functions has taken many directions. One direction studies which graphs are distinguished by their chromatic symmetric function or not [20]. Though all trees have the same chromatic polynomial Stanley [24, page 170] conjectures that non-isomorphic trees are distinguished by the chromatic symmetric function, which has been studied but not fully resolved [1,2,19,20]. Due to its connections with representation theory and algebraic geometry another direction has revolved around the ability to write a chromatic symmetric function as a non-negative linear combination of elementary symmetric functions or Schur symmetric functions, properties called e-positivity [5–7,11,12,14,15,29] and Schur-positivity [10,22,28] respectively. A conjecture by Stanley and Stembridge connected to immanants of Jacobi-Trudi matrices [27] has brought attention to a particular family of graphs, unit interval graphs. In this paper we particularly consider the family of lollipop graphs, which are unit interval graphs and include the families of path and complete graphs. Lollipop graphs have been proven to be e-positive [12] and have descriptive formulas [7,29]. They are important in the study of random walks [4,9,16].

In this paper we consider $\mathcal{G}_n$ the set of equivalence classes of connected graphs on n vertices determined by distinct chromatic symmetric functions. We form two posets on $\mathcal{G}_n$ with cover relations determined by the relative e-positivity and Schur-positivity between distinct chromatic symmetric functions. We find that this poset has many interesting properties including that the complete graph is a minimal element of the poset and that trees are maximal elements. We find that a graph is e-positive or Schur-positive, respectively, depending on the poset considered, if and only if that graph is weakly greater than the complete graph.

The paper is organized as follows. In Section 2 we introduce chromatic symmetric functions and our posets of interest. In particular, we show that differences of chromatic symmetric functions are neither e-positive nor Schur-positive in Theorems 2.17 and 2.16, respectively. Also we prove a biconditional criterion between e-positivity or Schur-positivity and a property of our poset, that is its relation to the complete graph in Theorem 2.22 and Theorem 2.23. Interestingly we find that a relation in one poset does not imply a relation in the other in general in Proposition 2.24. Section 3 discusses the poset related to the elementary basis and there we prove that the independence number increases as elements increase in the poset, trees form an anti-chain, trees are maximal elements, stars are independent elements, the poset is not a lattice and lollipops form a chain. In Section 4 we prove analogous results for the poset related to the Schur basis, however, the proof of lollipops being a chain is more intricate. It is instead presented in Section 5 with the analogous theorem given in Theorem 5.4.

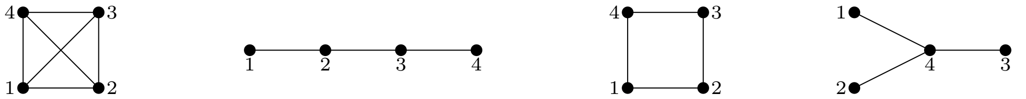


Fig. 1. From left to right we have K_4, P_4, C_4 and S_4 also called the claw.

2. Background

In this section we cover a lot of the background material needed in the rest of the paper including graphs, symmetric functions, posets and formulas for chromatic symmetric functions with sources provided for more details. After this material we will define our objects of interest, two families of posets that investigate the relative e-positivity or Schur-positivity between graphs as well as the motivation behind the relations we define. We also prove a biconditional criterion between e-positivity or Schur-positivity and a property of the poset.

A *graph* is a collection of vertices $V(G)$ and edges $E(G)$ between pairs of vertices. All throughout this paper when referring to a graph G we will be referring to simple graphs without multi-edges or loops. If not specified otherwise G is assumed to be a *connected* graph. There are a few families of graphs that we particularly refer to in this paper. The *complete graph*, K_n for $n \geq 1$, will have n vertices and all possible edges between all pairs of vertices. The *path graph*, P_n for $n \geq 1$, will be a graph on n vertices labeled by $[n] = \{1, 2, \ldots, n\}$ with edges between labels i and $i+1$. Note that $P_1 = K_1$. The *cycle graph*, C_n for $n \geq 3$, will be the path graph with the additional edge between 1 and n. The *star graph*, S_n for $n \geq 4$, will have n vertices labeled by $[n]$ with edges between n and j for all $j \in [n-1]$. The *lollipop graph*, $L_{m,n}$ for $m \geq 1, n \geq 0$, will be a graph on $m + n = N$ vertices labeled with $[N]$ and will have a complete graph on $[m]$ and edges between i and $i+1$ for $i \in [m, N-1] = \{m, m+1, \ldots, N-1\}$. See Figs. 1 and 4 for examples.

When restricting G to a subset of vertices $W \subseteq V(G)$ we are referring to the graph on vertices W with all edges that G has between vertices in W. The *independence number* of a graph G, $\alpha(G)$, is the maximal size of a subset $W \subseteq V(G)$ where G restricted to W has no edges. The *clique number* of a graph G, $\omega(G)$, is the maximal size of a subset $W \subseteq V(G)$ where G restricted to W is a complete graph. When restricting G to a subset of edges $F \subseteq E(G)$ we are referring to the graph G, but with the smaller edge set F.

In order to define the chromatic symmetric function we will need to define proper coloring. A *proper coloring* of a graph G is a map from the vertices $V(G) = \{v_1, v_2, \ldots, v_n\}$ to colors $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$,

$$\kappa : V(G) \to \mathbb{Z}^+,$$

so that if $\epsilon \in E(G)$ is an edge between vertices $u, v \in V(G)$ then $\kappa(u) \neq \kappa(v)$. The *chromatic symmetric function* is

$$X_G = \sum_{\kappa} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_n)}$$

summed over all proper colorings κ of G. The chromatic symmetric function generalizes the *chromatic polynomial* $\chi_G(k)$ that counts the number of proper colorings possible for G using at most k colors. Stanley [24, Proposition 2.2] showed that X_G is indeed a generalization of $\chi_G(k)$ because

$$X_G(1^k) = \chi_G(k)$$

where $X_G(1^k)$ means that we substitute in 1 for any k distinct variables and zero for the others. The chromatic polynomial generalizes the *chromatic number*, $\chi(G)$, the minimum number of colors required for a proper coloring.

The chromatic polynomial satisfies a very useful deletion-contraction property, though X_G does not. Given a graph G and an edge $\epsilon \in E(G)$ the *deletion* of ϵ, $G - \epsilon$, is the graph G with edge ϵ removed. The *contraction* along ϵ, G/ϵ, is the graph G but with the two vertices on ϵ merged with any multi-edges formed merged into a single edge and loops removed. The *deletion-contraction property* is that for any graph G and edge $\epsilon \in E(G)$ we have that

$$\chi_G(k) = \chi_{G-\epsilon}(k) - \chi_{G/\epsilon}(k). \tag{1}$$

The chromatic symmetric functions exist inside the algebra of symmetric functions, which is a subalgebra of $\mathbb{Q}[[x_1, x_2, \ldots]]$ in commuting variables where all bases are indexed by integer partitions. An *integer partition*, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{l(\lambda)})$, is a list of weakly decreasing positive integers λ_i called *parts* whose *length*, $l(\lambda)$, is the number of parts. If the sum of all λ_i is n we say that λ partitions n, denoted by $\lambda \vdash n$. At times we will write $\lambda = (1^{r_1}, 2^{r_2}, \ldots, n^{r_n})$ where r_i means that λ has r_i parts of size i. The *transpose* of λ, denoted by λ^t, is given by $\lambda^t = (r_1 + \cdots + r_n, r_2 + \cdots + r_n, \ldots, r_n)$ with zeros removed. Several change of bases formulas require the notion of dominance order. Given $\lambda, \nu \vdash n$ we say λ *dominates* ν, $\lambda \succeq \nu$, if the sum of the first j largest parts of λ is always at least the sum of the first j largest parts of ν.

Example 2.1. We have that $(4, 2, 2) \succeq (3, 3, 2)$ but $(4, 2, 2) \not\succeq (4, 3, 1)$.

To define the symmetric functions we define the *i-th elementary symmetric function*, which is

$$e_i = \sum_{j_1 < j_2 < \cdots < j_i} x_{j_1} x_{j_2} \cdots x_{j_i}$$

and the *elementary symmetric function* associated to λ is

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_{l(\lambda)}}.$$

Example 2.2. $e_{(2,1)} = (x_1x_2 + x_1x_3 + x_2x_3 + \cdots)(x_1 + x_2 + x_3 + \cdots)$

The *algebra of symmetric functions*, Λ, is the graded algebra

$$\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \cdots$$

where $\Lambda_0 = \mathrm{span}\{1 = e_0\} = \mathbb{Q}$ and for $n \geq 1$

$$\Lambda_n = \mathrm{span}\{e_\lambda : \lambda \vdash n\}.$$

Besides the basis of elementary symmetric functions, another classical basis that is of particular interest to us is the Schur symmetric functions. The *Schur symmetric function* associated to λ can be defined using a Jacobi-Trudi identity

$$s_\lambda = \det(e_{\lambda_i^t - i + j})_{1 \leq i,j \leq \lambda_1}$$

letting $e_0 = 1$ and $e_i = 0$ for $i < 0$. More information can be found in Macdonald's book [18] and Sagan's book [21].

Example 2.3. $s_{(2,1)} = e_{(2,1)} - e_{(3)} = x_1^2x_2 + x_1x_2^2 + \cdots + 2x_1x_2x_3 + \cdots$

We say a function $F \in \Lambda$ is *e-positive*, respectively *Schur-positive*, if F can be written as a non-negative sum of elementary symmetric functions, respectively Schur symmetric functions. We will often refer to a graph itself as e-positive or Schur-positive if its chromatic symmetric function is respectively e-positive or Schur-positive.

Example 2.4. The chromatic symmetric function for the complete graph is

$$X_{K_n} = n!e_{(n)} = n!s_{(1^n)},$$

so is both e-positive and Schur-positive.

Remark 2.5. The paths [24, Proposition 5.3], cycles [24, Proposition 5.4] and lollipops [12, Corollary 7.7] are well-known families of e-positive graphs.

Stanley [24, Proposition 2.3] found that if $G \cup H$ is the disjoint union of two graphs G and H then

$$X_{G \cup H} = X_G X_H.$$

Remark 2.6. Because the elementary basis is multiplicative if graphs G and H are e-positive then so is $G \cup H$.

There are two other bases of symmetric functions that will be especially useful. These two bases are called the power-sum basis and the monomial basis. The *i-th power-sum symmetric function* is

$$p_i = x_1^i + x_2^i + x_3^i + \cdots$$

and the *power-sum symmetric function* associated to λ is

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_{l(\lambda)}}.$$

Example 2.7. $p_{(2,1)} = (x_1^2 + x_2^2 + x_3^2 + \cdots)(x_1 + x_2 + x_3 + \cdots)$

The *monomial symmetric function* associated to λ is

$$m_\lambda = \sum_{j_1, j_2, \ldots, j_{l(\lambda)}} x_{j_1}^{\lambda_1} x_{j_2}^{\lambda_2} \cdots x_{j_{l(\lambda)}}^{\lambda_{l(\lambda)}}$$

summed over distinct monomials and the j_i are also distinct.

Example 2.8. $m_{(2,1)} = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + \cdots$

We will call a function *m-positive* if it can be written as a non-negative sum of monomial symmetric functions.

Remark 2.9. It is well known that all elementary and Schur symmetric functions can be written as non-negative sums of monomial symmetric functions. Hence, if a symmetric function F is e-positive or Schur-positive, then F is m-positive. As result, if F is not m-positive, then F is not e-positive and not Schur-positive. For details see [18,21].

Given any basis $\mathcal{B} = \{b_\lambda : \lambda \vdash n, n \geq 0\}$ of Λ we define for $F \in \Lambda$ the notation $[b_\lambda]F$ to be the coefficient of b_λ when F is fully expanded in basis $\mathcal{B}$. At times we will even talk about coefficients of polynomials $p(k)$ in variable k, so let $[k^j]p(k)$ be the coefficient of k^j in $p(k)$.

Stanley has several useful formulas for X_G in terms of the power-sum and monomial bases. A *partition* of the vertices $V(G)$ is a collection of disjoint non-empty subsets of vertices, called *blocks*, whose full union is $V(G)$. We say the partition is of *type* $\lambda \vdash \#V(G)$ if the relative sizes of the blocks form λ. A *stable partition* is a partition of the vertices so that each block is an *independent set* of G, meaning that G restricted to each block has no edges. A *connected partition* is a partition of the vertices so that G restricted to each block is connected. For an integer partition $\lambda = (1^{r_1}, 2^{r_2}, \ldots, n^{r_n})$ define $\lambda^! = r_1! r_2! \cdots r_n!$.

Theorem 2.10 *(Stanley [24] Proposition 2.4). For a graph G,*

$$X_G = \sum_{\lambda \vdash n} a_\lambda \lambda^! m_\lambda$$

where a_λ is the number of stable partitions of G of type λ.

Example 2.11. $X_{P_3} = m_{(2,1)} + 6m_{(1,1,1)}$

Theorem 2.12 *(Stanley [24] Theorem 2.5). For a graph G*

$$X_G = \sum_{S \subseteq E(G)} (-1)^{\#S} p_{\lambda(S)}$$

where $\lambda(S)$ is the integer partition formed from the sizes of the connected components formed by restricting the graph G to S.

Example 2.13. $X_{P_3} = p_{(3)} - 2p_{(2,1)} + p_{(1,1,1)}$

The goal of this paper is to investigate two posets on $\mathcal{G}_n$, equivalence classes of connected graphs on n vertices decided by equivalent chromatic symmetric functions. The relations will reflect relative e-positivity or Schur-positivity of the chromatic symmetric functions of graphs. A *poset*, or partially ordered set, is a collection of objects and a relation $\leq$ between some of these objects that is reflexive, transitive and antisymmetric. For more information see [25]. There are several elements or sets of elements that are studied in posets because of their particular properties. One group of elements are *maximal elements*, which are those $x \in P$ where $y \leq x$ for all comparable $y \in P$. Similarly, *minimal elements* are those $x \in P$ where $y \geq x$ for all comparable $y \in P$. A *chain* in a poset is a collection of elements $Q \subseteq P$ such that all elements $x, y \in Q$ are related with $x \leq y$ or $y \leq x$. An *antichain* is a collection of elements $Q \subseteq P$ such that all distinct $x, y \in Q$ are unrelated with $x \nleq y$ and $y \nleq x$. We call $x \in P$ *independent* if x is not related to any other element in P. An *interval* of a poset is $[x, y] = \{z \in P : x \leq z \leq y\}$. Every poset has an associated *Möbius function*, which is the map $\mu : P \times P \to \mathbb{Z}$ such that $\mu(x, x) = 1$, for $x < y$

$$\sum_{z \in [x,y]} \mu(x, z) = 0,$$

and for incomparable x and y let $\mu(x, y) = 0$.

One natural way to define relative e-positivity or Schur-positivity of the chromatic symmetric functions is by considering the e-positivity or Schur-positivity of the difference between two chromatic symmetric functions, $X_G - X_H$. However, we will find that $X_G - X_H$ is never e-positive or Schur-positive unless $X_G = X_H$. To show this we need the following lemma that will use the following conversion formula going from the Schur basis to the power-sum basis [21, Theorem 4.6.4],

$$s_\lambda = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \chi^\lambda(w) p_w \tag{2}$$

where $\mathfrak{S}_n$ is the symmetric group, $\chi^\lambda(w)$ is the irreducible character usually indexed by λ evaluated at w and if w is of cycle type $\mu \vdash n$ then $p_w = p_\mu$.

Lemma 2.14. *If $F \in \Lambda^n$ is a non-zero Schur-positive function then $[p_{(1^n)}]F > 0$.*

Proof. Note that in the conversion formula from the Schur basis to the power-sum basis in equation (2) the only time $p_{(1^n)}$ appears is for the identity permutation, $\pi = \mathrm{id}$. It is a fact that for any irreducible character $\chi^\lambda(\mathrm{id}) > 0$ because $\chi^\lambda(\mathrm{id})$ is the dimension of the character [21, Proposition 1.8.5]. This means that $[p_{(1^n)}]s_\lambda > 0$ for all $\lambda \vdash n$. Further if we are given a non-zero Schur-positive function $F \in \Lambda^n$ then $[p_{(1^n)}]F > 0$. It follows that if $F \in \Lambda^n$ is Schur-positive and $[p_{(1^n)}]F = 0$ then $F = 0$. $\square$

Lemma 2.15. *For all graphs G we have*

$$[p_{(1^n)}]X_G = 1.$$

Proof. Using the formula in Theorem 2.12 we can see that for a graph G the only way to get a term $p_{(1^n)}$ is to disconnect every vertex. This is only possible when using $S = \emptyset$, the empty edge subset. This shows that $[p_{(1^n)}]X_G = 1$ for all graphs G. $\square$

Theorem 2.16. *For all graphs G and H on n vertices we have either $X_G - X_H = 0$ or $X_G - X_H$ is not Schur-positive.*

Proof. Lemma 2.15 states that $[p_{(1^n)}]X_G = 1$ for all graphs G, which implies that for two graphs G and H on n vertices that $[p_{(1^n)}](X_G - X_H) = 0$. By Lemma 2.14 we can then see that if $F \in \Lambda^n$ is Schur-positive and $[p_{(1^n)}]F = 0$ then $F = 0$. So, if $X_G - X_H$ is a Schur-positive function then we can conclude that $X_G - X_H = 0$. Thus, $X_G - X_H$ is either not Schur-positive or $X_G - X_H = 0$. $\square$

We can similarly get the same result for e-positivity.

Theorem 2.17. *For all graphs G and H on n vertices we have either $X_G - X_H = 0$ or $X_G - X_H$ is not e-positive.*

Proof. This follows from Theorem 2.16: If $X_G - X_H$ is e-positive, then since all e-positive functions are Schur-positive, we then know that $X_G - X_H$ is Schur-positive, which implies that $X_G - X_H = 0$. $\square$

Because direct subtraction between two distinct chromatic symmetric functions is never e-positive or Schur-positive, defining the relation between G and H based on direct subtraction gives a poset with no relations. Hence, we base our relations in our posets on the following weighted subtractions. The goal of these subtractions is to zero-out the

$e_{(n)}$ or $s_{(1^n)}$ term, which is never zero in a chromatic symmetric function. This is a fact that will be evident later from theorems presented further on in this section. Define

$$X_e(G,H) = X_G - \frac{[e_{(n)}]X_G}{[e_{(n)}]X_H} X_H$$

and

$$X_s(G,H) = X_G - \frac{[s_{(1^n)}]X_G}{[s_{(1^n)}]X_H} X_H.$$

Define $\mathcal{E}_n$ to be the poset on $\mathcal{G}_n$ related to the elementary basis. We will say $G \geq_e H$ if and only if $X_e(G,H)$ is e-positive. Similarly, $\mathcal{S}_n$ will be the poset on $\mathcal{G}_n$ related to the Schur basis. We will say $G \geq_s H$ if and only if $X_s(G,H)$ is Schur-positive. See Fig. 2 for examples.

Because our relations depend on weighted subtractions determined by the coefficients of $e_{(n)}$ and $s_{(1^n)}$ we will need some machinery to determine these coefficients. The following propositions give a way to calculate these coefficients via their chromatic polynomial and an interpretation in terms of acyclic orientations. An *orientation* of a graph G is an assignment for each edge between u and v a direction from u to v or from v to u. We call an orientation *acyclic* if there are no directed cycles. Given an orientation on G we call a vertex v a *sink* if all edges incident to v are directed towards v and a *source* if all edges incident to v are directed away from v. Let $S(G,j)$ count the number of acyclic orientations of G with exactly j sinks. Stanley [24, Theorem 3.3] found that for any connected graph G on n vertices with $[e_\lambda]X_G = c_\lambda$ that

$$\sum_{\substack{\lambda \vdash n \\ l(\lambda)=j}} c_\lambda = S(G,j), \tag{3}$$

which is a refinement of a result in [23] that the total number of acyclic orientations is

$$\sum_{j=0}^{n} S(G,j) = (-1)^n \chi_G(-1). \tag{4}$$

Theorem 2.18 *(Greene and Zaslavsky [13] Theorem 7.3). Given a graph G on n vertices and $v \in V(G)$ the number of acyclic orientations with v as a unique sink is $(-1)^{n-1}[k]\chi_G(k)$. Also,*

$$S(G,1) = (-1)^{n-1} n \cdot [k]\chi_G(k)$$

and unless G has no edges $S(G,1) > 0$.

Putting together equation (3) and Theorem 2.18 we have

$$[e_{(n)}]X_G = (-1)^{n-1} n \cdot [k]\chi_G(k) = S(G,1). \tag{5}$$

Theorem 2.19 *(Kaliszewski [17] Theorem 1.1). In X_G the coefficient of the hook shape $(k, 1^{n-k})$ in the Schur basis is*

$$[s_{(k,1^{n-k})}]X_G = \sum_{j=1}^{n} \binom{j-1}{k-1} S(G, j)$$

for $k \in [n]$. Specifically $[s_{(1^n)}]X_G$ counts the total number of acyclic orientations of G.

Using equation (4) we can see from Theorem 2.19 that

$$[s_{(1^n)}]X_G = (-1)^n \chi_G(-1). \tag{6}$$

Using these methods on trees we get the following formulas for two coefficients.

Corollary 2.20. *If T is a tree on n vertices then $[e_{(n)}]X_T = n$ and $[s_{(1^n)}]X_T = 2^{n-1}$.*

Proof. It is known that the chromatic polynomial for a tree on n vertices is $\chi_T(k) = k(k-1)^{n-1}$. From equation (5) we can see that $[e_{(n)}]X_T = n$ and from equation (6) we can see that $[s_{(1^n)}]X_T = 2^{n-1}$. $\square$

Also using these methods we get the following formulas for the two coefficients for complete graphs.

Proposition 2.21. *We have $[e_{(n)}]X_{K_n} = [s_{(1^n)}]X_{K_n} = n!$.*

Proof. This follows immediately from Example 2.4. $\square$

One motivating property for our posets $\mathcal{E}_n$ and $\mathcal{S}_n$ is that they give a new equivalent condition for e-positivity and Schur-positivity.

Theorem 2.22. *We have $G \geq_e K_n$ if and only if G is e-positive.*

Proof. Note that because of Proposition 2.21

$$X_e(G, K_n) = X_G - \frac{[e_{(n)}]X_G}{[e_{(n)}]X_{K_n}} X_{K_n} = X_G - [e_{(n)}]X_G \cdot e_{(n)},$$

which is X_G without its $e_{(n)}$ term. We can see from Theorem 2.18 and equation (5) that $[e_{(n)}]X_G > 0$. Because of this X_G is not e-positive if there is a $\lambda \vdash n$ with $\lambda \neq (n)$ such that $[e_\lambda]X_G < 0$. Altogether X_G is e-positive if and only if $X_e(G, K_n) = X_G - [e_{(n)}]X_G \cdot e_{(n)}$ is e-positive. $\square$

We have a similar condition for our other poset.

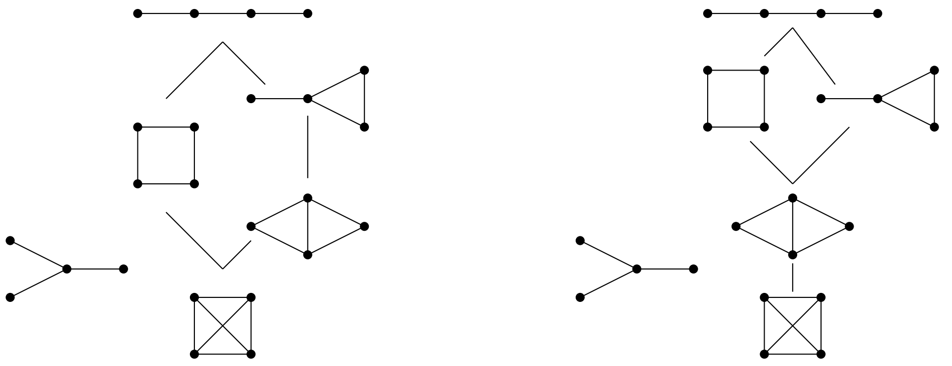


Fig. 2. On the left we have $\mathcal{E}_4$ and on the right we have $\mathcal{S}_4$.

Theorem 2.23. *We have $G \geq_s K_n$ if and only if G is Schur-positive.*

Proof. This is similar to the proof of Theorem 2.22. Note that because of Proposition 2.21

$$X_s(G, K_n) = X_G - \frac{[s_{(1^n)}]X_G}{[s_{(1^n)}]X_{K_n}} X_{K_n} = X_G - [s_{(1^n)}]X_G \cdot s_{(1^n)},$$

which is X_G without its $s_{(1^n)}$ term. We can see from Theorem 2.19 and Theorem 2.18 that $[s_{(1^n)}]X_G > 0$. Because of this X_G is not Schur-positive if there is a $\lambda \vdash n$ with $\lambda \neq (1^n)$ such that $[s_\lambda]X_G < 0$. Altogether X_G is Schur-positive if and only if $X_s(G, K_n) = X_G - [s_{(1^n)}]X_G \cdot s_{(1^n)}$ is Schur-positive. $\square$

Normally e-positivity implies Schur-positivity, but in general a relation in either poset does not imply a relation in the other.

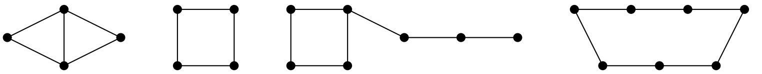
Proposition 2.24. *The relation $G <_e H$ does not necessarily imply $G <_s H$. The relation $G <_s H$ does not necessarily imply $G <_e H$.*

Proof. See Fig. 3 for the pictures of the graphs in this proof. We can see that the diamond D is less than C_4 in $\mathcal{S}_4$, however D is not less than C_4 in $\mathcal{E}_4$, which can be seen in Fig. 2. We also find that for G_1 and G_2 given in Fig. 3 that $G_1 >_e G_2$ in $\mathcal{E}_7$ but $G_1 \not>_s G_2$ in $\mathcal{S}_7$, looking at the coefficient of $s_{(2^3,1)}$. $\square$

However, this is true for trees. Furthermore, we will find later that trees form an anti-chain in both families of posets.

Proposition 2.25. *For two trees if $T_1 >_e T_2$ then $T_1 >_s T_2$.*

Proof. Let T_1 and T_2 be two trees on n vertices. By Corollary 2.20 we have $X_e(T_1, T_2) = X_{T_1} - X_{T_2}$ and $X_s(T_1, T_2) = X_{T_1} - X_{T_2}$, which are equal. Hence, if $X_{T_1} - X_{T_2}$ is e-positive, $T_1 >_e T_2$, then $X_{T_1} - X_{T_2}$ is Schur-positive, $T_1 >_s T_2$. $\square$



$$
\begin{aligned}
X_D &= 16e_{(4)} + 2e_{(3,1)} = 18s_{(1^4)} + 2s_{(2,1,1)} \\
X_{C_4} &= 12e_{(4)} + 2e_{(2,2)} = 14s_{(1^4)} + 2s_{(2,1,1)} + 2s_{(2,2)} \\
X_{G_1} &= 21e_{(7)} + 15e_{(6,1)} + 29e_{(5,2)} + 27e_{(4,3)} + 12e_{(4,2,1)} + 7e_{(3,2^2)} + e_{(2^3,1)} \\
&= 112s_{(1^7)} + 112s_{(2,1^5)} + 106s_{(2^2,1^3)} + 57s_{(2^3,1)} + 22s_{(3,14)} + 10s_{(3,2^2)} \\
&\quad + 32s_{(3,2,1^2)} + 10s_{(3^2,1)} + s_{(4,1^3)} + 2s_{(4,3,1)} + s_{(4,3)} \\
X_{G_2} &= 42e_{(7)} + 28e_{(5,2)} + 42e_{(4,3)} + 14e_{(3,2^2)} \\
&= 126s_{(1^7)} + 98s_{(2,1^5)} + 112s_{(2^2,1^3)} + 70s_{(2^3,1)} + 14s_{(3,1^4)} + 28s_{(3,2,1^2)} \\
&\quad + 14s_{(3,2^2)} + 14s_{(3^2,1)}
\end{aligned}
$$

Fig. 3. From left to right we have the diamond D, C_4, G_1 and G_2.

3. Properties of the chromatic e-positivity poset

In this section we show several properties of $\mathcal{E}_n$. In Theorem 2.22 we proved that $G \geq_e K_n$ if and only if G is e-positive, and now we will prove that K_n is a minimal element. We will find that as we increase in $\mathcal{E}_n$ the independence number, $\alpha(G)$, increases, the number of acyclic orientations with one sink, $S(G,1)$, decreases, and the chromatic number, $\chi(G)$, decreases. Also, we will show that trees with distinct chromatic symmetric functions form an anti-chain and are maximal elements. In particular, the stars S_n are independent elements, so $\mathcal{E}_n$ cannot be a lattice. Lastly, we show that the collection of lollipops forms a chain with the complete graph as the minimal element and the path as the maximal element.

Though it will be shown later in Corollary 3.5 that the complete graph is a minimal element and in Proposition 3.10 that trees are maximal elements in $\mathcal{E}_n$, we consider several common graph statistics that have the complete graph and trees at the two extremal bounds of the statistic. We determine if the statistic increases or decreases with relations in the poset. The statistics we consider here are the independence number, $\alpha(G)$, the number of acyclic orientations with one sink, $S(G,1)$, the chromatic number $\chi(G)$, and the clique number, $\omega(G)$.

Proposition 3.1. *If $G \geq_e H$ then $\alpha(G) \geq \alpha(H)$.*

Proof. Suppose $\alpha(G) < \alpha(H)$ and let λ be the partition of n given by $(\alpha(H), 1^{n-\alpha(H)})$. By definition, H has a stable partition of type λ but G does not. Hence, by the expansion of X_G and X_H into the monomial basis, Theorem 2.10, we have $[m_\lambda]X_G = 0$ but $[m_\lambda]X_H \geq 1$ since the coefficient of m_λ in X_G is a multiple of the number of stable partitions of type λ in G. Therefore, $[m_\lambda]X_e(G,H) < 0$ since equation (5) implies that the scaling factor is always positive. Hence by Remark 2.9, $X_e(G,H)$ cannot be e-positive as $X_e(G,H)$ is not m-positive. □

Proposition 3.2. *If $G >_e H$ then $S(G,1) < S(H,1)$.*

Proof. Let $G >_e H$ so $X_e(G,H) \neq 0$ is e-positive. It is well-known that e-positive functions are Schur-positive, so $X_e(G,H) \neq 0$ is Schur-positive. By Lemma 2.15 $[p_{(1^n)}]X_G = [p_{(1^n)}]X_H = 1$ so we have that

$$[p_{(1^n)}]X_e(G,H) = 1 - \frac{[e_{(n)}]X_G}{[e_{(n)}]X_H}.$$

By Lemma 2.14 this coefficient is positive so $[e_{(n)}]X_G < [e_{(n)}]X_H$. By equation (5) we have our result. □

Proposition 3.3. *If $G \geq_e H$, then $\chi(G) \leq \chi(H)$.*

Proof. Suppose $\chi(G) > \chi(H)$. Since there is a coloring of H with $\chi(H)$ colors, then H has a stable partition of some type $\lambda \vdash n$ with length $\chi(H)$ and hence, we have $[m_\lambda]X_H \geq 1$ by Theorem 2.10. Furthermore, $[m_\lambda]X_G = 0$ since by definition, G cannot be colored with fewer than $\chi(G)$ colors and by assumption $\chi(G) > \chi(H)$. Therefore, $[m_\lambda]X_e(G,H) < 0$ since equation (5) implies that the scaling factor is always positive. Hence by Remark 2.9, $X_e(G,H)$ cannot be e-positive as $X_e(G,H)$ is not m-positive. □

Remark 3.4. Note that there is no consistent relationship between clique numbers $\omega(G)$ and relations in the poset $\mathcal{E}_n$. We find that $P_5 >_e L_{3,2} >_e C_5$ but $\omega(P_5) < \omega(L_{3,2}) > \omega(C_5)$.

Earlier in Theorem 2.22 we proved that G is e-positive if and only if $G \geq_e K_n$. We can further show that K_n is a minimal element in our poset $\mathcal{E}_n$.

Corollary 3.5. *The complete graph K_n is a minimal element in $\mathcal{E}_n$.*

Proof. If $G \neq K_n$ is a connected graph on n vertices then there are two vertices u and v without an edge between them. The set $\{u,v\}$ is an independent set and so $\alpha(G) \geq 2$. By Proposition 3.1 we know $K_n \ngeq_e G$ because $1 = \alpha(K_n) < \alpha(G)$. Hence K_n is a minimal element. □

The next proposition gives us an elegant condition that can generate independent sets in $\mathcal{E}_n$.

Theorem 3.6. *Let $\{G_1, G_2, \ldots, G_k\}$ be some set of connected graphs on n vertices with equal $e_{(n)}$ coefficients, $[e_{(n)}]X_{G_i} = [e_{(n)}]X_{G_j}$, and distinct chromatic symmetric functions. Then $\{G_1, G_2, \ldots, G_k\}$ is an anti-chain in $\mathcal{E}_n$.*

Proof. Let $\{G_1, G_2, \dots, G_k\}$ satisfy all the assumptions stated in the proposition. This means $X_e(G_i, G_j) = X_{G_i} - X_{G_j}$. By Theorem 2.17 we know that either $X_{G_i} - X_{G_j} = 0$ so $i = j$ or $X_{G_i} - X_{G_j}$ is not e-positive. This implies that G_i is not related to G_j for all $i \neq j$, which means that $\{G_1, G_2, \dots, G_k\}$ is an anti-chain in $\mathcal{E}_n$. $\square$

Remark 3.7. By Theorem 3.6 given any integer $z \in \mathbb{Z}$ the collection of graphs on n vertices $\{G : [e_{(n)}]X_G = z\}$ is an anti-chain in $\mathcal{E}_n$ under the assumption that we are grouping graphs together in $\mathcal{E}_n$ if they have equal chromatic symmetric function.

Corollary 3.8. *Trees on n vertices with distinct chromatic symmetric functions form an anti-chain in $\mathcal{E}_n$.*

Proof. By Corollary 2.20 all trees T on n vertices have $[e_{(n)}]X_T = n$, so by Theorem 3.6 we have our result. $\square$

Next we will show that trees not only form an anti-chain, but also that they are actually maximal elements in $\mathcal{E}_n$. In order to do this we will need a lower bound on $[e_{(n)}]X_G$.

Lemma 3.9. *For any connected graph G on n vertices with $\epsilon \in E(G)$*

(i) $[e_{(n)}]X_G = [e_{(n)}]\left(X_{G-\epsilon} + \frac{n}{n-1}X_{G/\epsilon}\right)$,
(ii) $[e_{(n)}]X_G \geq n(\#E(G) - n + 2)$ *and*
(iii) $[e_{(n)}]X_G \geq n$.

Proof. We can use the chromatic polynomial to calculate $[e_{(n)}]X_G$ via equation (5). Using deletion-contraction from equation (1) on the chromatic polynomial in this formula gives us

$$\begin{aligned}
[e_{(n)}]X_G &= (-1)^{n-1}n \cdot [k](\chi_{G-\epsilon}(k) - \chi_{G/\epsilon}(k)) \\
&= (-1)^{n-1}n \cdot [k]\chi_{G-\epsilon}(k) + \frac{n}{n-1}(-1)^{n-2}(n-1) \cdot [k]\chi_{G/\epsilon}(k) \\
&= [e_{(n)}]\left(X_{G-\epsilon} + \frac{n}{n-1}X_{G/\epsilon}\right).
\end{aligned}$$

To prove (ii) we will induct on the number of edges. If G has the minimal number of edges for a connected graph then G is a tree and $[e_{(n)}]X_G = n$ by Corollary 2.20, which satisfies the inequality. Let $\#E(G) > n - 1$. This means that G is not a tree and there exists a cycle. Let ϵ be an edge on one of these cycles, so $G - \epsilon$ is connected. Using part (i), induction and the fact that $\#E(G/\epsilon) \geq n - 2$ we have

$$[e_{(n)}]X_G = [e_{(n)}]\left(X_{G-\epsilon} + \frac{n}{n-1}X_{G/\epsilon}\right)$$

$$\geq n(\#E(G-\epsilon)-n+2)+n(\#E(G/\epsilon)-n+3)$$
$$\geq n(\#E(G)-n+2).$$

Because G is a connected graph, so $\#E(G)\geq n-1$, we can conclude from part (ii) that $[e_{(n)}]X_G \geq n$. □

Proposition 3.10. *All trees on n vertices are maximal elements in $\mathcal{E}_n$.*

Proof. Suppose that $G >_e T$ for some tree T and a connected graph G each on n vertices. By assumption, $X_e(G,T)$ is non-zero and e-positive, using Proposition 3.2 and equation (5), we have $[e_{(n)}]X_G < [e_{(n)}]X_T$. Hence, $[e_{(n)}]X_G < n$ by Corollary 2.20 since T was a tree. However, Lemma 3.9 (iii) implies that $[e_{(n)}]X_G \geq n$ for any connected graph G, so we have reached a contradiction. Hence, T is maximal. □

While all trees are maximal elements there are many that are specifically independent elements. One such family of trees are the star graphs S_n. To show this we will need several lemmas and well-known results. One well-known result we need for our lemma is Stanley's formula for X_G in the power-sum basis that is summed over the bond lattice of the graph. The *bond lattice*, L_G, of a graph is a poset with vertices formed from connected partitions of the vertices $V(G)$. Let π_1 and π_2 be two connected partitions of $V(G)$. We say $\pi_1 \leq_L \pi_2$ if all blocks in π_1 are a subset of some block of π_2. Let μ_L be the Möbius function of the poset L_G. Stanley's formula [24, Theorem 2.6] is

$$X_G = \sum_{\pi\in L_G} \mu_L(\hat{0},\pi)p_\pi \tag{7}$$

where $\hat{0}$ is the partition with all vertices in their own block and p_π is the power-sum function associated to the sizes of the blocks in π. It is known, for example [24], that $(-1)^{n-1}\mu_L(\hat{0},\hat{1}) > 0$ where $\hat{1}$ is the partition with one block and that $(-1)^{n-l(\pi)}\mu_L(\hat{0},\pi) > 0$. From this we know $(-1)^{n-1}[p_{(n)}]X_G > 0$ and the sign of any coefficient of p_λ in X_G.

Another result we will need is the Newtonian identity [26, Proposition 7.7.6]

$$p_n = \sum_{\lambda=(1^{m_1},2^{m_2},\ldots,n^{m_n})} \frac{(-1)^{n-l(\lambda)}n(l(\lambda)-1)!}{m_1!m_2!\cdots m_n!}e_\lambda. \tag{8}$$

Lemma 3.11. *Let G be a connected graph with $\#V(G)\geq 4$ that is not a tree. Then*

(i) *G has a connected partition of type $\lambda=(\lambda_1,\lambda_2)$ with all parts more than one and*
(ii) *$[e_\lambda]X_G + [e_{(n)}]X_G > 0$ if $\lambda_1 \neq \lambda_2$ or*
(iii) *$[e_\lambda]X_G + \frac{[e_{(n)}]X_G}{2} > 0$ if $\lambda_1 = \lambda_2$.*

Proof. First we will show part (i) by inducting on the number of edges. The minimal number of edges for connected non-tree graphs G is n where G is a unicyclic graph, meaning it has only one cycle. Note that if we remove any two edges from the cycle we have disconnected the graph into two connected components, and that the sizes of these components form a connected partition. However, we are not guaranteed that the sizes are more than one. First consider the case where our unique cycle has three vertices. Because $\#V(G) \geq 4$ there is some vertex v in the cycle connected to another vertex u outside the cycle. If we remove the two edges in the cycle incident to v then we will have two components of size more than one. Next consider the case where the unique cycle has four or more vertices. Removing any two edges that do not share a vertex will guarantee that the two connected components have size more than one. This completes the base case.

Assume that $\#E(G) > n$ and that any connected non-tree graph H with $\#E(H) < \#E(G)$ has a connected partition of type $\lambda = (\lambda_1, \lambda_2)$ where all parts are more than one. There will exist some edge $\epsilon \in E(G)$ whose removal does not disconnect G. Because $\#E(G - \epsilon) \geq n$ we know that $G - \epsilon$ is a connected non-tree graph. By induction $G - \epsilon$ has a connected partition $\lambda = (\lambda_1, \lambda_2)$ with the required conditions, so we can conclude that G does as well. This completes our proof of part (i).

Let $\lambda = (\lambda_1, \lambda_2)$ be a connected partition of G where all parts are more than one. We will do some calculations for $[e_\lambda]X_G$ considering X_G written in the power-sum basis. Because the power-sum and elementary bases are multiplicative and because of equation (8) we can see that $e_{(n)}$ will only appear in p_ν when $\nu = (n)$. This means that

$$[e_{(n)}]X_G = [p_{(n)}]X_G \cdot [e_{(n)}]p_{(n)} = (-1)^{n-1} n \cdot [p_{(n)}]X_G. \tag{9}$$

By equation (8) we can also see that e_λ will only appear in p_ν when $\nu = (n)$ or $\nu = \lambda$ so

$$[e_\lambda]X_G = [p_{(n)}]X_G \cdot [e_\lambda]p_{(n)} + [p_\lambda]X_G \cdot [e_\lambda]p_\lambda = [p_{(n)}]X_G \cdot [e_\lambda]p_{(n)} + [p_\lambda]X_G \cdot (-1)^{n-2}\lambda_1\lambda_2. \tag{10}$$

By the signs of the terms in equation (7) we know that $(-1)^{n-2}[p_\lambda]X_G > 0$ so

$$[e_\lambda]X_G > [p_{(n)}]X_G \cdot [e_\lambda]p_{(n)}.$$

Using equation (8) and equation (9) this gives

$$[e_\lambda]X_G > -[e_{(n)}]X_G$$

when $\lambda_1 \neq \lambda_2$ and

$$[e_\lambda]X_G > -\frac{[e_{(n)}]X_G}{2}$$

when $\lambda_1 = \lambda_2$, which completes the proof. □

Now we just need some coefficients for the chromatic symmetric functions for stars S_n.

Lemma 3.12. *Let $\lambda = (\lambda_1, \lambda_2)$ be an integer partition with parts greater than one. Then,*

(i) $[e_\lambda]X_{S_n} = -n$ *if* $\lambda_1 \neq \lambda_2$ *or*
(ii) $[e_\lambda]X_{S_n} = -\frac{n}{2}$ *if* $\lambda_1 = \lambda_2$.

Proof. Note that there does not exist a connected partition for S_n of type $\lambda = (\lambda_1, \lambda_2)$ as described, so by equation (7) we can see that $[p_\lambda]X_{S_n} = 0$. Because equation (10) is always true, using this equation and equations (8) and (9) we can calculate

$$[e_\lambda]X_{S_n} = -n$$

when $\lambda_1 \neq \lambda_2$ and

$$[e_\lambda]X_{S_n} = -\frac{n}{2}$$

when $\lambda_1 = \lambda_2$. $\square$

Now we have all the tools we need to show that stars are independent in $\mathcal{E}_n$.

Proposition 3.13. *The star S_n is an independent element in $\mathcal{E}_n$ for $n \geq 4$.*

Proof. Since all trees are maximal elements by Proposition 3.10 it suffices to show that stars are minimal elements. Assume that G is a connected graph on n vertices with $S_n >_e G$. We will show $S_n >_e G$ is impossible by showing $X_e(S_n, G)$ is not e-positive. The graph G cannot be a tree because all trees are maximal, so by Lemma 3.11 we know that G has a connected partition of type $\lambda = (\lambda_1, \lambda_2)$ with all parts greater than one. Let us first consider the case when $\lambda_1 \neq \lambda_2$. We have in this case using the coefficients calculated in Lemma 3.12 and Corollary 2.20 that

$$[e_\lambda]X_e(S_n, G) = -n - \frac{n}{[e_{(n)}]X_G} \cdot [e_\lambda]X_G.$$

Lemma 3.11 implies that this coefficient is negative so $X_e(S_n, G)$ is not e-positive in this case and we have a contradiction. Next let us consider the case when $\lambda_1 = \lambda_2$. Using the coefficients calculated in Lemma 3.12 and Corollary 2.20 we have that

$$[e_\lambda]X_e(S_n, G) = -\frac{n}{2} - \frac{n}{[e_{(n)}]X_G} \cdot [e_\lambda]X_G.$$

Lemma 3.11 implies that this coefficient is negative so $X_e(S_n, G)$ is not e-positive in all cases. Thus, stars are minimal and further are independent elements in $\mathcal{E}_n$. $\square$

Corollary 3.14. *The poset $\mathcal{E}_n$ is not a lattice for $n \geq 4$.*

Proof. Lattices do not have independent elements, and we have shown by Proposition 3.13 that the poset $\mathcal{E}_n$ for $n \geq 4$ has an independent element. $\square$

Our last result about $\mathcal{E}_n$ we will prove in this section is that the family of lollipop graphs forms a chain with the complete graph as the minimal element and the path as the maximal element. To prove this we will use the following formulas for the chromatic symmetric functions and polynomials of lollipop graphs.

Theorem 3.15 *([7] Theorem 7 and Lemma 11). The chromatic polynomial of a lollipop $L_{m,n}$ is*

$$\chi_{L_{m,n}}(k) = k(k-1)^{n+1}(k-2)(k-3)\cdots(k-(m-1))$$

and the chromatic symmetric function satisfies

$$X_{L_{m,n}} = (m-1)X_{L_{m-1,n+1}} - (m-2)X_{K_{m-1}}X_{P_{n+1}}.$$

The above theorem can be applied to give us the coefficient of $e_{(m+n)}$ for a lollipop graph.

Proposition 3.16. *The coefficient of $e_{(m+n)}$ for the lollipop graph is*

$$[e_{(m+n)}]X_{L_{m,n}} = (m+n)(m-1)!.$$

Proof. By equation (5) we can use the chromatic polynomial of $L_{m,n}$ given in Theorem 3.15 to calculate the coefficient as

$$\begin{aligned}[e_{(m+n)}]X_{L_{m,n}} &= (-1)^{m+n-1}(m+n)\cdot[k]\left(k(k-1)^{n+1}(k-2)(k-3)\cdots(k-(m-1))\right)\\ &= (m+n)(m-1)!\end{aligned}$$

so we are done. $\square$

Theorem 3.17. *The family of lollipop graphs $\{L_{m,n}\}$ on $m+n=N$ vertices forms a chain in $\mathcal{E}_N$. In particular, for $m \geq 3$ and $n \geq 0$,*

$$L_{m-1,n+1} \geq_e L_{m,n}.$$

Hence, the path P_N is the maximal element and the complete graph K_N is the minimal element of the chain.

Proof. It suffices to show $L_{m-1,n+1} \geq_e L_{m,n}$ by transitivity for $m \geq 3$ and $n \geq 0$. Using the coefficients calculated in Proposition 3.16 and the formula in Theorem 3.15 we have

$$\begin{aligned} X_e(L_{m-1,n+1}, L_{m,n}) &= X_{L_{m-1,n+1}} - \frac{1}{m-1} X_{L_{m,n}} \\ &= X_{L_{m-1,n+1}} - \frac{1}{m-1} \left((m-1) X_{L_{m-1,n+1}} - (m-2) X_{K_{m-1}} X_{P_{n+1}}\right) \\ &= \frac{m-2}{m-1} X_{K_{m-1}} X_{P_{n+1}}, \end{aligned}$$

which is known to be e-positive by Example 2.4, Remark 2.5 and that the multiplication of two e-positive functions is e-positive. □

4. Properties of the chromatic Schur-positivity poset

In this section we determine properties of $\mathcal{S}_n$. These properties parallel the properties we have already proven for $\mathcal{E}_n$. In Theorem 2.23 we proved that $G \geq_s K_n$ if and only if G is Schur-positive, and we will further prove that K_n is a minimal element. We will find that as we increase in $\mathcal{S}_n$ that the independence number, $\alpha(G)$, increases, the number of acyclic orientations decreases, and the chromatic number decreases. Also, we will find that the trees with distinct chromatic symmetric functions form an anti-chain and are maximal elements. In particular, the stars S_n are independent elements, so $\mathcal{S}_n$ also cannot be a lattice. Lastly, we prove that the family of lollipop graphs is a chain with the complete graph as the minimal element and the path as the maximal element, however, the proof is more complex than in the case of $\mathcal{E}_n$. We present the theorem and proof in Section 5. The proofs for Proposition 4.1 to Corollary 4.7 are relatively similar to the analogous results in the previous section, so may be skipped if so desired. However, we include them for completeness and because some of the differences are quite subtle.

First we consider several common statistics on graphs and discover if there is a consistent relationship to the relations in $\mathcal{S}_n$.

Proposition 4.1. *If $G \geq_s H$ then $\alpha(G) \geq \alpha(H)$.*

Proof. Suppose $\alpha(G) < \alpha(H)$ and let λ be the partition of n given by $(\alpha(H), 1^{n-\alpha(H)})$. By definition, H has a stable partition of type λ but G does not. Hence, by the expansion of X_G and X_H into the monomial basis, Theorem 2.10, we have $[m_\lambda]X_G = 0$ but $[m_\lambda]X_H \geq 1$ since the coefficient of m_λ in X_G is a multiple of the number of stable partitions of type λ in G. Therefore, $[m_\lambda]X_s(G,H) < 0$ since Theorem 2.19 implies that the scaling factor is always positive. Hence by Remark 2.9, $X_s(G,H)$ cannot be Schur-positive as $X_s(G,H)$ is not m-positive. □

Proposition 4.2. *If $G >_s H$ then the number of acyclic orientations for G is less than the number of acyclic orientations for H.*

Proof. Let $G >_s H$ so $X_s(G,H) \neq 0$ is Schur-positive. By Lemma 2.15 $[p_{(1^n)}]X_G = [p_{(1^n)}]X_H = 1$ so we have that

$$[p_{(1^n)}]X_s(G,H) = 1 - \frac{[s_{(1^n)}]X_G}{[s_{(1^n)}]X_H}.$$

By Lemma 2.14 this coefficient is positive so $[s_{(1^n)}]X_G < [s_{(1^n)}]X_H$. By Theorem 2.19 we have our result. □

Proposition 4.3. *If $G \geq_s H$, then $\chi(G) \leq \chi(H)$.*

Proof. Suppose $\chi(G) > \chi(H)$. Since there is a coloring of H with $\chi(H)$ colors, then H has a stable partition of some type $\lambda \vdash n$ with length $\chi(H)$ and hence, we have $[m_\lambda]X_H \geq 1$ by Theorem 2.10. Furthermore, $[m_\lambda]X_G = 0$ since by definition, G cannot be colored with fewer than $\chi(G)$ colors and by assumption $\chi(G) > \chi(H)$. Therefore, $[m_\lambda]X_s(G,H) < 0$ since Theorem 2.19 implies that the scaling factor is always positive. Hence by Remark 2.9, $X_s(G,H)$ cannot be Schur-positive as $X_s(G,H)$ is not m-positive. □

Remark 4.4. Note that there is no consistent relationship between clique numbers $\omega(G)$ and relations in the poset $\mathcal{S}_n$. We find that $P_5 >_s L_{3,2} >_s C_5$ but $\omega(P_5) < \omega(L_{3,2}) > \omega(C_5)$.

We have proven in Theorem 2.23 that G is Schur-positive if and only if $G \geq_s K_n$. Now we will prove that the complete graph is a minimal element.

Corollary 4.5. *The complete graph K_n is a minimal element in $\mathcal{S}_n$.*

Proof. If $G \neq K_n$ is a connected graph on n vertices then there are two vertices u and v without an edge between them. The set $\{u, v\}$ is an independent set and $\alpha(G) \geq 2$. By Proposition 4.1 we know $K_n \not\geq_s G$ because $1 = \alpha(K_n) < \alpha(G)$. Hence K_n is a minimal element. □

Similar to Theorem 3.6 there is an elegant condition that can generate many anti-chains in $\mathcal{S}_n$.

Theorem 4.6. *Let $\{G_1, G_2, \ldots, G_k\}$ be some set of connected graphs on n vertices with equal $s_{(1^n)}$ coefficients, $[s_{(1^n)}]X_{G_i} = [s_{(1^n)}]X_{G_j}$, and distinct chromatic symmetric functions. Then $\{G_1, G_2, \ldots, G_k\}$ is an anti-chain in $\mathcal{S}_n$.*

Proof. Let $\{G_1, G_2, \ldots, G_k\}$ satisfy all the assumptions stated in the proposition. This means $X_s(G_i, G_j) = X_{G_i} - X_{G_j}$. By Theorem 2.16 we know that either that $X_{G_i} - X_{G_j} = 0$ so $i = j$ or that $X_{G_i} - X_{G_j}$ is not Schur-positive. This implies that G_i is not related to G_j for all $i \neq j$, which means that $\{G_1, G_2, \ldots, G_k\}$ is an anti-chain in $\mathcal{S}_n$. □

Using this proposition we can prove that trees form an anti-chain.

Corollary 4.7. *Trees on n vertices with distinct chromatic symmetric functions form an anti-chain in $\mathcal{S}_n$.*

Proof. By Corollary 2.20 all trees T on n vertices have $[s_{(1^n)}]X_T = 2^{n-1}$, so by Theorem 4.6 we have our result. □

Remark 4.8. By Theorem 4.6 given any integer $z \in \mathbb{Z}$ the collection of graphs on n vertices $\{G : [s_{(1^n)}]X_G = z\}$ is an anti-chain in $\mathcal{S}_n$ under the assumption that we are grouping graphs together in $\mathcal{S}_n$ if they have equal chromatic symmetric function.

Trees are maximal elements in $\mathcal{S}_n$ just like they are in $\mathcal{E}_n$. Before we prove this we will prove a lemma on the number of acyclic orientations of non-tree graphs compared with trees.

Lemma 4.9. *Trees T on n vertices have 2^{n-1} acyclic orientations. Graphs G on n vertices that are not trees have more than 2^{n-1} acyclic orientations.*

Proof. This can be proved using deletion-contraction and induction on the number of vertices and edges in graphs. Our base case is any tree on $n \geq 1$ vertices, which will have $n-1$ edges. Since there are no cycles there are 2^{n-1} acyclic orientations. Now let G be a non-tree connected graph on $n \geq 2$ vertices, so G has at least n edges. There exists an edge $\epsilon \in E(G)$ where $G - \epsilon$ is still connected. By deletion-contraction we have that $\chi_G(k) = \chi_{G-\epsilon}(k) - \chi_{G/\epsilon}(k)$. Because equation (4) tells us that $(-1)^n\chi_G(-1)$ is the total number of acyclic orientations for G, using induction we get that the total number of acyclic orientations for G is

$$(-1)^n\chi_G(-1) = (-1)^n\chi_{G-\epsilon}(-1) + (-1)^{n-1}\chi_{G/\epsilon}(-1) \geq 2^{n-1} + 2^{n-2},$$

which is certainly greater than 2^{n-1}. □

Proposition 4.10. *All trees on n vertices are maximal elements in $\mathcal{S}_n$.*

Proof. Suppose that $G >_s T$ for some tree T and connected graph G on n vertices. We will show that this leads to a contradiction. By Lemma 2.14 if $F \in \Lambda^n$ is a non-zero Schur-positive function then $[p_{(1^n)}]F > 0$. Since $G >_s T$ we know that $X_s(G, T)$ is a non-zero Schur-positive function so $[p_{(1^n)}]X_s(G, T) > 0$. By Lemma 2.15 $[p_{(1^n)}]X_G = [p_{(1^n)}]X_T = 1$ so using Corollary 2.20 we have that

$$[p_{(1^n)}]X_s(G, T) = 1 - \frac{[s_{(1^n)}]X_G}{2^{n-1}} > 0.$$

This means that $2^{n-1} > [s_{(1^n)}]X_G$ where $[s_{(1^n)}]X_G$ is the total number of acyclic orientations of G by Theorem 2.19. By Lemma 4.9 this is a contradiction. $\square$

Just like in $\mathcal{E}_n$, while all trees are maximal in $\mathcal{S}_n$ some are actually independent elements. The stars S_n are a family of independent elements. In order to prove this we need to study some specific coefficients of X_{S_n} and X_G for a general graph G. We will need the conversion formula from monomial symmetric functions to Schur symmetric functions. In Macdonald's book [18, page 105] the transition formula is

$$m_\lambda = \sum_{\nu \vdash n} K^{-1}_{\lambda,\nu} s_\nu.$$

The coefficients $K^{-1}_{\lambda,\nu}$ are the *inverse Kostka numbers* defined by

$$K^{-1}_{\lambda,\nu} = \sum_T (-1)^{\mathrm{ht}(T)}, \tag{11}$$

which is summed over special rim hooks T with underlying Young diagram ν. See [18, page 107] for full details. Particularly, Macdonald's book notes that $K^{-1}_{\lambda,\nu} = 0$ unless $\lambda \succeq \nu$ in dominance order.

Lemma 4.11. *Let $n \geq 4$. Then*

(i) $[s_{(n-2,2)}]X_G = [m_{(n-2,2)}]X_G - [m_{(n-1,1)}]X_G$ *and*
(ii) $[s_{(n-2,2)}]X_{S_n} = -1$.
(iii) *If $\alpha(G) \leq n-2$ then $[s_{(n-2,2)}]X_G \geq 0$.*
(iv) *If $\alpha(G) = n-1$ then $G = S_n$.*

Proof. From the definition of inverse Kostka numbers given in equation (11) we can see that $K^{-1}_{(n-2,2),(n-2,2)} = 1$, $K^{-1}_{(n-1,1),(n-2,2)} = -1$ and $K^{-1}_{(n),(n-2,2)} = 0$. Since $K^{-1}_{\lambda,\nu} = 0$ unless $\lambda \succeq \nu$ we know that $s_{(n-2,2)}$ only appears in the expansion of m_λ in the Schur basis when $\lambda \succeq (n-2,2)$. Such λ are (n), $(n-1,1)$ and $(n-2,2)$. So using the expansion of X_G in the monomial basis we can calculate the coefficient of $s_{(n-2,2)}$, which is

$$\begin{aligned}[s_{(n-2,2)}]X_G &= K^{-1}_{(n-2,2),(n-2,2)} \cdot [m_{(n-2,2)}]X_G + K^{-1}_{(n-1,1),(n-2,2)} \cdot [m_{(n-1,1)}]X_G \\ &\quad + K^{-1}_{(n),(n-2,2)} \cdot [m_{(n)}]X_G \\ &= [m_{(n-2,2)}]X_G - [m_{(n-1,1)}]X_G.\end{aligned}$$

Note that the star S_n has exactly one stable partition of type $(n-1,1)$ and does not have any stable partitions of type $(n-2,2)$. By Theorem 2.10 we have $[m_{(n-2,2)}]X_{S_n} = 0$ and $[m_{(n-1,1)}]X_{S_n} = 1$ so $[s_{(n-2,2)}]X_{S_n} = -1$. Further note that if $\alpha(G) \leq n-2$ there is no stable partition of G of type $(n-1,1)$, so $[s_{(n-2,2)}]X_G = [m_{(n-2,2)}]X_G \geq 0$. Lastly note that in the case where G is a connected graph with $\alpha(G) = n-1$ there are $n-1$ vertices with no edges between them and at most $n-1$ edges from these $n-1$ vertices to the last nth vertex. For the graph to be connected we need all $n-1$ edges, which describes the star S_n. □

We now have everything we need to prove stars are independent elements.

Proposition 4.12. *The star S_n is an independent element in $\mathcal{S}_n$ for $n \geq 4$.*

Proof. Let G be a connected graph that is not a star. By Proposition 4.10 we know that S_n is a maximal element in $\mathcal{S}_n$ so we only have to show that $S_n \not\geq_s G$ for any $G \neq S_n$. Because connected graphs can have at most $n-1$ independent vertices, by Lemma 4.11 we know that $\alpha(G) \leq n-2$ and so $[s_{(n-2,2)}]X_G \geq 0$. Recall that by Theorem 2.19 that $[s_{(1^n)}]X_G > 0$ because $[s_{(1^n)}]X_G$ counts the number of acyclic orientations of G. Using Corollary 2.20 and Lemma 4.11 we have that

$$[s_{(n-2,2)}]X_s(S_n, G) = -1 - 2^{n-1}\frac{[s_{(n-2,2)}]X_G}{[s_{(1^n)}]X_G} < 0,$$

which shows that $S_n \not\geq_s G$. □

Corollary 4.13. *The poset $\mathcal{S}_n$ is not a lattice for $n \geq 4$.*

Proof. Lattices do not have independent elements, which by Proposition 4.12 the poset $\mathcal{S}_n$ has. □

In $\mathcal{S}_n$, similar to $\mathcal{E}_n$, we have that the family of lollipop graphs on n vertices forms a chain with the complete graph as the minimal element and the path as the maximal element. The proof is more complex than in the case of $\mathcal{E}_n$, so we present the proof in its own section, concluding with the theorem in Theorem 5.4.

5. Lollipops in the chromatic Schur-positivity poset

In this section we prove that the set of lollipops $\{L_{m,n} : m+n = N\}$ forms a chain in the poset $\mathcal{S}_N$. However, the proof will not be as straightforward as it was in the case of $\mathcal{E}_N$. We will use Gasharov's [10] interpretation for the coefficients of X_G in the Schur basis. This interpretation is in terms of P-tableau in the case when G is an incomparability graph of a $(3+1)$-free poset. All lollipop graphs are examples of incomparability graphs of $(3+1)$-free posets. Before we present the needed background on P-tableaux we will set up our proof. First we will need the coefficient $[s_{(1^{m+n})}]X_{L_{m,n}}$.

Proposition 5.1. *We have $[s_{(1^{m+n})}]X_{L_{m,n}} = 2^n m!$.*

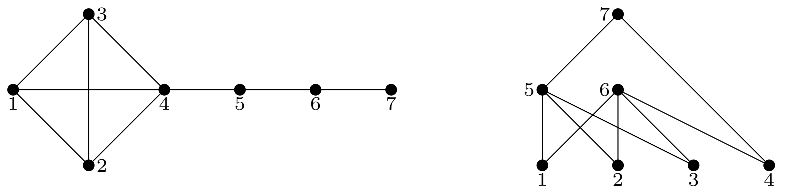


Fig. 4. On the left we have $L_{4,3}$ and on the right we have the Hasse diagram of its associated poset $\mathcal{P}_{4,3}$.

Proof. Using Theorem 3.15 and equation (6) we have that

$$[s_{(1^{m+n})}]X_{L_{m,n}} = (-1)^{m+n}\chi_{L_{m,n}}(-1) = 2^n m!,$$

which completes the proof. $\square$

To prove that the lollipops form a chain it suffices to show that $L_{m,n} \geq_s L_{m+1,n-1}$ for any $m \geq 2$ and $n \geq 1$. Using Proposition 5.1 this is equivalent to showing that

$$X_s(L_{m,n}, L_{m+1,n-1}) = X_{L_{m,n}} - \frac{2}{m+1}X_{L_{m+1,n-1}}$$

is Schur-positive. It suffices to show that

$$2 \cdot [s_\lambda]X_{L_{m+1,n-1}} \leq (m+1) \cdot [s_\lambda]X_{L_{m,n}}.$$

Now we will introduce P-tableaux in the case of lollipops. See [10] for more details. Consider the poset $\mathcal{P}_{m,n}$ on $[m+n]$ where $i \leq_{\mathcal{P}_{m,n}} j$ if and only if $i \leq j$ and $ij \notin E(L_{m,n})$. The lollipop $L_{m,n}$ is the incomparability graph of $\mathcal{P}_{m,n}$. See Fig. 4. A *P-tableau* of shape $\lambda \vdash m+n$ for $L_{m,n}$ is a filling of the Young diagram of λ. We will use the convention of drawing our Young diagrams of shape $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{l(\lambda)})$ so that row 1 is at the top with λ_1 boxes, row $l(\lambda)$ is at the bottom with $\lambda_{l(\lambda)}$ boxes and all rows are left-justified. We will number columns from left to right. We fill the Young diagram with $[m+n]$ so that:

1. The rows are increasing with respect to the poset $\mathcal{P}_{m,n}$.
2. There are no adjacent decreases along the columns with respect to the poset $\mathcal{P}_{m,n}$.

This means that if i appears to the left of j in the same row then $i <_{\mathcal{P}_{m,n}} j$ and if i appears immediately above j in the same column then $i \not>_{\mathcal{P}_{m,n}} j$. Let $\mathcal{T}_{\lambda,m,n}$ be the collection of all P-tableau for $L_{m,n}$ of shape λ. Gasharov's result states that

$$[s_\lambda]X_{L_{m,n}} = \#\mathcal{T}_{\lambda,m,n}$$

so we are done if we can show that

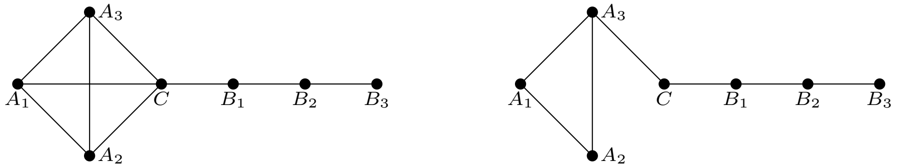


Fig. 5. For $m = 3$ and $n = 4$ on the left we have $L_{m+1,n-1}$ and on the right we have $L_{m,n}$ using the vertex labels $\mathcal{A}$, $\mathcal{B}$ and C.

$$2 \cdot \#\mathcal{T}_{\lambda,m+1,n-1} \leq (m+1) \cdot \#\mathcal{T}_{\lambda,m,n}.$$

We will show the inequality above by defining an injection

$$f_\lambda : [2] \times \mathcal{T}_{\lambda,m+1,n-1} \to [m+1] \times \mathcal{T}_{\lambda,m,n}.$$

To more easily compare P-tableaux for $L_{m,n}$ and $L_{m+1,n-1}$ we will relabel the vertices $[m+n]$ with $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$, C and $\mathcal{B} = \{B_1, B_2, \ldots, B_{n-1}\}$. In $L_{m+1,n-1}$ the vertices in $\mathcal{A}$ are those in K_{m+1} not adjacent to the path, C is the vertex in K_{m+1} adjacent to the path and the path is labeled with $\mathcal{B}$ so that the smaller subscripts are closer to C. In $L_{m,n}$ the vertices in $\mathcal{A} - \{A_m\}$ are those in K_m not adjacent to the path, A_m is the vertex in K_m adjacent to the path and the path is labeled with $\mathcal{B} \cup \{C\}$ so that C is adjacent to A_m and the smaller subscripts in $\mathcal{B}$ are closer to C. See Fig. 5.

With these labels we can now more specifically describe the P-tableaux, whose proof follows by definition.

Lemma 5.2. *The following are the rules for adjacent cells in the P-tableaux of $L_{m+1,n-1}$.*

1. *Let X be immediately left of Y.*
   (i) *If $X \in \mathcal{A}$ then $Y \in \mathcal{B}$.*
   (ii) *If $X = C$ then $Y \in \mathcal{B} - \{B_1\}$.*
   (iii) *If $X = B_i$ then $Y = B_j$ with $j \geq i + 2$.*
2. *Let X be immediately above of Y.*
   (i) *If $X \in \mathcal{A}$ then Y can have any label.*
   (ii) *If $X = C$ then Y can have any label.*
   (iii) *If $X = B_i$, $i \geq 1$, then $Y = B_j$ with $j \geq i - 1$ considering $C = B_0$.*

We now have all the background we need to define our injection f_λ, but in order to make our argument smoother we will first establish some facts about P-tableaux in the following structure lemma. See Fig. 7 for a concrete illustration.

Lemma 5.3. *We have the following for $T \in \mathcal{T}_{\lambda,m+1,n-1}$.*

(a) *All of the fillings from $\mathcal{A} \cup \{C\}$ appear in the first column.*

(b) *The tableau* $T \notin \mathcal{T}_{\lambda,m,n}$ *if and only if* C *appears directly above some* A_j *with* $j \in [m-1]$.
(c) *Column 1 of* T *is composed of, reading from top to bottom,* $\mathcal{A}'$, $\mathcal{B}'$, C, $\mathcal{A}''$ *and* $\mathcal{B}''$ *where* $\mathcal{A}'$ *and* $\mathcal{A}''$ *are contiguous blocks of cells with fillings from* $\mathcal{A}$ *and* $\mathcal{B}'$ *and* $\mathcal{B}''$ *are contiguous blocks of cells containing fillings from* $\mathcal{B}$*, any of which could possibly be empty. Particularly, if* $\mathcal{B}'$ *is non-empty then* $\mathcal{B}'$ *contains* $\{B_1, B_2, \ldots, B_s\}$ *with subscripts increasing as you go up in* T.
(d) *If* B_s *appears in column* c *of* T *and* B_{s+1} *appears in column* $c+1$ *below* B_s *then above and including* B_{s+1} *appear all of* B_i, $i \in [s+1, s+L]$ *for some* $L > 1$*, in a contiguous block of cells with subscripts increasing as we go up in* T*. To the immediate right of* B_s *we have one of the* B_i, $i \in [s+1, s+L]$*. Additionally, any* B_i *above* B_{s+L} *in column* $c+1$ *has* $i < s$ *and any* B_i *below* B_{s+1} *in column* $c+1$ *has* $i > s+L$.

Proof. Part (a) follows from the fact that $\mathcal{A} \cup \{C\}$ are minimal elements in $\mathcal{P}_{m+1,n-1}$.

Part (b) comes from the fact that $L_{m,n}$ and $L_{m+1,n-1}$ have the exact same edges except $L_{m,n}$ is missing the edges between the vertices in $\mathcal{A}-\{A_m\}$ and C. This means that if $T \in \mathcal{T}_{\lambda,m+1,n-1}$ then the only relations that could disrupt $T \in \mathcal{T}_{\lambda,m,n}$ is $C >_{\mathcal{P}_{m,n}} A_i$ for all $i \in [m-1]$. Meaning that $T \notin \mathcal{T}_{\lambda,m,n}$ if and only if C is above A_j for some $j \in [m-1]$.

Part (c) follows from Lemma 5.2. The breakdown into contiguous blocks follows from the fact that directly below a filling from $\mathcal{B}$ we can only have another filling from $\mathcal{B}$ or the filling C. After this filling C we could have some fillings from $\mathcal{A}$, but as soon as there is one more filling from $\mathcal{B}$ then we only have fillings from $\mathcal{B}$. Now we will use the labels for the contiguous blocks of cells described in part (c) of this lemma. From the information we have we can conclude exactly what the fillings of $\mathcal{B}'$ are if $\mathcal{B}'$ is non-empty. Because $\mathcal{B}'$ is above the filling C by Lemma 5.2 the only possibility is that it contains $\{B_1, B_2, \ldots, B_s\}$ with subscripts increasing as you go up in T.

For part (d) assume that B_s appears in column c of T and B_{s+1} appears in column $c+1$ below B_s in a lower row. We know for sure by Lemma 5.2 that to the right of B_s we have B_t where $t \geq s+2$. The only way to have B_t in column $c+1$ somewhere above B_{s+1} is to have $t > s+1$ and for there to be all of the B_i, $i \in [s+1, t]$, between B_{s+1} and B_t with subscripts increasing as we go up in T. This pattern of fillings from $\mathcal{B}$ with increasing subscripts may continue beyond B_t in consecutive cells until some highest B_{s+L}. Because we know the placement for all B_i for $i \in [s, s+L]$ by Lemma 5.2 we can conclude that all B_i above B_{s+L} in column $c+1$ have $i < s$ and any B_i below B_{s+1} in column $c+1$ have $i > s+L$. □

Now we will define our injection $f_\lambda : [2] \times \mathcal{T}_{\lambda,m+1,n-1} \to [m+1] \times \mathcal{T}_{\lambda,m,n}$. Let $T \in \mathcal{T}_{\lambda,m+1,n-1}$ and $k \in [2]$.

Case 1: Say $T \in \mathcal{T}_{\lambda,m,n}$ and $k \in [2]$. We map

$$f_\lambda(k, T) = (m-1+k, T).$$

$$\left(2,\ \begin{array}{|c|c|}\hline C & B_2 \\ \hline A_3 & B_1 \\ \hline A_1 & B_3 \\ \hline A_2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\right) \mapsto \left(4,\ \begin{array}{|c|c|}\hline C & B_2 \\ \hline A_3 & B_1 \\ \hline A_1 & B_3 \\ \hline A_2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\right) \qquad \left(1,\ \begin{array}{|c|c|}\hline C & B_2 \\ \hline A_2 & B_1 \\ \hline A_1 & B_3 \\ \hline A_3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\right) \mapsto \left(2,\ \begin{array}{|c|c|}\hline C & B_2 \\ \hline A_3 & B_1 \\ \hline A_1 & B_3 \\ \hline A_2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\right)$$

Fig. 6. We have two examples for $f_\lambda : [2] \times \mathcal{T}_{\lambda,4,3} \to [4] \times \mathcal{T}_{\lambda,3,4}$ where $\lambda = (2, 2, 2, 1)$. On the left we have an example from Case 1 and on the right we have an example from Case 2.

See Fig. 6 for an example. This case is clearly well defined and injective. Additionally, note that the first coordinate of the output is either m or $m + 1$. We will see in Case 2 and Case 3 of our map that the first coordinate of the output will be at most $m - 1$, so will not intersect Case 1.

Case 2: Say that instead $T \notin \mathcal{T}_{\lambda,m,n}$ and the first coordinate of our input is $k = 1$. By Lemma 5.3 (a) and (b) we know that C must appear in the first column of T directly above A_j for some $j \in [m - 1]$. Let T' be T but we switch A_j and A_m. Because A_j and A_m have the same relations in $\mathcal{P}_{m+1,n-1}$ certainly $T' \in \mathcal{T}_{\lambda,m+1,n-1}$. Since C is now immediately above A_m we can conclude by Lemma 5.3 (b) that $T' \in \mathcal{T}_{\lambda,m,n}$. We map

$$f_\lambda(1, T) = (j, T').$$

See Fig. 6 for an example. Note in this case that the first coordinate of the output is at most $m - 1$, which means outputs from Case 2 do not intersect with outputs from Case 1. Also note in all outputs from this case that C appears directly above A_m in T'. This guarantees that we are injective. We will show in Case 3 that our outputs do not have C immediately above A_m.

Case 3: Say that we still have $T \notin \mathcal{T}_{\lambda,m,n}$ but the first coordinate of our input is now $k = 2$. Again by Lemma 5.3 (a) and (b) we know that C appears in the first column of T directly above A_j for some $j \in [m - 1]$. This case will be more complicated, but we will be mapping

$$f_\lambda(2, T) = (j, T'')$$

for some P-tableau $T'' \in \mathcal{T}_{\lambda,m,n}$ whose construction we describe next. We will describe the construction in steps: first what we will call an $\mathcal{A}$-shift, then some column $\mathcal{B}$-shifts.

The $\mathcal{A}$-shift: According to Lemma 5.3 (c) we can decompose the first column of T as in Fig. 7 where $\mathcal{A}'$ and $\mathcal{A}''$ are contiguous blocks of cells containing vertices from $\mathcal{A}$ and $\mathcal{B}'$ and $\mathcal{B}''$ are contiguous blocks of cells containing vertices from $\mathcal{B}$, any of which could possibly be empty. We will shift the blocks of cells containing $\mathcal{B}'$ and C down below A_j and $\mathcal{A}''$ which we shift up as displayed in Fig. 7. We will call this move the $\mathcal{A}$-*shift* and the new tableau formed T'. We can see that column 1 in T' satisfies the conditions necessary in order to be a P-tableau for $L_{m+1,n-1}$ by Lemma 5.3 (c). However, we are not guaranteed that T' is a P-tableau for $L_{m+1,n-1}$ because of possible issues between columns 1 and 2. From Lemma 5.3 (c) we know that $\mathcal{B}'$, if nonempty, contains fillings

$B_1, B_2, \ldots, B_s$ with subscripts increasing as we go up. So our $\mathcal{A}$-shift moves around fillings from $\mathcal{A} \cup \{C, B_1, B_2, \ldots, B_s\}$. Because all $B_1, B_2, \ldots, B_s$ are in column 1 we can conclude that all of $\mathcal{A} \cup \{C, B_1, B_2, \ldots, B_s\}$ share the same relations with any possible filling from column 2 (except the relation between B_s and B_{s+1}), so the $\mathcal{A}$-shift preserves all properties we need in order to be a P-tableau for $L_{m+1,n-1}$ except in the case where B_s gets shifted down to be left of B_{s+1}. If that is not the case let $T' = T''$ and we are done.

The column 2 $\mathcal{B}$-shift: Now consider the unfortunate case where B_s gets shifted down to be left of B_{s+1}. In this case we know in T that B_s is in column 1 row r and B_{s+1} is in column 2 below row r in row $r + a$ where $a = \#(\mathcal{A}'' \cup \{A_j\})$. By Lemma 5.3 (d) we can conclude that above and including B_{s+1} in column 2 we have all of B_i for $i \in [s + 1, s + r + a]$ in a contiguous block of cells with subscripts increasing as we go up and B_{s+r+a} in row 1. We will vertically cycle the block of cells containing B_i for $i \in [s + 1, s + r + a]$ so that B_{s+1} is in row r (immediately right of where B_s was originally in T). Call this the *column 2 $\mathcal{B}$-shift*. Using the last parts of Lemma 5.3 (d) we are guaranteed that the first two columns satisfying the conditions needed to be a P-tableau for $L_{m+1,n-1}$. If the resulting tableau happens to additionally be a P-tableau for $L_{m+1,n-1}$ then this P-tableau is our T''.

The column 3 $\mathcal{B}$-shift and further column $\mathcal{B}$-shifts: Otherwise by similar reasons as before to the right of B_{s+r+a} after the column 2 $\mathcal{B}$-shift we have $B_{s+r+a+1}$, which is in row $r + 1$. Also similar to before, above and including $B_{s+r+a+1}$ we have all of B_i for $i \in [s + r + a + 1, s + 2r + a + 1]$ with subscripts increasing as we go up and $B_{s+2r+a+1}$ is in row 1. We will vertically cycle the block of cells containing B_i for $i \in [s + r + a + 1, s + 2r + a + 1]$ so that $B_{s+r+a+1}$ is in row 1 (immediately right of where B_{s+r+a} was originally in T). Call this the *column 3 $\mathcal{B}$-shift*. We will continue doing these column $\mathcal{B}$-shifts until we arrive at a P-tableau of $L_{m+1,n-1}$. Note that it is straightforward to see that this will always terminate successfully as follows. We can always do a $\mathcal{B}$-shift unless some column c only contains one cell, which is in the top row. Let this c be minimal. In this case, the previous $\mathcal{B}$-shift in column $c - 1$ will have replaced the B_i that was in its top row with some $B_{i'}$ where $i' < i$ and hence we will not need to perform a $\mathcal{B}$-shift in column c. This P-tableau we created is T''. See Fig. 7 for an example.

During our construction of T'' we have shown that $T'' \in \mathcal{T}_{\lambda,m+1,n-1}$. Because the filling immediately below C in T'' is either non-existent or a filling from $\mathcal{B}$ we know by Lemma 5.3 (b) that $T'' \in \mathcal{T}_{\lambda,m,n}$, so our map is well defined. Recall that in this case we mapped $f_\lambda(2, T) = (j, T'')$ where $j \leq m - 1$, so we have in the output the filling C in T'' not immediately above A_m. This means that Case 3 outputs do not intersect with those from Case 1 or Case 2.

Lastly, we only have to argue why Case 3 is injective. Case 3 is injective because if $f_\lambda(2, T) = (j, T'')$ comes from Case 3 we can recover T from (j, T''). Note that because of the $\mathcal{A}$-shift T'' has its first column as follows reading from top to bottom: a contiguous block of cells containing all of $\mathcal{A}$ in some order, a contiguous block of cells $\mathcal{B}'$ with fillings

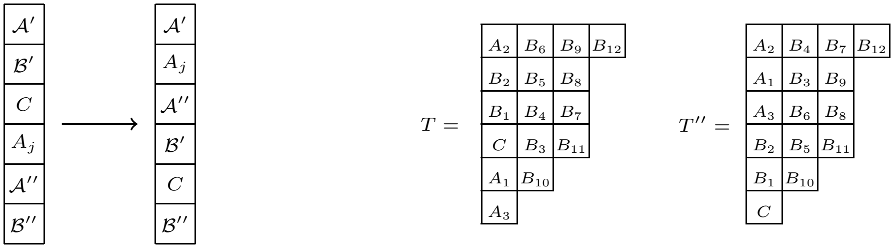


Fig. 7. On the left we illustrate the $\mathcal{A}$-shift in general by drawing only the first column. On the right we display T and T'' in an example for Case 3 in the map $f_\lambda : [2] \times \mathcal{T}_{\lambda,4,12} \to [4] \times \mathcal{T}_{\lambda,3,13}$ where $\lambda = (4,3,3,3,2,1)$ and $f_\lambda(2,T) = (1,T'')$.

from $\mathcal{B}$, C and finally another contiguous block of cells $\mathcal{B}''$ with fillings from $\mathcal{B}$. Because j is specified we can split the top contiguous block of cells containing everything from $\mathcal{A}$ into $\mathcal{A}'$, those above A_j, and $\mathcal{A}''$, those below A_j. This allows us to undo the $\mathcal{A}$-shift. Let $\tilde{T}'$ be T'' with the $\mathcal{A}$-shift undone. Because of how we defined the column 2 $\mathcal{B}$-shift, we know we did a column 2 $\mathcal{B}$-shift if there is some B_s in column 1 followed immediately to its right by B_{s+1}. Note that in $\tilde{T}'$ the occurrence of a B_s in column 1 followed immediately to its right by B_{s+1} will only happen because of a $\mathcal{B}$-shift. If we can identify the block of cells we cycled in the column 2 $\mathcal{B}$-shift then we will be able to undo this shift. This block will contain the cell B_{s+1}, the cell below B_{s+1} filled with some B_{s+t}, $t > 1$, and all other cells containing B_i, $i \in [s+l, s+t]$ for some smallest $l \leq t$ with subscripts decreasing consecutively as we go down. The block of cells continues above B_{s+1} up until we reach row 1 with subscripts increasing consecutively as we go up. Now that we have identified all cells in the block that we cycled in the column 2 $\mathcal{B}$-shift we can undo the column 2 $\mathcal{B}$-shift by cycling vertically until B_{s+1} is at the bottom. We can similarly undo all other column $\mathcal{B}$-shifts, each of which will be indicated by some B_i in column c with a B_{i+1} to its immediate right. Since we have recovered T from (j, T'') we have proven that Case 3 is injective and can further conclude that f_λ is injective.

Theorem 5.4. *The family of lollipop graphs $\{L_{m,n}\}$ on $m+n = N$ vertices forms a chain in $\mathcal{S}_N$. In particular, for $m \geq 3$ and $n \geq 0$,*

$$L_{m-1,n+1} \geq_s L_{m,n}.$$

Hence, the path P_N is the maximal element and the complete graph K_N is the minimal element of the chain.

6. Further directions

There are many more properties to study in these posets including a description for all maximal, minimal and independent elements. While we have shown both in $\mathcal{E}_n$ and $\mathcal{S}_n$ that the complete graph K_n is a minimal element and that trees are maximal elements,

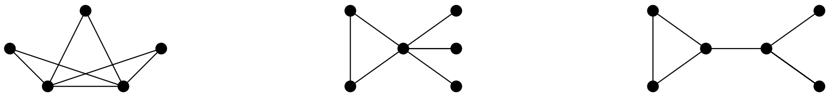


Fig. 8. In both $\mathcal{E}_5$ and $\mathcal{S}_5$ the graph on the left is a non-independent minimum. In both $\mathcal{E}_6$ and $\mathcal{S}_6$ the graph in the middle is independent and the graph on the right is a non-independent maximum.

these are not the only minimal and maximal elements in these posets. Three particular examples are given in Fig. 8.

One motivation behind setting up these posets was to study e-positivity and Schur-positivity of chromatic symmetric functions. More is known about Schur-positivity of chromatic symmetric functions since Gasharov [10] has proven that incomparability graphs of $(3+1)$-free posets are Schur-positive, but it has yet to be proven that the same class of graphs are all e-positive. Guay-Paquet [14] has reduced the question to only needing to show that incomparability graphs of $(2+2)$ and $(3+1)$-free posets are e-positive, a family of graphs better known as unit interval graphs. Since we have proven that G is e-positive if and only if $G \geq_e K_n$ in $\mathcal{E}_n$ in Theorem 2.22 the poset $\mathcal{E}_n$ gives us another approach to e-positivity. We can now show G is e-positive by finding and proving a sequence of inequalities $G \geq_e G_1 \geq_e G_2 \geq_e \cdots \geq_e G_l \geq_e K_n$. For example the inequalities proven in Theorem 3.17 prove that lollipop graphs are e-positive, although, lollipops have been proven to be e-positive previously by other methods [7,12].

Below we conjecture many inequalities between connected unit interval graphs in $\mathcal{E}_n$ and in $\mathcal{S}_n$. These inequalities can be placed in series to show any connected unit interval graph is e-positive and Schur-positive, though Gasharov [10] has proven all unit interval graphs are Schur-positive. First we need a description of unit interval graphs. There are many equivalent definitions for unit interval graphs with some equivalences proven in [8]. Here we will describe *unit interval graphs* on vertices in $[n]$ using a weakly-increasing sequence $\mathbf{m} = (m_1, m_2, \ldots, m_{n-1})$ where $i \leq m_i \leq n$ for all $i \in [n-1]$. The graph will have an edge between a and b whenever $a, b \in [i, m_i]$ for some i.

Conjecture 6.1. *Let G be a connected unit interval graph defined by the weakly-increasing sequence $\mathbf{m} = (m_1, m_2, \ldots, m_{n-1})$, $i \leq m_i \leq n$ for all $i \in [n-1]$. Let G' be the unit interval graph defined by the sequence $\mathbf{m}' = (m_1 + 1, \ldots, m_r + 1, m_{r+1}, \ldots, m_{n-1})$ where $m_r < m_{r+1}$ is the first increase. Then,*

$$G \geq_e G' \text{ and } G \geq_s G'.$$

Using deletion-contraction we can compute the chromatic polynomial for all unit interval graphs and the coefficients $[e_{(n)}]X_G$ and $[s_{(1^n)}]X_G$.

Proposition 6.2. *Let G be a unit interval graph defined by the weakly-increasing sequence $\mathbf{m} = (m_1, m_2, \ldots, m_{n-1})$, $i \leq m_i \leq n$ for all $i \in [n-1]$. Then,*

$$\chi_G(k) = k \prod_{i=1}^{n-1} (k - (m_i - i)).$$

Also,

$$[e_{(n)}]X_G = n \prod_{i=1}^{n-1} (m_i - i) \text{ and } [s_{(1^n)}]X_G = \prod_{i=1}^{n-1} (m_i - i + 1).$$

Proof. Let G be a unit interval graph defined by the weakly-increasing sequence $\mathbf{m} = (m_1, m_2, \ldots, m_{n-1})$, $i \leq m_i \leq n$ for all $i \in [n-1]$. The formula for the chromatic polynomial will follow from deletion-contraction and induction. Our base case is when $n = 1$ and $\mathbf{m}$ is an empty list. In this case $\chi_G = k$, which matches the formula. Using deletion-contraction repeatedly on all edges connected to vertex 1 we get

$$\chi_G = \chi_{G'} - (m_1 - 1)\chi_{G''},$$

where G' and G'' are the graphs associated to $\mathbf{m}' = (1, m_2, \ldots, m_{n-1})$ and $\mathbf{m}'' = (m_2 - 1, m_3 - 1, \ldots, m_{n-1} - 1)$, respectively. Note that G' is G'', but with an additional disjoint vertex. This means $\chi_{G'} = k\chi_{G''}$. Then,

$$\chi_G = \chi_{G''}(k - (m_1 - 1)),$$

so by induction we have our formula.

By equation (5) we get that $[e_{(n)}]X_G = n \prod_{i=1}^{n-1} (m_i - i)$. By equation (6) we get that $[s_{(1^n)}]X_G = \prod_{i=1}^{n-1} (m_i - i + 1)$. $\square$

Remark 6.3. Since we can form a sequence of connected unit interval graphs from any connected unit interval graph to the complete graph, as described in Conjecture 6.1, proving Conjecture 6.1 would imply that all unit interval graphs are e-positive. The conjecture has been confirmed up until $n = 7$.

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