# Minimal spanning trees, Travelling salespeople, and together

Steph van Willigenburg
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# Minimal spanning tree (MST)

## Definition

Let $G$ be a connected weighted graph. A spanning tree of $G$ with smallest sum of edge weights is called a minimal spanning tree of $G$.

Example

$G =$

The minimal spanning tree of $G$ is

# Kruskal's algorithm for MST

Given a connected weighted graph $G$.

1. Write down the vertices of $G$. Add an edge of lowest weight.
2. Continue to add edges of lowest weight, never making a cycle.
3. Stop when all vertices are connected.

Note: Break ties arbitrarily.

Example

# Travelling salesperson problem (TSP)

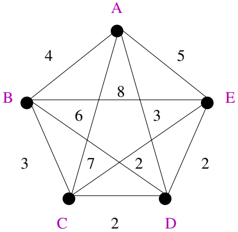
Given a network of $n$ connected cities, to visit once and only once and then return home, what is the minimum distance to travel?

OR

Given a weighted complete graph $K_n$, which Hamiltonian cycle has the sum of edge weights being minimal?

Note: No algorithm exists that does better than check all cycles.

Example



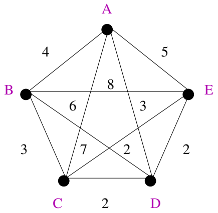

A minimal weight of   is given by

# Computing a lower bound: TSP meets MST

Given a weighted complete graph, $K_n$.

1. Choose a vertex $v$ and delete it.
2. Compute an MST for $K_{n-1} = K_n - v$.
3. Note the two smallest edge weights on edges coming from $v$.
4. Sum the edge weights found in 2 and 3.

Example $\quad 1+2$ : Delete $D$ and get MST




3 : Two smallest weights coming from $D$:

4 : Minimal weight $\geq$

# Why does this work?

If we take any Hamiltonian cycle in a weighted $K_n$ and delete a vertex v, then we are left with a spanning tree of $K_{n-1} = K_n - v$.

So

(Sum of weights in MST for $K_{n-1}$)
+ (2 smallest edge weights meeting at $v$)

$$\leq$$

(Sum of weights in span. tree for $K_{n-1}$, part of TSP solution)
+ (2 edge weights meeting at $v$, part of TSP solution).

# Bonus - directed graphs

## Definition

A directed graph (or digraph) $D$ has vertices $V(D)$ and directed arcs $A(D)$ connecting the vertices. It is simple if

- finite vertices
- no arc joining vertex to itself
- not multiple arcs going in same direction.

Note: Take directions from arcs in $D$ gives underlying graph $U(D)$. Add directions to edges in graph $G$ gives orientation of $G$, $D_G$.

Example

$D =$ is a simple digraph with $U(D) =$

# Bonus - directed graphs

Many concepts are similar to graphs, e.g. a directed path/cycle is a path/cycle whose directions you follow.

## Definition

Let $D$ be a digraph. The number of arc ends meeting at a vertex $v$ is called the degree, $deg(v)$. Arcs coming in is the indegree, $indeg(v)$. Arcs going out is the outdegree, $outdeg(v)$.

Note: $indeg(v) = 0$ is a source, $outdeg(v) = 0$ is a sink.

Example

$D =$ has $indeg(v) =$ , $outdeg(v) =$ , $deg(v) =$

# In summary

- Minimal spanning tree
- Kruskal's algorithm
- Travelling salesperson problem
- Applying minimal spanning trees to find a lower bound for the travelling salesperson problem.
- Bonus - primer on directed graphs

Thanks! See you next time!