# Spanning trees, Cayley's theorem, and Prüfer sequences

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# Spanning trees

## Definition

Let $G$ be a connected graph. A subgraph of $G$ that involves all the vertices of $G$ and is a tree is called a spanning tree of $G$. The number of spanning trees is $\tau(G)$.

Example

$G =$

$\tau(G) =$

# Labelled trees

## Definition

A tree $T$ is a labelled tree if each vertex has a label attached to it. The labels on $n$ vertices are usually $1, 2, \ldots n$. Two labelled trees are isomorphic if the isomorphism preserves the labelling.

Example

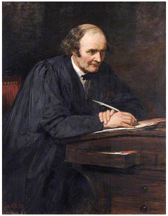
# Cayley's theorem

### Theorem

*There are*

$$n^{(n-2)}$$

*non-isomorphic labelled trees with* $n \geq 2$ *vertices.*



Wikimedia

Example

To prove Cayley's theorem we need Prüfer sequences.

# Prüfer sequences from trees

Given a labelled tree $T$.

1. Choose leaf with smallest label, $v$.
2. Put the label adjacent to $v$ in the sequence.
3. Delete $v$ and repeat until only two vertices remain.

Note: A labelled tree has $n$ vertices iff its Prüfer sequence has length $n-2$.

Example ( , , , , , , )

# Trees from Prüfer sequences

Given a Prüfer sequence $P$ length $n-2$.

1. Make a list $L = (1, 2, \dots n)$. Draw $n$ vertices $(1, 2, \dots n)$.
2. Find smallest number $k$ in $L$ not in $P$.
3. Join vertex $k$ to the first number in $P$.
4. Delete and repeat. When 2 numbers are left in $L$, join them.

Note: This works for any sequence!

Example

$P = (4, 9, 4, 9, 5, 6, 5)$

$L = (1, 2, 3, 4, 5, 6, 7, 8, 9)$

# Trees from Prüfer sequences

Given a Prüfer sequence $P$ length $n-2$.

1. Make a list $L = (1, 2, \ldots n)$. Draw $n$ vertices $(1, 2, \ldots n)$.
2. Find smallest number $k$ in $L$ not in $P$.
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4. Delete and repeat. When 2 numbers are left in $L$, join them.

Note: This works for any sequence!

Example

# Proof of Cayley's theorem

We give a sketch. Our real proof takes about 45 mins.

1. We have generating Prüfer sequences from trees.
2. We have generating trees from Püfer sequences, which reverses it (small proof).
3. We then do an induction to show that any sequence of length $n-2$ produces a labelled tree on $n$ vertices (big proof).
4. The number of such sequences, length $n-2$, on labels $1, 2, \ldots n$ is then $n^{(n-2)}$ since

$$( \quad , \quad , \ldots , \quad ).$$

# In summary

- Spanning trees
- Labelled trees and graphs
- Cayley's theorem
- Prüfer sequences from trees
- Trees from Prüfer sequences

Thanks! See you next time!